Compiled Nonlocal Games

Sum of Squares Optimization Meets Cryptography?

Anand Natarajan & Tina Zhang

arXiv:2303.01545

Nonlocal Games

Invented by John Bell to test quantum mechanics

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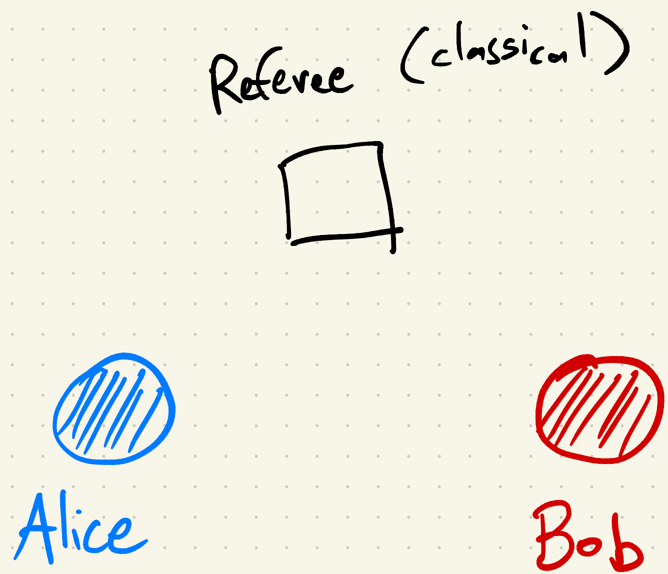

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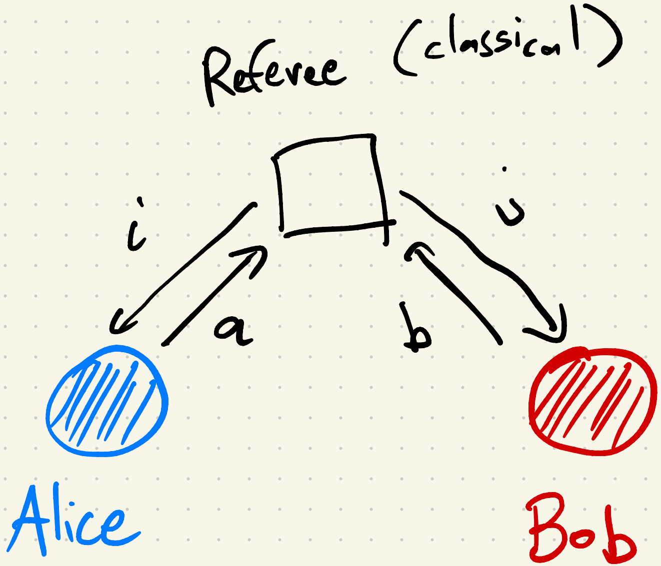

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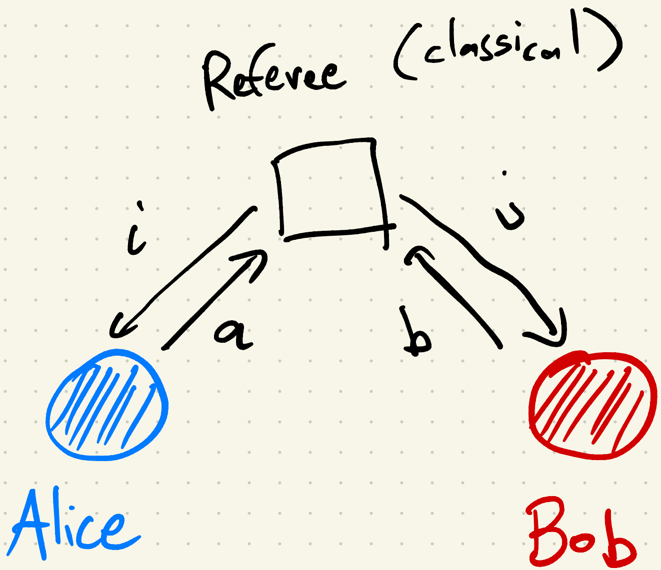

$$P_{win} = \Pr_{\substack{i,j \\ a \sim i \\ b \sim j}} \left[\text{Answers } a,b \text{ are "correct" for questions } i,j \right]$$

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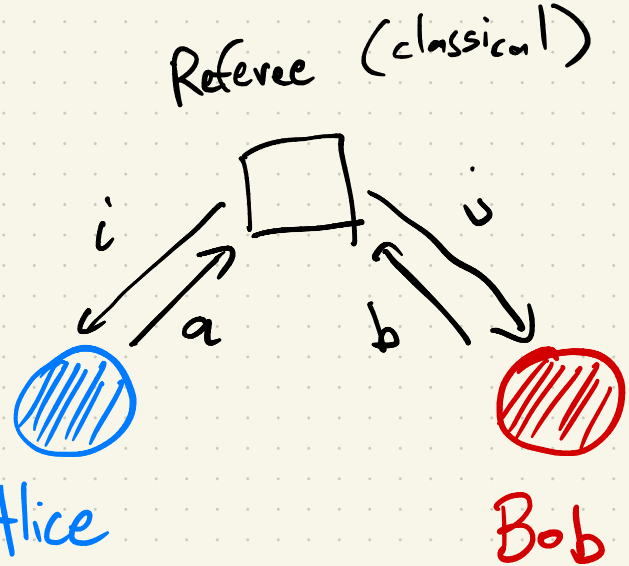
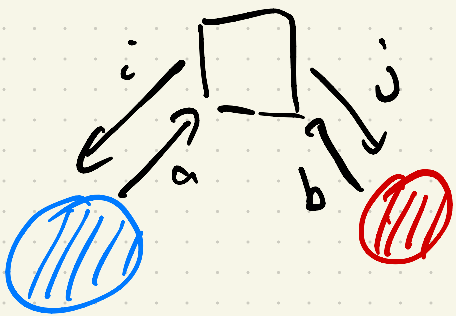


$$P_{win} = \Pr_{\substack{i,j \\ a \sim i \\ b \sim j}} \left[\text{Answers } a, b \text{ are "correct" for questions } i, j \right]$$

