Convex hull of bounded monomials on two-variable cones

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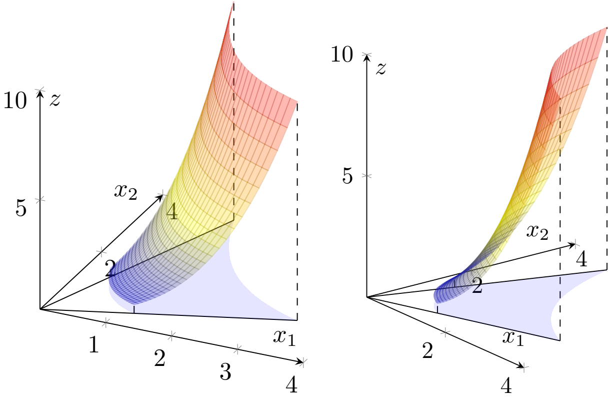
Problem

Find the convex hull of

$$F = \{(x_1, x_2, z) \in \mathbb{R}^3_+ : z = x_1^{a_1} x_2^{a_2}, \ell \le z \le u, p x_1 \le x_2 \le q x_1\},$$

with a_1, a_2 positive, $0 \le \ell < u$, and $0 < p < q$.



Motivation

$$
\begin{array}{lll}
\min & f_0(\boldsymbol{x}) & \\
\text{s.t.} & f_j(\boldsymbol{x}) \leq b_j & \forall j = 1, 2, \ldots, m \\
& \ell_i \leq x_i \leq u_i & \forall i = 1, 2, \ldots, n \\
& \boldsymbol{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p} &
\end{array}
$$

with f_j nonlinear, possibly nonconvex but *factorable* (i.e., "non-blackbox"), are solved to global optimality by

1. reformulation
2. branch-and-bound

Reformulation of a single (polynomial) constraint

Reformulate $S = \{\boldsymbol{x} : \sum_{j \in J} b_j \prod_{i \in I_j} x_i^{a_{ji}} \le b_0\}$: add *auxiliaries* y, t.

$$\sum_{j \in J} b_j y_j \le b_0$$
$$y_j = \prod_{i \in I_j} t_{ji}$$
$$t_{ji} = x_i^{a_{ji}}$$

Then a convex relaxation is $R = R' \cap \left(\bigcap_{j \in J} R''_j\right) \cap \left(\bigcap_{i \in I_j, j \in J} R'''_{ji}\right)$

$$\begin{array}{lcll} R' & = & \{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) : \sum_{j \in J} b_j y_j \le b_0\} & \\ R''_j & \supseteq & \mathrm{conv}(\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) : y_j = \prod_{i \in I_j} t_{ji}\}) & \forall j \in J \\ R'''_{ji} & \supseteq & \mathrm{conv}(\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}) : t_{ji} = x_i^{a_{ji}}\}) & \forall i \in I_j, \quad \forall j \in J \end{array}$$

R may not be the convex hull of S even if R'' and R''' are tightest

Example: $x_1^2 + 2x_1x_2 + x_2^2 \le 1$.

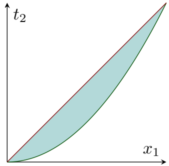
$$2x_1x_2^3 + 3x_1x_2 - 4x_1^2x_2^6 \leq 2$$

with $x_1 \in [0,2]$ and $x_2 \in [0,1]$, is reformulated as

$$\begin{array}{lll} 2y_1 + 3y_2 - 4y_3 \leq 2 & & \\ y_1 = x_1 t_1 & y_2 = x_1 x_2 & y_3 = t_2 t_3 \\ t_1 = x_2^3 & t_2 = x_1^2 & t_3 = x_2^6 \end{array}$$

A convex relaxation is

$$\begin{array}{ll} 2y_1 + 3y_2 - 4y_3 \leq 2 & \\ t_1 \geq x_2^3 & t_1 \leq x_2 \\ t_2 \geq x_1^2 & t_2 \leq 2x_1 \\ t_3 \geq x_2^6 & t_3 \leq x_2 \end{array}$$



plus all *McCormick inequalities* for $y_1 = x_1 t_1$, $y_2 = x_1 x_2$, and $y_3 = t_2 t_3$.

