Measuring the Importance of Facets of Cyclic Group Polyhedra Using Solid Angles

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Joint work with Allison Fitisone
Overview

1. Cyclic group relaxation of IP
2. Measure relative sizes of the facets
3. Compute solid angle measure in $\mathbb{R}^n$
General purpose cutting planes

Many widely used general-purpose cuts are derived from a single row of the optimal simplex tableau:

\[ x_B = A_B^{-1} b + (-A_B^{-1} A_N) x_N, \quad x_B \in \mathbb{Z}_+^B, \ x_N \in \mathbb{Z}_+^N. \]

Gomory Mixed Integer Cut Example:

- **Tableau row:**
  \[ x = -\left(\frac{-4}{5}\right) + \frac{2}{5} x_1 - \frac{1}{5} x_2 + \frac{11}{5} x_3, \]
  where \( x, x_1, x_2, x_3 \in \mathbb{Z}_+. \)

- **Determine the cut:**
  \[ ?x_1 + ?x_2 + ?x_3 \geq 1. \]

- **Cyclic group order** \( q = 5 \)
- \( (-\frac{4}{5}) \equiv \frac{1}{5} \) (mod 1)
- **GMIC function** \( \pi_{\text{mic}} \)

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Yuan Zhou (U. Kentucky)
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  where \( x, x_1, x_2, x_3 \in \mathbb{Z}_+ \).
- Determine the cut:
  \[ \frac{3}{4}x_1 + \frac{1}{4}x_2 + x_3 \geq 1. \]
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- Determine the cut:
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- **Cyclic group order** $$q = 5$$  
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Cyclic group relaxation

- Tableau row: \( x = -(\frac{4}{5}) + \frac{2}{5}x_1 - \frac{1}{5}x_2 + \frac{11}{5}x_3, \quad x, x_1, x_2, x_3 \in \mathbb{Z}_+ \)
  - Relax \( x \) from \( \mathbb{Z}_+ \) to \( \mathbb{Z} \):
    \[
    0 \equiv -\frac{1}{5} + \frac{2}{5}x_1 + \frac{4}{5}x_2 + \frac{1}{5}x_3 \pmod{1}
    \]
  - Introduce \( z \in \mathbb{Z}_+^{q-1} \):
    \[
    \frac{1}{5} \equiv \frac{1}{5}z_1 + \frac{2}{5}z_2 + \frac{3}{5}z_3 + \frac{4}{5}z_4 \pmod{1}
    \]
  - \( q = 5 \) and \( f = 1 \)

- Master cyclic group polyhedron
  \[
  P(q, f) := \text{conv} \left\{ z \in \mathbb{Z}_+^{q-1} \mid \sum_{i=1}^{q-1} \left( \frac{i}{q} \right) z_i \equiv \frac{f}{q} \pmod{1} \right\}
  \]

- We are interested in \( \pi \in \mathbb{R}_+^{q-1} \) s.t. \( \pi \cdot z \geq 1 \) for all \( z \in P(q, f) \), and in particular, the facet-defining ones.
Master cyclic group polyhedron and Group-facet polytope

- **Blocker** \( \mathcal{B}(P(q, f)) = \{ \pi \in \mathbb{R}^{q-1}_+ \mid \pi \cdot z \geq 1 \ \forall \ z \in P(q, f) \} \)

- \( P(q, f) \subseteq \mathbb{R}^{q-1}_+ \) is full-dimensional, and has \( \mathbb{R}^{q-1}_+ \) as recession cone.

    \[ \implies \mathcal{B}(P(q, f)) = \Pi(q, f) + \mathbb{R}^{q-1}_+ \]

- **Group-facet polytope** \( \Pi(q, f) \) consists of \( \pi = (\pi_1, \ldots, \pi_{q-1}) \) with \( \pi_0 = 0, \pi_f = 1 \) and satisfies
  - (nonnegativity) \( \pi_i \geq 0 \) for \( i = 1, \ldots, q - 1 \)
  - (symmetry) \( \pi_i + \pi_j = 1, \ i + j \equiv f \ (\text{mod } q) \)
  - (subadditivity) \( \pi_i + \pi_j \geq \pi_k, \ i + j \equiv k \ (\text{mod } q) \)

- Non-trivial facet of \( P(q, f) \iff \) vertex of \( \Pi(q, f) \)

- Number of facets / vertices \( \pi \) grows exponentially with \( q \).