In general

$$P^{classical}_{win} \leq P^{quantum}_{win}$$

The CHSH Game



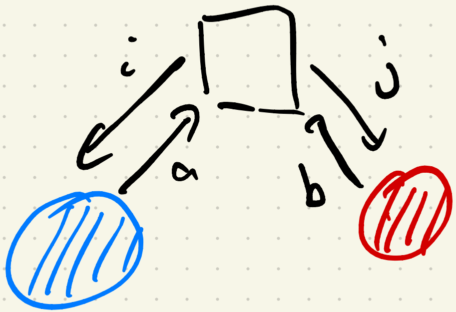
$i, j \in \{0, 1\}$

$a, b \in \{\pm 1\}$

Win if $ab = (-1)^{s_{ij}}$

i	j	$ab \stackrel{?}{=}$	s_{ij}
0	0	+1	0
0	1	+1	0
1	0	+1	0
1	1	−1	1

The CHSH Game



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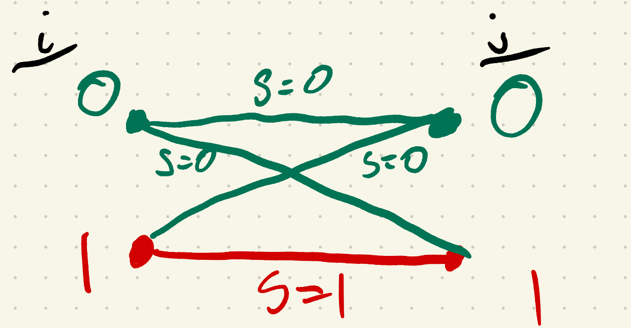
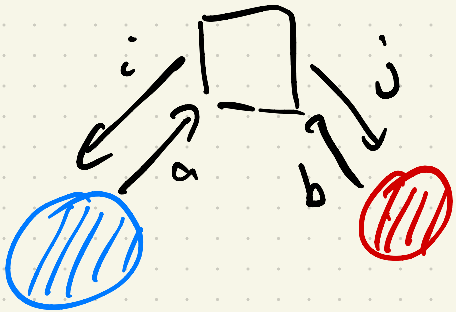


The CHSH Game

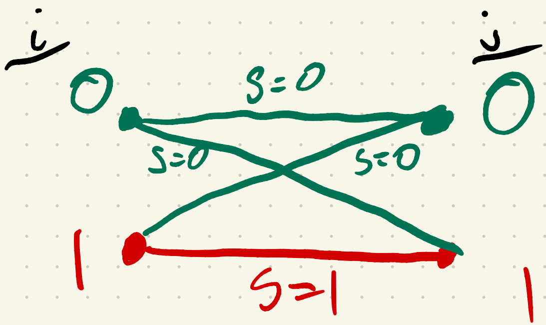


$$p_{win} = \frac{1}{2} + \frac{1}{2}\beta$$

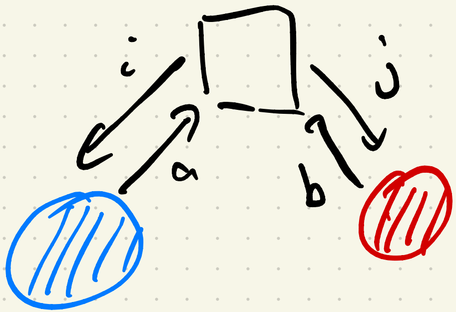
$$\beta^{\text{classical}} = \max \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} a_i b_j$$