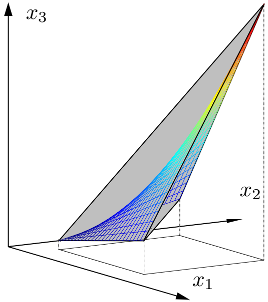
Known convex hulls

McCormick inequalities

$$B = \{(x_1, x_2, z) : z = x_1 x_2, x_1 \in [\ell_1, u_1], x_2 \in [\ell_2, u_2]\}$$

The convex hull is (McCormick '76, Al-Khayyal & Falk '83)

$$\begin{aligned} H = \{(x_1, x_2, z) :& \\ z \geq \ell_2 x_1 + \ell_1 x_2 - \ell_2 \ell_1 & \\ z \geq u_2 x_1 + u_1 x_2 - u_2 u_1 & \\ z \leq \ell_2 x_1 + u_1 x_2 - \ell_2 u_1 & \\ z \leq u_2 x_1 + \ell_1 x_2 - u_2 \ell_1 \} & \end{aligned}$$



With bounds on z in a bilinear set, i.e.,

$$B' = \{(x_1, x_2, z) : z = x_1 x_2, x_1 \in [\ell_1, u_1], x_2 \in [\ell_2, u_2], z \in [\ell, u]\},$$

the McCormick inequalities alone don't give the convex hull.

- There are infinitely many additional inequalities[1] for the *upper envelope* of B', i.e. of the form $z \leq c_1 x_1 + c_2 x_2 + c_0$;
- The convex hull of B' is the union of three sets[2], each a subset of a second-order cone, i.e. $\boldsymbol{c}^T \boldsymbol{x} + c_0 \geq ||G\boldsymbol{x} + \boldsymbol{g}||_2$.

[1] P. B., A. J. Miller, and M. Namazifar. Valid inequalities and convex hulls for multilinear functions. Electronic Notes in Discrete Math. 36:805–812, 2010.

[2] K. M. Anstreicher, S. Burer, and K. Park. Convex hull representations for bounded products of variables. J. of Global Optimization, 80:757–778, 2021.

It would be really nice to find the convex hull of

$$B'' = \{(x_1, x_2, z) \in \mathbb{R}^3_+ : z = x_1^{a_1} x_2^{a_2}, x_1 \in [\ell_1, u_1], x_2 \in [\ell_2, u_2], z \in [\ell, u]\},$$

for general a_1, a_2 (for $a_1, a_2 \geq 1$ see Nguyen et al. 2018[3]). Instead we'll talk about the convex hull of

$$F = \{(x_1, x_2, z) \in \mathbb{R}^3_+ : z = x_1^{a_1} x_2^{a_2}, px_1 \leq x_2 \leq qx_1, z \in [\ell, u]\}.$$

for $0 \leq \ell < u$ and $0 < p < q$.

[3] Nguyen, Trang T., Jean-Philippe P. Richard, and Mohit Tawarmalani, "Deriving convex hulls through lifting and projection," *Mathematical Programming* 169.2 (2018): 377-415.

Upper envelopes in $n \geq 2$ dimensions

(The easy part)

Definition

Given $T \subseteq \mathbb{R}^n$ and a function $f : T \to \mathbb{R}$,

- The *epigraph* of f in T is $\mathrm{epi}(f, T) = \{(\boldsymbol{x}, z) \in T \times \mathbb{R} : z \geq f(\boldsymbol{x})\}$
- The *hypograph* of f in T is $\mathrm{hyp}(f, T) = \{(\boldsymbol{x}, z) \in T \times \mathbb{R} : z \leq f(\boldsymbol{x})\}$.

Definition

Given $T \subseteq \mathbb{R}^n$ and a function $f : T \to \mathbb{R}$, the *lower envelope* $E_L(f, T)$ (resp. *upper envelope* $E_U(f, T)$) of f over T is the convex hull of the epigraph (resp. hypograph) of f in T.

Consider $n \geq 2$, $N = \{1, 2, \ldots, n\}$, and $f(\boldsymbol{x}) = \prod_{k \in N} x_k^{a_k}$. Define

$$F_0 = \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : z = f(\boldsymbol{x}), z \in [\ell, u]\}.$$

Also, define $\beta := \sum_{k \in N} a_i$ and

$$F_0^{\leq} = \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : z \leq f(\boldsymbol{x}), z \in [\ell, u]\}.$$

Fact: $f(\boldsymbol{x})$ is concave if $\beta \leq 1$, nonconvex & nonconcave otherwise.