- Which \( \pi \)'s are more important?
Choose random direction $\mathbf{v} \in \mathbb{R}^{q-1}$.

Move from the origin along the ray $\lambda \mathbf{v}$ ($\lambda > 0$) until hitting $P(q, f)$.

Record which facet $\pi^*$ is hit: $\pi^* = \arg \min_\pi \{ \pi \cdot \mathbf{v} \mid \pi \in \Pi(q, f) \}$.

Facets that receive more hits are deemed more important.
Relative sizes of facets — Shooting experiment

Proportion of hits received by facets of $P(7, 6)$ according to [Hunsaker, 2003] (out of 1000 shots) and [Shim, 2009] (geometry)

<table>
<thead>
<tr>
<th>Facets of $P(7, 6)$</th>
<th>Hunsaker</th>
<th>Shim</th>
</tr>
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<tbody>
<tr>
<td>1 2 3 4 5 6</td>
<td>0.332</td>
<td>0.3238</td>
</tr>
<tr>
<td>9 4 6 8 3 12</td>
<td>0.253</td>
<td>0.2500</td>
</tr>
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</tr>
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<td>6 5 4 3 2 8</td>
<td>0.170</td>
<td>0.1762</td>
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Discrepancies?
Relative sizes of facets — Shooting experiment

Proportion of hits received by facets of $P(11, 10)$ according to [Hunsaker, 2003] (out of 1000 shots) and [Shim, 2009] (out of 10000 shots)

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<tr>
<td>1 2 3 4 5 6 7 8 9 10</td>
<td>0.134</td>
<td>0.1354</td>
</tr>
<tr>
<td>4 8 12 16 9 2 6 10 14 18</td>
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<td>0.1291</td>
</tr>
<tr>
<td>9 18 16 3 12 21 8 6 15 24</td>
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<td>10 9 8 7 6 5 4 3 2 12</td>
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<tr>
<td>16 21 4 20 14 8 24 7 12 28</td>
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<td>0.0816</td>
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<td>25 6 20 12 15 18 10 24 5 30</td>
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Proposition

Shooting experiment size of a facet $\pi \cdot \mathbf{z} \geq 1$ of $P(q, f)$

$\simeq$ solid angle subtended by the facet of $P(q, f)$ at the origin

$=$ solid angle of the normal cone at the vertex $\pi$ of $\mathcal{B}(P(q, f))$

$=$ (normalized) solid angle of the normal cone at the vertex $\pi$ of $\Pi(q, f)$
Relative sizes of facets — Solid angle

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Motivations for studying solid angles

- Draw more accurate conclusions regarding the importance of facets of master cyclic group polyhedra
- Estimate the number of simplices in a triangulation of an $n$-cube
- Solid angle sum over rational polytope:
  - Ehrhart polynomial and lattice counting
- Approximate the relative volume in many applications:
  - Pixel purity index (PPI) scores
  - Feasibility domain of an ecological community
  - ......
### Normalized solid angle measure

**Definition (Property)**

The normalized solid angle measure of a cone $C \subseteq \mathbb{R}^n$ with respect to $\mathbb{R}^n$ is defined as

$$\tilde{\Omega}_n(C) = \frac{\text{vol}_{n-1}(C \cap S_{n-1})}{\text{vol}_{n-1}(S_{n-1})} \left( = \frac{\text{vol}_n(C \cap B_n)}{\text{vol}_n(B_n)} = \frac{\int_C f(x) \, dx}{\int_{\mathbb{R}^n} f(x) \, dx} \right),$$

where

- $\text{vol}_n$ is the usual volume form in $\mathbb{R}^n$,
- $S_{n-1}$ is the unit $(n-1)$-sphere (residing in $\mathbb{R}^n$) centered at $0$,
- $B_n$ is the unit ball in $\mathbb{R}^n$ centered at $0$,
- $f : \mathbb{R}^n \to \mathbb{R}$ is any function that is invariant under rotations around $0$, e.g. $f = e^{-\|x\|^2}$