$$\text{s.t.} \quad \forall i \quad a_i^2 = I$$

$$\forall j \quad b_j^2 = I$$



The CHSH Game

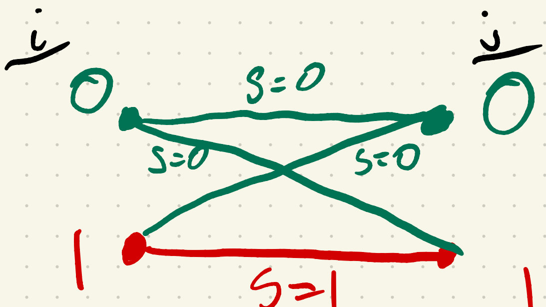


$$P_{win} = \frac{1}{2} + \frac{1}{2}\beta$$

$$\beta^{classical} = \max \mathbb{E}_{ij} (-1)^{s_{ij}} a_i b_j$$

s.t. $\forall i \quad a_i^2 = 1$

$\forall j \quad b_j^2 = 1$



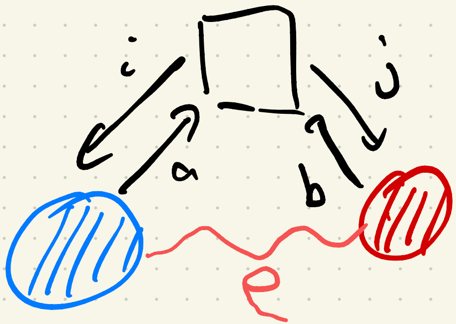
$$\beta^{classical} = \frac{1}{2}$$

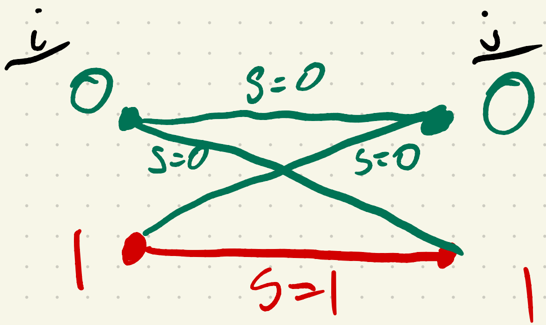
Set $a_0 = a_1 = 1$

$b_0 = b_1 = 1$

$\Rightarrow P_{win}^{classical} = 3/4$

The CHSH Game





$$p_{win} = \frac{1}{2} + \frac{1}{2}\beta$$

matrices!

quantum

$$\beta = \max_{A_i, B_j, \rho} \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} \mathrm{tr}[A_i B_j \rho]$$

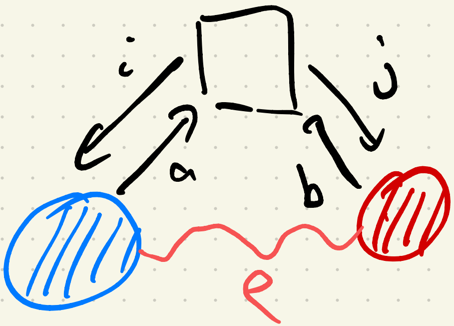
s.t. $\rho \succeq 0$, $\mathrm{tr}\,\rho = 1$

$\forall i,\ A_i^2 = I$

$\forall j\ \ B_j^2 = I$

$\forall i,j\ \ A_i B_j = B_j A_i$

The CHSH Game



$$p_{win} = \frac{1}{2} + \frac{1}{2}\beta$$

$$\beta^{quantum} = \max_{A_i, B_j, \rho} \mathbb{E}_{ij} (-1)^{s_{ij}} \operatorname{tr}[A_i B_j \rho]$$

s.t. $\rho \succeq 0$, $\operatorname{tr}\rho = 1$

$\forall i, \; A_i^2 = I$

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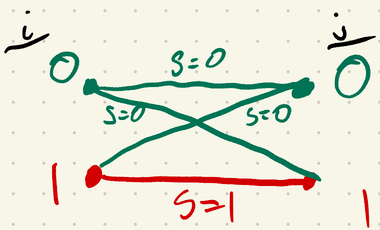
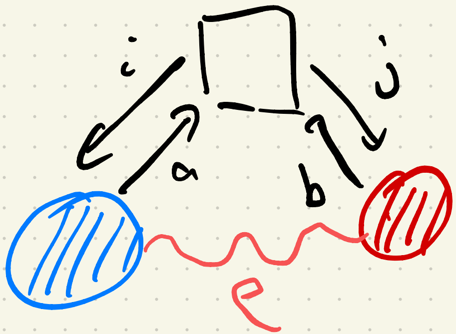
$\forall i,j \;\; A_i B_j = B_j A_i$



$$\beta^{quantum} = \frac{1}{2} \cdot \sqrt{2} > \beta^{classical}$$

The CHSH Game



$$p_{win} = \frac{1}{2} + \frac{1}{2}\beta$$

$$\beta^{\text{quantum}} = \max_{A_i, B_j, \rho} \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} \operatorname{tr}[A_i B_j \rho]$$

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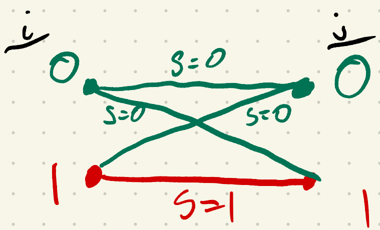

$$\beta^{\text{quantum}} = \frac{1}{2} \cdot \sqrt{2}$$

$$p_{win}^{\text{quantum}} \approx 0.85 > \frac{3}{4}$$

The CHSH Game

- $P^{\text{quantum}}_{\text{win}} > P^{\text{classical}}_{\text{win}}$ demonstrated in lab [2022 Nobel Prize!]