Lemma

If $\beta \leq 1$, then $\mathit{conv}(F_0) = F_0^{\leq}$.

Upper envelope for F_0 if $\beta \geq 1$

For $\beta \geq 1$, consider the **cone**

$$\mathcal{K} = \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : (z - z_0)^\beta \leq \gamma \prod_{k \in N} x_k^{a_k}\}$$

(its vertex is $(\mathbf{0}, z_0)$) with z_0, γ such that

$$\{(\boldsymbol{x}, z) \in \mathcal{K} : z = \ell\} = \{(\boldsymbol{x}, z) \in F_0 : z = \ell\}$$
$$\{(\boldsymbol{x}, z) \in \mathcal{K} : z = u\} = \{(\boldsymbol{x}, z) \in F_0 : z = u\}$$

$$\Rightarrow \qquad z_0 = \frac{u^{\frac{1}{\beta}} \ell - \ell^{\frac{1}{\beta}} u}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}}, \quad \gamma = \left(\frac{u - \ell}{u^{\frac{1}{\beta}} - \ell^{\frac{1}{\beta}}} \right)^\beta .$$

Lemma

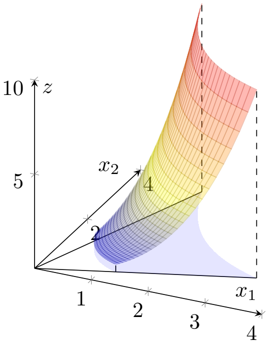
If $\beta \geq 1$, then $conv(F_0) = \{(\boldsymbol{x}, z) \in \mathcal{K} : \ell \leq z \leq u\}$.

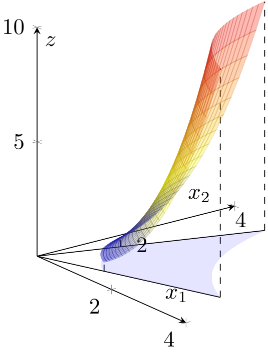
A two-variable cone appears!

For any $i, j \in N$ with $i \neq j$, consider the set

$$F = \{(\boldsymbol{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : z = f(\boldsymbol{x}), z \in [\ell, u], px_i \leq x_j \leq qx_i\}.$$

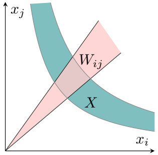
(bounded for $n = 2$, unbounded for $n > 2$).





$f(x_1, x_2) = x_1^{1.7} x_2^{1.5} \in [0.4, 10]$ with $p = 0.35,\ q = 3$.

Upper envelope for $n > 2$



Recall $f(\mathbf{x}) = \prod_{i \in N} x_i^{a_i}$.

Define $W_{ij} := \{\mathbf{x} \in \mathbb{R}^n_+ : p x_i \leq x_j \leq q x_i\}$

Define $X := \{\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) \in [\ell, u]\}$.

Lemma ($\beta \geq 1$)

The upper envelope of f on $W_{ij} \cap X$ is

$$H = \{(\mathbf{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : (z - z_0)^\beta \leq \gamma \prod_{k \in N} x_k^{a_k}, \; z \in [\ell, u], \; p x_i \leq x_j \leq q x_i\}.$$

Lemma ($\beta \leq 1$)

The upper envelope of f on $W_{ij} \cap X$ is

$$H = \{(\mathbf{x}, z) \in \mathbb{R}^n_+ \times \mathbb{R} : z \leq \prod_{k \in N} x_k^{a_k}, \; z \in [\ell, u], \; p x_i \leq x_j \leq q x_i\}.$$

$\Rightarrow$ Same as upper envelope of F_0 with $\{(\mathbf{x}, z) : p x_i \leq x_j \leq q x_i\}$ (unlike bounds on x_1, x_2).

Lower envelopes ($n \geq 2$)

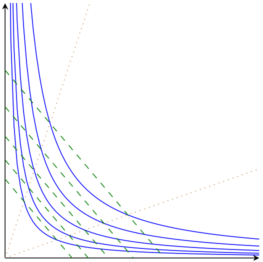
Level sets and two-variable cones

For any $\xi \in \mathbb{R}$, consider the level set $C_\xi := \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 x_2 = \xi\}$.