In the following, we consider simplicial cones $C = c(v_1, v_2, \ldots, v_n)$. 
Solid angles in $\mathbb{R}^2$

In $\mathbb{R}^2$, the normalized solid (or plane) angle measure can be computed via the standard inner product.

$$\tilde{\Omega}_2(c(v_1, v_2)) = \frac{1}{2\pi} \cos^{-1} \left( \frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|} \right)$$

$$\tilde{\Omega}_2(c(v_1, v_2)) = \frac{1}{2\pi} \cos^{-1} \left( \frac{\sqrt{2}}{2} \right) = \frac{1}{8}$$
Solid angles in $\mathbb{R}^3$

The normalized solid angle measure is the ratio of the area of the spherical triangle formed by the unit vectors to the surface area of the unit sphere.

$$\tilde{\Omega}_3(c(v_1, v_2, v_3)) = \frac{1}{4\pi} \left( 2 \tan^{-1} \left( \frac{|v_1 \cdot (v_2 \times v_3)|}{1 + v_2 \cdot v_3 + v_2 \cdot v_1 + v_1 \cdot v_3} \right) \right).$$
Solid angles in $\mathbb{R}^n$

$$\tilde{\Omega}_n(C) = \frac{\text{vol}_{n-1}(C \cap S_{n-1})}{\text{vol}_{n-1}(S_{n-1})} = \frac{\text{vol}_n(C \cap B_n)}{\text{vol}_n(B_n)} = \frac{\int_C f(x)dx}{\int_{\mathbb{R}^n} f(x)dx}$$

- Computing $\tilde{\Omega}_n(c(v_1, \ldots, v_n))$ amounts to computing the volume of a “spherically faced simplex”
- Closed formula not known
- Probabilistic methods — require large samples
- Numerical integration methods — high in computational cost
- We consider the multivariate power series method independently discovered by [Aomoto, 1977] and [Ribando, 2006]
Theorem (Ribando, 2006)

Let $C \subseteq \mathbb{R}^n$ be the simplicial cone generated by the unit vectors $v_1, \ldots, v_n$. Let $V \in \mathbb{R}^{n \times n}$ be the matrix whose $i^{th}$ column is $v_i$. Let $\alpha_{ij} = v_i \cdot v_j$ for $1 \leq i, j \leq n$. Define

$$
T_\alpha = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[ \frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma \left( \frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^\mathbf{a}.
$$

Then, $T_\alpha$ converges absolutely to $\tilde{\Omega}_n(C)$, if and only if its associated matrix $M_n(C)$ is positive definite.

$$
M_n(C) = 
\begin{bmatrix}
1 & -|v_1 \cdot v_2| & \cdots & -|v_1 \cdot v_n| \\
-|v_2 \cdot v_1| & 1 & \cdots & -|v_2 \cdot v_n| \\
\vdots & \vdots & \ddots & \vdots \\
-|v_n \cdot v_1| & -|v_n \cdot v_2| & \cdots & 1 \\
\end{bmatrix}
$$
Ribando’s hypergeometric series

\[
T_\alpha = \frac{|\det V|}{(4\pi)^{n/2}} \sum_{\mathbf{a} \in \mathbb{N}^{(n/2)}} \left[ \frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma \left( \frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^\mathbf{a}.
\]

- \( \alpha = (\alpha_{12}, \alpha_{13}, \ldots , \alpha_{n-1,n}) \) is a multivariable in \( \binom{n}{2} \) variables;
- \( \mathbf{a} = (a_{12}, a_{13}, \ldots , a_{n-1,n}) \) is a multiexponent;
- \( \alpha^\mathbf{a} := \prod_{i<j} \alpha_{ij} \);
- \( \sum_{m \neq i} a_{im} = \sum_{j<i} a_{ji} + \sum_{j>i} a_{ij} \);
- \( \Gamma \) is the Euler-Gamma function.
- \( T_\alpha \) is a hypergeometric series as the ratio of the neighboring coefficients is a rational function of the index.
Issue with $T_\alpha$: convergence condition not met

Let $v_1 = [1, 1, 0]$, $v_2 = [1, 0, 1]$, $v_3 = [1, 0, 0]$, and $C = c(v_1, v_2, v_3)$.