- Q. optimizer is "rigid"

$$\beta^Q(A, B) \approx \tfrac{1}{2}\sqrt{2} \Rightarrow \underbrace{\begin{matrix} A_0 A_1 \approx -A_1 A_0 \\ B_0 B_1 \approx -B_1 B_0 \end{matrix}}$$

Anticommutation ⇒ "complementary measurements" in physics

The CHSH Game

- Q. optimizer is "rigid"

$$\beta^Q(A, B) \approx \tfrac{1}{2}\sqrt{2} \Rightarrow \underbrace{\begin{array}{l} A_0A_1 \approx -A_1A_0 \\ B_0B_1 \approx -B_1B_0 \end{array}}$$

Anticommutation $\Rightarrow$ "complementary measurements" in physics

$\Longrightarrow$ Using CHSH, can build protocols to test quantum computers "gate by gate" [RUV'13]

Drawbacks of nonlocal games

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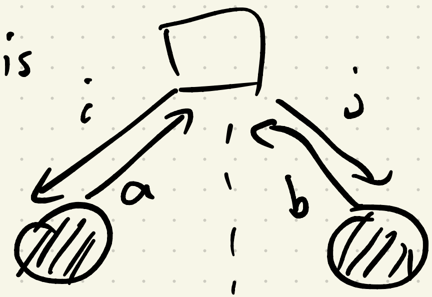
- Non-communication between players is <u>essential</u>



a must only depend on i

b must only depend on j

Drawbacks of nonlocal games

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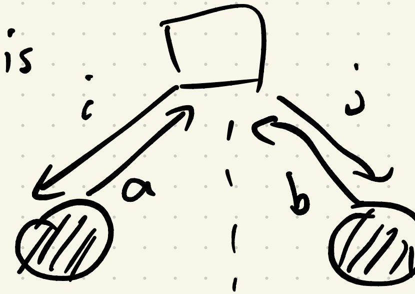

a must only depend on i

b must only depend on j

Quantum: A_i and B_j must be commuting matrices

Drawbacks of nonlocal games

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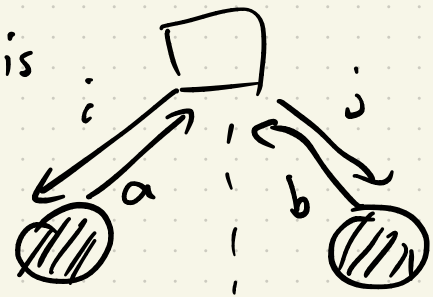


a must only depend on i

b must only depend on j

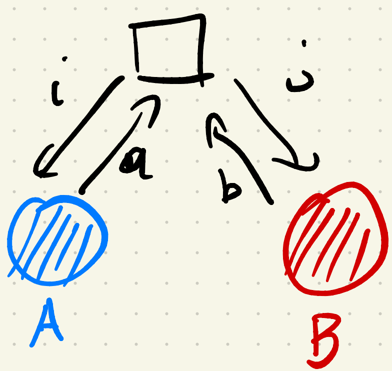
Quantum: A_i and B_j must be commuting matrices

Experimentally very difficult to realize!

Cryptographic compilers

- **Idea:** Use cryptography so one player can "simulate" 2 separated players

[Long history in crypto, e.g. Kilian '92, BMW'98, KRR'14, ...]



Cryptographic compilers

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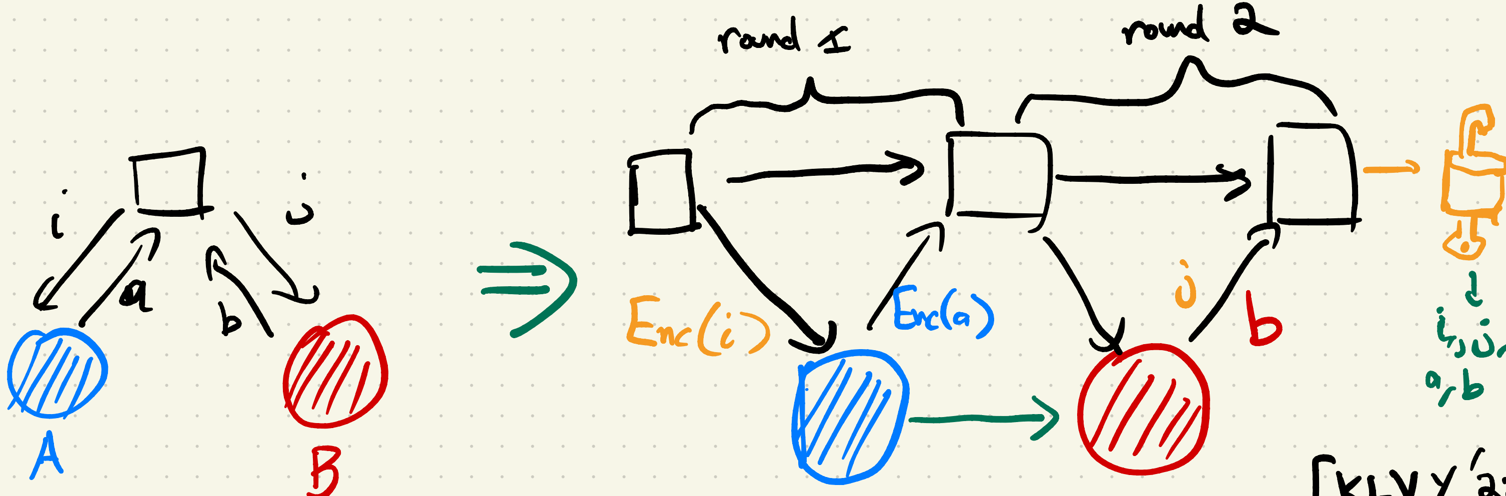
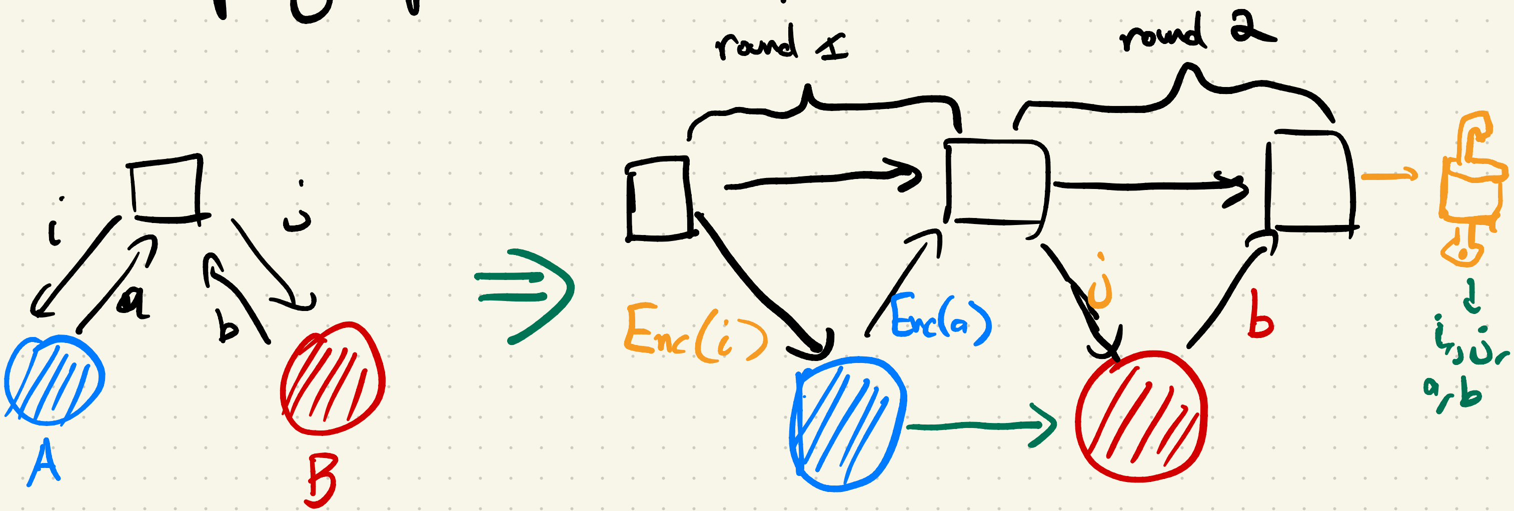