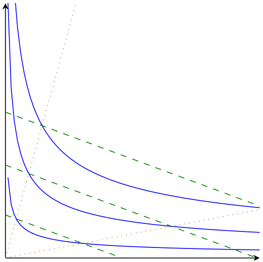
For p, q with $0 < p < q$, define $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(q)}$ as the intersections of C_ξ with $x_2 = p x_1$ and with $x_2 = q x_1$ respectively.

Property

The slope of the line through $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(q)}$ is independent of ξ.



$x_1 x_2, p = 0.35, q = 3$



$x_1^{0.3} x_2^{0.6}, p = 0.19, q = 3.6$

This generalizes to $n \geq 2$

Proposition

Given $\boldsymbol{a} \in \mathbb{R}^n_+$, $i, j \in N$ with $i \neq j$, and $p, q \in \mathbb{R}_+$ with $0 < p < q$, there exist $d_i < 0$ and $d_j > 0$ such that for any $\check{\boldsymbol{x}}$ that satisfies $\check{x}_j = p\check{x}_i$ and $\prod_{i \in N} \check{x}_i^{a_i} = \xi$, there exists a solution $(\bar{s}, \bar{\boldsymbol{x}}) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ to the nonlinear system

$$
\begin{aligned}
&\bar{x}_j = q\bar{x}_i \\
&(\bar{x}_i, \bar{x}_j) = (\check{x}_i + \bar{s}d_i, \check{x}_j + \bar{s}d_j) \\
&\bar{x}_k = \check{x}_k \quad \forall k \notin \{i, j\} \\
&\textstyle\prod_{k \in N} \bar{x}_k^{a_k} = \xi.
\end{aligned}
$$

$\Rightarrow$ **Bijection**: $S_p := \{\boldsymbol{x} : x_j = px_i, f(\boldsymbol{x}) = \xi\} \quad \longleftrightarrow \quad S_q := \{\boldsymbol{x} : x_j = qx_i, f(\boldsymbol{x}) = \xi\}$

Every pair $(\boldsymbol{x}^{(p)}, \boldsymbol{x}^{(q)}) \in S_p \times S_q$ is on a line with slope $(0, 0, \ldots, d_i, 0, \ldots, d_j, 0, \ldots, 0)$ independent of ξ.

Proof sketch

1. Replace $(\bar{x}_i, \bar{x}_j) = (\check{x}_i + \bar{s} d_i, \check{x}_j + \bar{s} d_j)$
   with $(\bar{x}_i, \bar{x}_j) = (\eta_i \check{x}_i, \eta_j \check{x}_j)$
2. η_i and η_j are such that $\frac{\eta_j}{\eta_i} = \frac{q}{p}$ and $\eta_i^{a_i} \eta_j^{a_j} = 1$

$\Rightarrow$ Find $\eta_i = (q/p)^{-\frac{a_j}{a_i + a_j}}$, $\eta_j = (q/p)^{\frac{a_i}{a_i + a_j}}$

3. Solving for $\bar{s}$ yields $\frac{d_i}{d_j} \frac{\eta_j - 1}{\eta_i - 1} = \frac{1}{p}$

$\Rightarrow$ Can find d_i and d_j independent of ξ $\square$

$$
\begin{aligned}
d_i &= q^{-a_j/(a_i+a_j)} - p^{-a_j/(a_i+a_j)} \\
d_j &= q^{a_i/(a_i+a_j)} - p^{a_i/(a_i+a_j)}.
\end{aligned}
$$

On the direction (d_i, d_j), the value of $f(\boldsymbol{x})$ is the same across the two half-lines $x_j = px_i$ and $x_j = qx_i$.

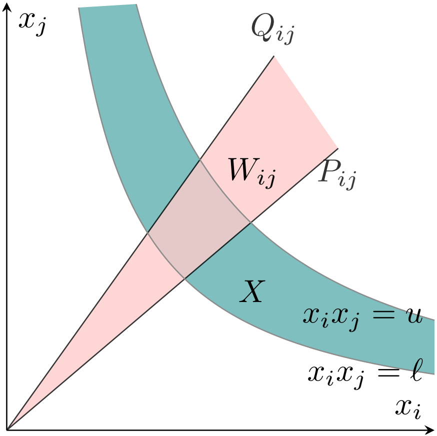
$\Rightarrow$ On an orthogonal direction $(d_j, -d_i)$, we can define a function whose level curves are precisely the lines with direction (d_i, d_j).

Some more notation

$$P_{ij} := \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_j = px_i\}$$
$$Q_{ij} := \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_j = qx_i\}$$
$$W_{ij} := \{\boldsymbol{x} \in \mathbb{R}^n_+ : px_i \leq x_j \leq qx_i\}$$

$$X := \{\boldsymbol{x} \in \mathbb{R}^n_+ : f(\boldsymbol{x}) \in [\ell, u]\}.$$

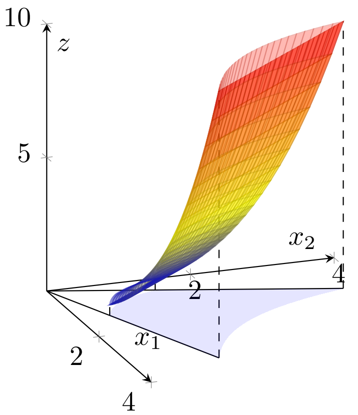
$$C_\xi := \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i x_j = \xi\}$$



Proposition

The function
$f'_\ell(\boldsymbol{x}) = \lambda(d_j x_i - d_i x_j)^{a_i + a_j} \prod_{k \in N \setminus \{i,j\}} x_k^{a_k}$, *with*
$\lambda = p^{a_j} / (d_j - d_i p)^{a_i + a_j}$,

1. *matches the value of* $f(\boldsymbol{x})$ *for* $\boldsymbol{x} \in P_{ij} \cup Q_{ij}$;
2. *is* $\leq f(\boldsymbol{x})$ *for* $\boldsymbol{x} \in W_{ij}$;
3. *is the lower envelope of* $f(\boldsymbol{x})$ *in* W_{ij} *for* $n = 2$ *and* $\beta = a_1 + a_2 \geq 1$.



Lower envelope

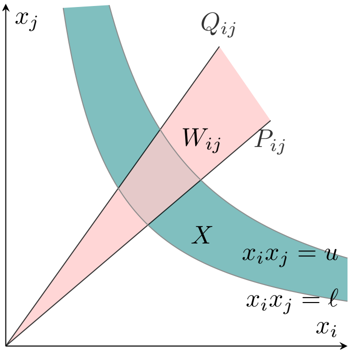
Proposition

For $\beta \leq 1$, the function

$$f''_\ell(\boldsymbol{x}) = \zeta(d_j x_i - d_i x_j)^{\frac{a_i+a_j}{\beta}} \left(\prod_{k\in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} + \zeta_0,$$

with $\zeta = \lambda^{1/\beta}\frac{u-\ell}{u^{1/\beta}-\ell^{1/\beta}}$ and ζ_0 equal to z_0 defined for $\mathcal{K}$,

1. *matches $f(\boldsymbol{x})$ at $(P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u)$;*
2. *is $\leq f(\boldsymbol{x})$ in $X \cap W_{ij}$;*
3. *is the lower envelope of f over $conv((P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u))$ for $n = 2$.*

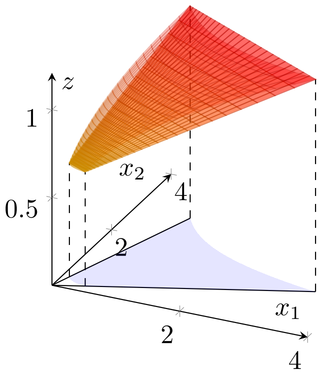


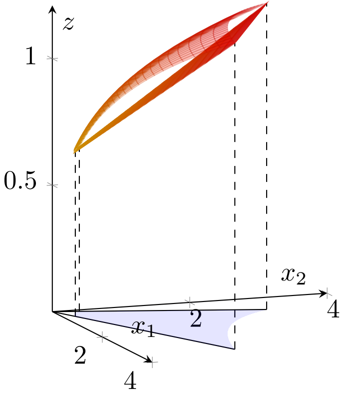
Proof sketch ($\beta \leq 1$)