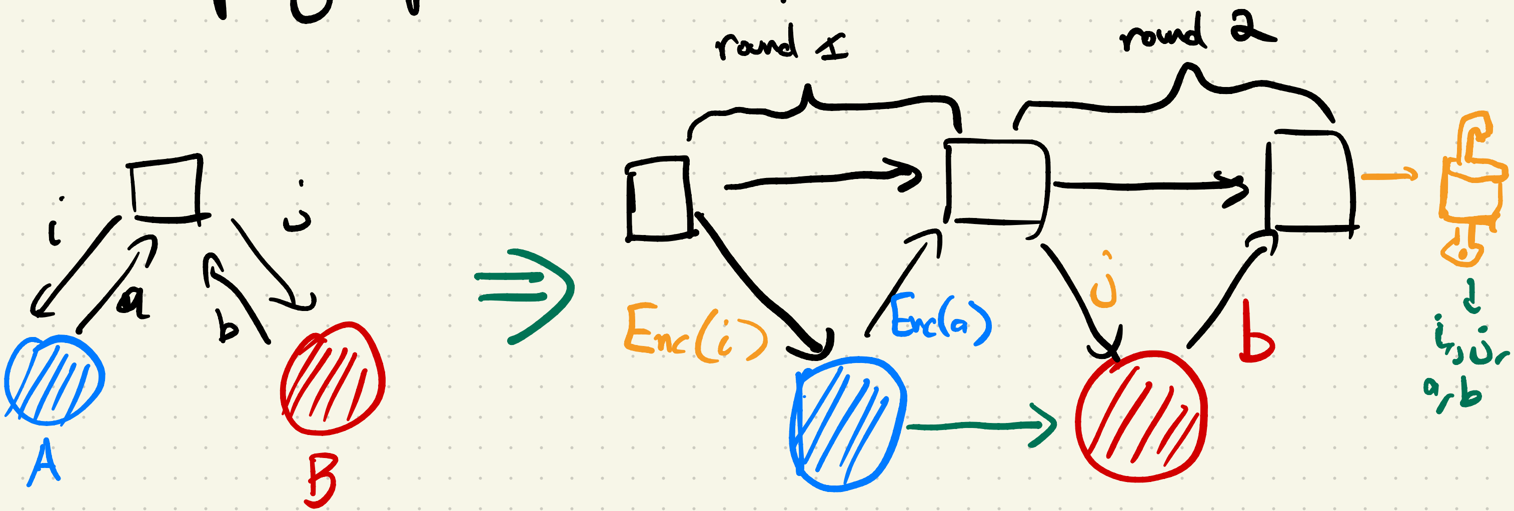
[KLVY'22]

Cryptographic compilers



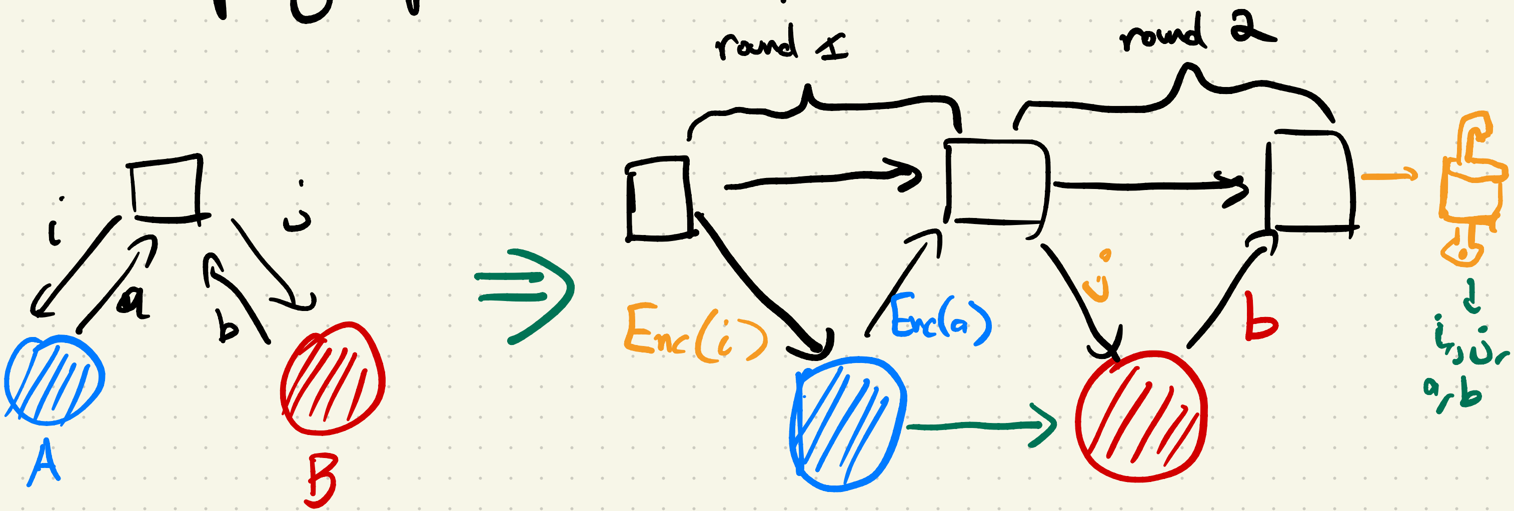
[KLVY'22]: $P_{win}^{\text{classical, compiled}} \leq P_{win}^{\text{classical nonlocal}} + \text{negl.}$

Cryptographic compilers



[KLVY'22]: $P_{\text{win}}^{\text{quantum compiled}} \overset{???}{\leq} P_{\text{win}}^{\text{quantum nonlocal}} + \text{negl.}$???

Cryptographic compilers: CHSH



	classical	quantum
nonlocal	$3/4$	0.85 $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)$
compiled	$3/4$ + neg!	????

(is it even < 1?)

Our results

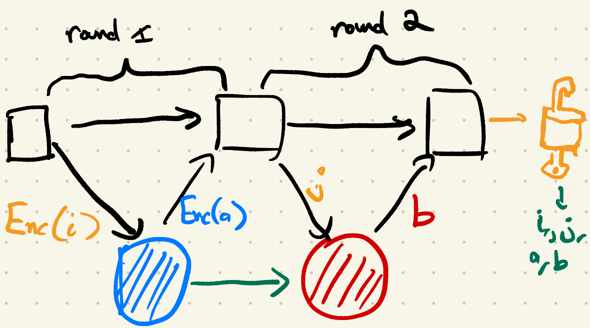
	classical	quantum
nonlocal	$3/4$	0.85 $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)$
compiled	$3/4$ + neg!	0.85 + neg ! $\left(\frac{1}{2}+\frac{\sqrt{2}}{4} + \text{neg}!\right)$

Our results: rigidity

	classical	quantum
nonlocal	3/4	0.85 $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)$
compiled	3/4 + negl	0.85 + negl $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}+\text{negl}\right)$

→ Any strategy that is near optimal must have

$$B_0 B_1 \approx -B_1 B_0$$

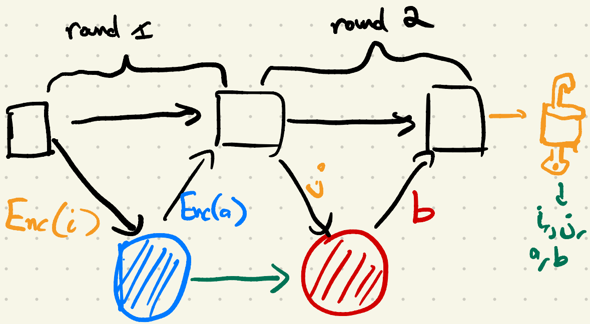


Our results: delegation

	classical	quantum
nonlocal	$3/4$	0.85 $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)$
compiled	$3/4 + \text{negl}$	$0.85 + \text{negl}$ $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}+\text{negl}\right)$

$\rightarrow$ A delegation scheme for poly time quantum computation w/ classical client

(assuming quantum fully homomorphic encryption $\Leftarrow$ LWE [Mahadev'17])