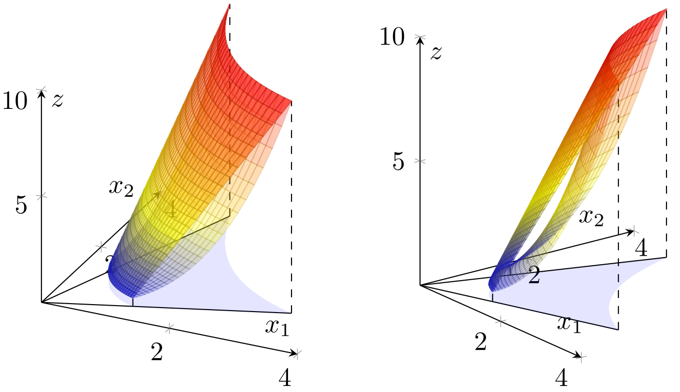
$$f''_\ell(\boldsymbol{x}) = \zeta(d_j x_i - d_i x_j)^{\frac{a_i+a_j}{\beta}} \left(\prod_{k \in N\setminus\{i,j\}} x_k^{a_k}\right)^{\frac{1}{\beta}} + \zeta_0.$$

1. $f''_\ell(\boldsymbol{x})$ matches $f(\boldsymbol{x})$ at $(P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u)$:
   Reduces to solving a linear system in ζ, ζ_0;
2. $f''_\ell(\boldsymbol{x}) \leq f(\boldsymbol{x})$ in $X \cap W_{ij}$:
   The conic function $f_u(\boldsymbol{x}) = z_0 + \left(\prod_{i \in N} x_i^{a_i}\right)^{\frac{1}{\beta}}$ giving the upper envelope for $\beta \geq 1$ is actually a lower bounding function for $f(\boldsymbol{x})$ when $\beta \leq 1$, so $f_u(\boldsymbol{x}) \leq f(\boldsymbol{x})$.
   Proving that $f''_\ell(\boldsymbol{x}) \leq f_u(\boldsymbol{x})$ is easy as they have the same structure.
3. $f''_\ell(\boldsymbol{x})$ is the lower envelope of f over $\mathrm{conv}((P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u))$ for $n = 2$:
   For $n = 2$ the function is **linear**; it matches the (concave) function $f(\boldsymbol{x})$ at the extreme points of $\mathrm{conv}((P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u))$, i.e. the four points composing $(P_{ij} \cup Q_{ij}) \cap (C_\ell \cup C_u)$ itself.

Lower envelope for $\beta \leq 1$






10
z
5
x2
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2
x1
2
4
10
z
5
x2
4
2
x1
2
4

The above gives us the convex hull of F for $n = 2$ (next slides).

For $\beta \geq 1$,

- the conic function $f_U(\boldsymbol{x}) := z_0 + (\gamma f(\boldsymbol{x}))^{\frac{1}{\beta}}$ is the upper envelope;
- f_U matches f for $f(\boldsymbol{x}) \in \{\ell, u\}$;
- the non-linear function $f'_L(\boldsymbol{x}) = \lambda(d_2 x_1 - d_1 x_2)^{\beta}$ matches f for $\frac{x_2}{x_1} \in \{p, q\}$.

For $\beta \leq 1$,

- the function $f(\boldsymbol{x})$ is concave, so it forms its own upper envelope;
- the linear function $f''_L(\boldsymbol{x}) = \zeta(d_2 x_1 - d_1 x_2) + \zeta_0$ only matches f in four points: the intersections of $x_2 = p x_1 \vee x_2 = q x_1$ with $f(\boldsymbol{x}) \in \{\ell, u\}$.

Convex hull for $n = 2$

Theorem

The convex hull of

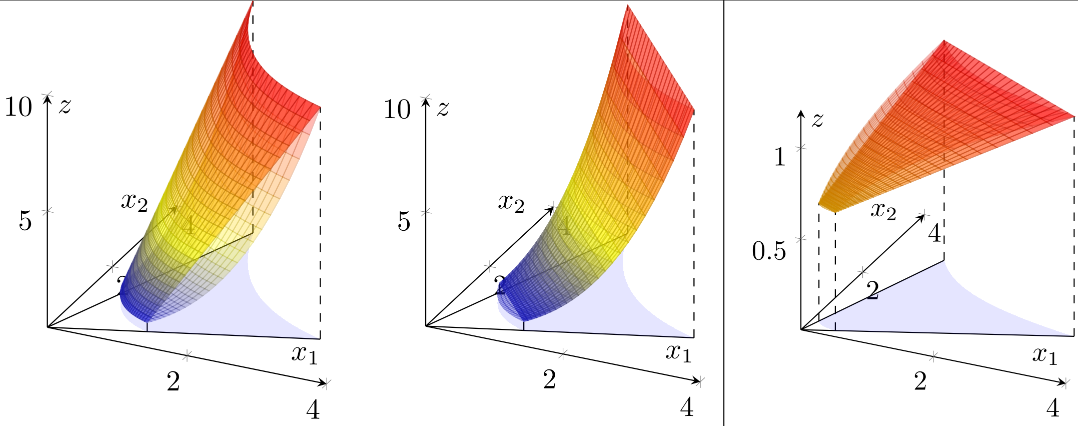
$$F = \{(x_1, x_2, z) \in \mathbb{R}^3_+ : z = x_1^{a_1} x_2^{a_2}, z \in [\ell, u], px_1 \le x_2 \le qx_1\}$$