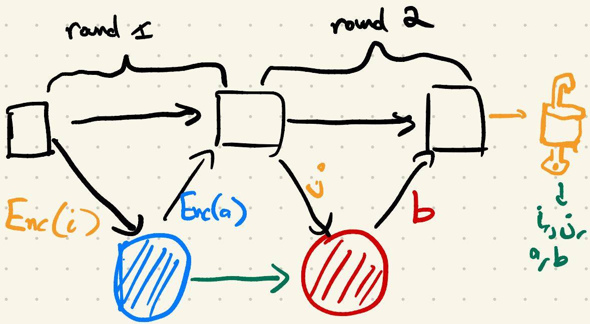
Our results: delegation

	classical	quantum
nonlocal	3/4	0.85 $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)$
compiled	3/4 + negl	0.85 + negl $\left(\frac{1}{2}+\frac{\sqrt{2}}{4}+\text{negl}\right)$

→ A delegation scheme for poly time quantum computation w/ classical client

(assuming quantum fully homomorphic encryption $\Leftarrow$ LWE [Mahadev '17])

Matches [Mahadev '18] by new techniques



Techniques: outline

1) Prove $\beta^Q \leq \frac{1}{2} \cdot \sqrt{2}$ in nonlocal world using noncommutative Sum-of-Squares

2) Modify this proof to show

$$\beta^Q_{\text{compiled}} \leq \frac{1}{2} \cdot \sqrt{2} + \text{negl.}$$

SoS for β^Q

$$\beta^Q = \max \ \mathrm{tr}\Big[\overbrace{\mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} A_i B_j}^{\hat{G}(A_0, A_1, B_0, B_1)} \ \rho \Big]$$

s.t. $\quad A_i^2 = B_j^2 = I, \quad A_i B_j = B_j A_i$

$\rho \succeq 0, \quad \mathrm{tr}\,\rho = 1$

SoS for β^Q

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$$\text{s.t.} \quad A_i^2 = B_j^2 = I, \quad A_i B_j = B_j A_i \quad (*)$$

$$\rho \succeq 0, \quad \operatorname{tr}\rho = 1$$

To show $\beta^Q \leq \frac{1}{2} \cdot \sqrt{2}$, suffices to show

$$\frac{1}{2}\sqrt{2} - \hat{G} \succeq 0 \quad \text{whenever } (*) \text{ holds}$$

SoS for β^Q

To show $\beta^Q \leq \frac{1}{2}\cdot\sqrt{2}$

suffices to show

$\frac{1}{2}\sqrt{2} - \hat{G} \geq 0 \quad \Leftarrow (*)$

$\Uparrow$

$$\beta^Q = \max \operatorname{tr}\Big[\overbrace{\mathbb{E}_{i,j}(-1)^{s_{ij}} A_i B_j}^{\hat{G}(A_0, A_1, B_0, B_1)} \rho\Big]$$

s.t. $A_i^2 = B_j^2 = I, \quad A_i B_j = B_j A_i \quad (*)$

$\rho \geq 0, \quad \operatorname{tr}\rho = 1$

$$\frac{1}{2}\cdot\sqrt{2} - \hat{G}(A, B) = \underbrace{\sum_k p_k(A,B)^\dagger p_k(A,B)}_{\text{Sum of squares (must be } \geq 0)} \mod (*)$$

SoS for β^Q

To show $\beta^Q \le \frac{1}{2}\cdot\sqrt{2}$

suffices to show

$$\beta^Q = \max \operatorname{tr}\Big[\overbrace{\mathbb{E}_{i,j}(-1)^{s_{ij}} A_i \, B_j}^{\hat{G}(A_0,A_1,B_0,B_1)} \; \rho\Big]$$

s.t. $A_i^2 = B_j^2 = I, \quad A_i B_j = B_j A_i \quad (*)$

$\rho \geq 0, \quad \operatorname{tr}\rho = 1$

$$\frac{1}{2}\cdot\sqrt{2} - \hat{G}(A, B) = \sum_k P_k(A,B)^\dagger P_k(A,B) \mod (*)$$

Take $P_1 \propto \left(A_0 - \frac{B_0 + B_1}{\sqrt{2}}\right)$

$P_2 \propto \left(A_1 - \frac{B_0 - B_1}{\sqrt{2}}\right)$

SoS for β^Q

$$\beta^Q = \max \operatorname{tr}\Big[\overbrace{\mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} A_i B_j}^{\hat{G}(A_0,A_1,B_0,B_1)} \rho\Big]$$

s.t. $A_i^2 = B_j^2 = I, \quad A_i B_j = B_j A_i \quad (*)$

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Take $p_1 \propto \left(A_0 - \frac{B_0 + B_1}{\sqrt{2}}\right)$

$p_2 \propto \left(A_1 - \frac{B_0 - B_1}{\sqrt{2}}\right)$

$\Rightarrow$ rigidity (near optimal strategies have $p_k \approx 0$)

SoS for $\beta^{Q}_{\text{compiled}}$

$$\beta^{Q}_{\text{compiled}} = \max \operatorname{tr}\left[\sum_a \mathbb{E}_{ij} (-1)^{s_{ij}} A_i^{(a)} B_j A_i^{(a)} \rho \right]$$

s.t.

$$\forall i, \quad \sum_a A_i^{(a)\dagger} A_i^{(a)} = I$$

$$\forall j \qquad B_j^2 = I$$

$$\rho \succeq 0, \quad \operatorname{tr} \rho = 1$$

SoS for $\beta^{Q}_{\text{compiled}}$

sequential quantum measurement

$$\beta^{Q}_{\text{compiled}} = \max \operatorname{tr}\left[\sum_a \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} A_i^{(a)} B_j A_i^{(a)} \rho \right]$$

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$$\forall i, \quad \sum_a A_i^{(a)\dagger} A_i^{(a)} = I$$

$$\forall j \qquad B_j^2 = I$$

$$\boxed{\forall\, i, i', \ \forall \text{ "efficient" } f(B) \quad \operatorname{tr}\left[\sum_a A_i^{(a)} f(B) A_i^{(a)} \rho\right] \approx_{\text{negl}} \operatorname{tr}\left[\sum_a A_{i'}^{(a)} f(B) A_{i'}^{(a)} \rho\right]}$$

$$\rho \succeq 0, \quad \operatorname{tr}\rho = 1$$

SoS for $\beta^{Q}_{compiled}$

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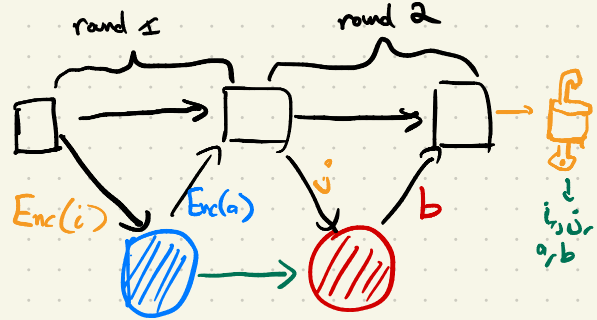
$\rho \geq 0, \ \operatorname{tr}\rho = 1$

from cryptography

replaces

$A_i B_j = B_j A_i$

SoS for $\beta^{Q}_{\text{compiled}}$



$$\beta^{Q}_{\text{compiled}} = \max \operatorname{tr}\left[\sum_a \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} A_i^{(a)} B_j A_i^{(a)} \rho \right]$$

s.t. $\quad \forall i, \quad \sum_a A_i^{(a)\dagger} A_i^{(a)} = I$

$\quad \forall j \quad B_j^2 = I$

"No measurement in round 2 can leak info about i (since it is encrypted)"

$\forall\, i, i', \ \forall$ "efficient" $f(B)$ $\quad \operatorname{tr}\left[\sum_a A_i^{(a)} f(B) A_i^{(a)} \rho\right] \approx_{\text{negl}} \operatorname{tr}\left[\sum_a A_{i'}^{(a)} f(B) A_{i'}^{(a)} \rho\right]$

$\rho \geq 0, \quad \operatorname{tr} \rho = 1$

SoS for $\beta^{Q}_{\text{compiled}}$

$$\beta^{Q}_{\text{compiled}} = \max \operatorname{tr}\Big[\overbrace{\sum_a \mathbb{E}_{i,j} (-1)^{s_{ij}} A_i^{(a)} B_j A_i^{(a)}}^{\mathcal{G}_{\text{compiled}}} \rho \Big]$$

s.t.

$$\forall i, \quad \sum_a A_i^{(a)\dagger} A_i^{(a)} = I$$

$$\forall j \quad B_j^2 = I$$

$$\forall i, i', \ \forall \text{ "efficient" } f(B) \quad \operatorname{tr}\Big[\sum_a A_i^{(a)} f(B) A_i^{(a)} \rho\Big] \approx_{negl} \operatorname{tr}\Big[\sum_a A_{i'}^{(a)} f(B) A_{i'}^{(a)} \rho\Big]$$

$$\rho \geq 0, \quad \operatorname{tr} \rho = 1$$

$(**)$

Can we show

$$\frac{1}{2}\cdot\sqrt{2} - \mathcal{G}_{\text{compiled}} = \sum p^\dagger p$$

mod $(**)$?

SoS for $\beta^{Q}_{compiled}$

$$\beta^{Q}_{compiled} = \max \operatorname{tr}\Big[\overbrace{\sum_a \mathop{\mathbb{E}}_{i,j} (-1)^{s_{ij}} A_i^{(a)} B_j A_i^{(a)}}^{\hat{\mathcal{G}}_{compiled}} \rho\Big]$$

s.t.

$$\left.\begin{array}{l} \forall i, \ \sum_a A_i^{(a)\dagger} A_i^{(a)} = I \\ \forall j \quad B_j^2 = I \\ \forall i, i', \ \forall \text{ "efficient" } f(B) \quad \operatorname{tr}\Big[\sum_a A_i^{(a)} f(B) A_i^{(a)} \rho\Big] \approx_{negl} \\ \qquad \operatorname{tr}\Big[\sum_a A_{i'}^{(a)} f(B) A_{i'}^{(a)} \rho\Big] \end{array}\right\} (**)$$

$\rho \geq 0, \ \operatorname{tr}\rho = 1$

We show

$$\operatorname{tr}\Big[\Big(\tfrac{1}{2}\cdot\sqrt{2} - \hat{\mathcal{G}}_{compiled}\Big)\rho\Big] = \operatorname{tr}\Big[\sum p^\dagger p \, \rho\Big]$$

mod $(**)$

which is sufficient!

Open questions

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  When is $p_{win}^{Q,\,compiled} \leq p_{win}^{Q} + \text{negl}$?

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  - Our SoS manipulations are very specific to CHSH
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- Can the cryptographic requirements be relaxed? (Less than full QFHE?)

Open questions

- Does quantum crypto give rise to interesting new <u>non-commutative</u> polynomial optimization problems?

Thank you!

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