is

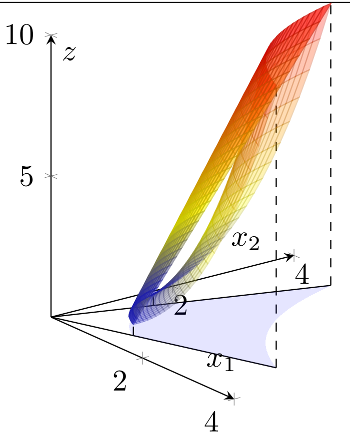
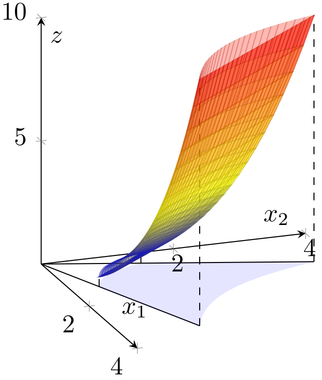
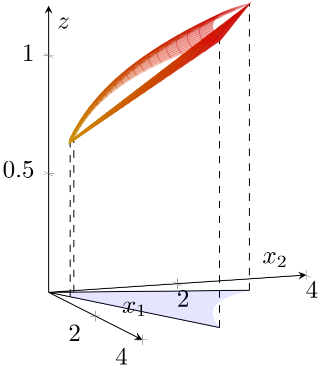
For $\beta \ge 1$	*For* $\beta \le 1$
$H = \{(x_1, x_2, z) \in \mathbb{R}^3_+ :$ $z \ge \max\{\ell, \lambda(d_2 x_1 - d_1 x_2)^{\beta}\},$ $z \le \min\{u, z_0 + (\gamma x_1^{a_1} x_2^{a_2})^{\frac{1}{\beta}}\},$ $px_1 \le x_2 \le qx_1\}$	$H = \{(x_1, x_2, z) \in \mathbb{R}^3_+ :$ $z \ge \max\{\ell, \zeta(d_2 x_1 - d_1 x_2) + \zeta_0\},$ $z \le \min\{u, x_1^{a_1} x_2^{a_2}\},$ $px_1 \le x_2 \le qx_1\}.$

In pictures

$\beta \geq 1$	$\beta \leq 1$



In pictures

$\beta \geq 1$		$\beta \leq 1$
		

Volume of the convex hull (for $n = 2$)

Volume of the convex hull

For both $\beta \geq 1$ and $\beta \leq 1$, the convex hull is defined by two inequalities:

$$\begin{aligned} H = \; & \{(\boldsymbol{x}, z) : \max\{\ell, g_1(\boldsymbol{x})\} \leq z \leq \min\{u, g_2(\boldsymbol{x})\}, \\ & px_1 \leq x_2 \leq qx_1\}. \end{aligned}$$

The volume could be computed as the integral of the difference:

$$\min\{u, g_2(\boldsymbol{x})\} - \max\{\ell, g_1(\boldsymbol{x})\},$$

but this would be a nightmare.

Volume of the convex hull

Key observation: the cross-section of H at $z = \xi$ has the same structure for $\xi \in [\ell, u]$, both for $\beta \geq 1$ and for $\beta \leq 1$.

Specifically, the structure only depends on whether $\beta \geq 1$ or $\beta \leq 1$.

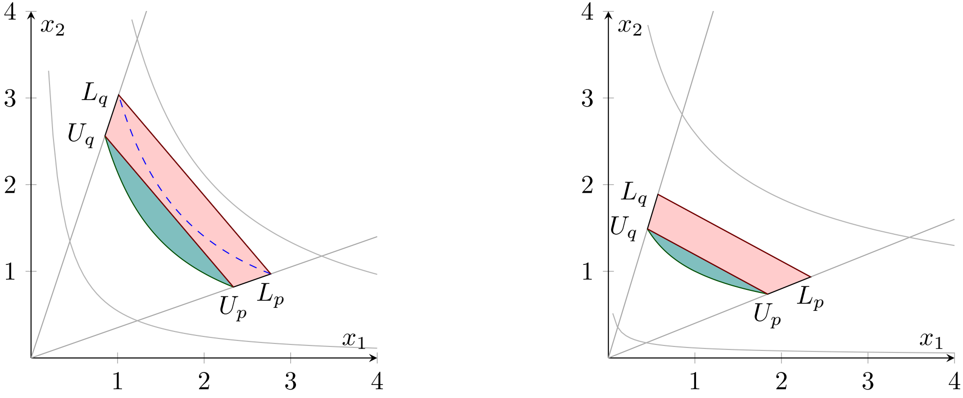
Idea: compute the area of

$$A(\xi) = \{(\boldsymbol{x}, z) \in H : z = \xi\};$$

then the volume of the convex hull is

$$V = \int_{\ell}^{u} A(\xi) d\xi.$$

Cross-section at $z = \xi$



$\beta \geq 1 : f(\boldsymbol{x}) = x_1^{1.7} x_2^{1.5}$

$\beta \leq 1 : f(\boldsymbol{x}) = x_1^{0.1} x_2^{0.2}$

Area:

$$A(\xi) = b_1 \xi^{\frac{2}{\beta}} + b_2(\xi - z_0)^2 + b_3(\xi - z_0)^{1+\frac{1}{a_2}}$$

for opportune values of b_1, b_2, b_3 which depend on a_1, a_2, p, q, ℓ, u.

Volume:

$$\begin{aligned} V = & \left[b_1 \tfrac{\beta}{\beta+2} z^{1+\frac{2}{\beta}} + \tfrac{1}{3} b_2 (z - z_0)^3 + \tfrac{a_2}{2a_2+1} b_3 (z - z_0)^{2+\frac{1}{a_2}} \right]_{\ell}^{u} \\ = & \; b_1 \tfrac{\beta}{\beta+2} (u^{1+\frac{2}{\beta}} - \ell^{1+\frac{2}{\beta}}) + \tfrac{1}{3} b_2 \left((u - z_0)^3 - (\ell - z_0)^3 \right) + \\ & \tfrac{a_2}{2a_2+1} b_3 \left((u - z_0)^{2+\frac{1}{a_2}} - (\ell - z_0)^{2+\frac{1}{a_2}} \right). \end{aligned}$$

Branching rules

Suppose we write a branch-and-bound algorithm that uses such convex hulls to find a lower bound at each node.

In order to maintain the structure, only two branching rules should be used for

$$\{(x_1, x_2, z) : z = f(\boldsymbol{x}), z \in [\ell, u], px_1 \le x_2 \le qx_1\}$$

- select $r \in (p, q)$, then branch on

  $$x_2 \le rx_1 \quad \vee \quad x_2 \ge rx_1;$$

  as this will create two subproblems, with constraints $px_1 \le x_2 \le rx_1$ and $rx_1 \le x_2 \le qx_1$, respectively;
- select $\xi \in (\ell, u)$, then branch on

  $$z \le \xi \quad \vee \quad z \ge \xi.$$

Branching rule questions

We know the closed form of the volume $V(\ell, u, p, q)$, so we can find ξ and r to satisfy either of the following two criteria:

- Minimize the total resulting volume: $V(\ell, \xi, p, q) + V(\xi, u, p, q)$ if we branch on z or $V(\ell, u, p, r) + V(\ell, u, r, q)$ if we branch on x_2/x_1.
- Make the volumes of the two branches equal (*balanced branching*), i.e., $V(\ell, \xi, p, q) = V(\xi, u, p, q)$ or $V(\ell, u, p, r) = V(\ell, u, r, q)$.

Final remarks

Using $px_1 \le x_2 \le qx_1$ instead of $(x_1, x_2) \in [\ell_1, u_1] \times [\ell_2, u_2]$ has its pros and cons.

+ we have a convex hull;
− we can't always find tight p or q.

Using the convex hull within a branch-and-bound requires us to maintain the structure via two branching rules:

- on the ratio x_2/x_1 (tricky);
- on z; closer to usual branching rules, though z is most likely auxiliary.

But there is no experimental support yet for whether these branching rules are effective or not.

References

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Thank you

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