- Scale to unit vectors;
- $V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

$\implies V^tV = \begin{bmatrix} 1 & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \end{bmatrix}$

- $M_3(C) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \end{bmatrix}$ is NOT positive-definite
Remedy: Cone decomposition

Let $\mathbf{v}_1 = [1, 1, 0]$, $\mathbf{v}_2 = [1, 0, 1]$, $\mathbf{v}_3 = [1, 0, 0]$, $\mathbf{v}_4 = [0, -1, 1]$.

\[
\begin{align*}
[\mathcal{C}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)] &= [\mathcal{C}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4)] - [\mathcal{C}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)] + [\mathcal{C}(\mathbf{v}_3, \mathbf{v}_4)] \\
\tilde{\Omega}_3 (\mathcal{C}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)) &= \tilde{\Omega}_3 (\mathcal{C}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4)) - \tilde{\Omega}_3 (\mathcal{C}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)) + 0
\end{align*}
\]

where $[\mathcal{C}]$ denotes the indicator function of the cone $\mathcal{C}$, and $\mathcal{C}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4), \mathcal{C}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ have positive-definite associated matrices.
Theorem (Fitisone–Zhou, 2023)

Given linearly independent unit vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in \mathbb{R}^n$, the cone $c(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ can be decomposed into a finite family of cones, each of which is either:

- a cone of affine dimension less than $n$, or
- a full-dimensional cone $c(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ such that
  - $\|\mathbf{v}_i\| = 1$ for $i = 1, 2, \ldots, n$
  - $\mathbf{v}_n = \mathbf{w}_n$
  - $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1} \rangle = \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1} \rangle$
  - $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $1 \leq i, j \leq n$ such that $j \neq i, j \neq i \pm 1$.

Remarks:

- A cone in 1 has solid angle 0.
- Properties b and c ensure induction works.
- Property d ensures $V^TV$ is tridiagonal, so $M_n(c)$ is positive-definite.
Decomposition Theorem

**Proof idea.** Use induction on dimension. Apply Brion–Vergne decomposition with respect to a line carefully chosen, to the dual cone. Then take the dual again.

![Diagram showing decomposition process]

**Corollary (Fitisone–Zhou, 2023)**

The decomposition theorem gives explicitly $N \leq (n - 1)!$ cones $C_i$ whose $\tilde{\Omega}_n(C_i)$ can be computed via the power series formula $T_\alpha$ and the signs $s_i \in \{\pm 1\}$, such that

$$\tilde{\Omega}_n(C) = \sum_{i=1}^{N} s_i \tilde{\Omega}_n(C_i).$$

We can make the cones $C_i$ to have tridiagonal associated matrices.
Issue with $T_\alpha$: too many terms $a \in \mathbb{N}^{\binom{n}{2}}$

$$T_\alpha = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{a \in \mathbb{N}^{\binom{n}{2}}} \left[ \frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma \left( \frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

Recall property ⚫ from the theorem: tridiagonal $V^T V$!

$$V^T V = \begin{bmatrix}
1 & \beta_1 & 0 & \ldots & 0 \\
\beta_1 & 1 & \beta_2 & \ldots & \vdots \\
0 & \ldots & \beta_{n-2} & 1 & \beta_{n-1} \\
\vdots & \ldots & 0 & \beta_{n-1} & 1 \\
0 & \ldots & \ldots & \ldots & 1
\end{bmatrix}$$

Remedy: series $T_\alpha$ in $\binom{n}{2}$ variables simplifies to series in $(n - 1)$ variables

$$T_\beta = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{\mathbf{b} \in \mathbb{N}^{n-1}} A_{\mathbf{b}} \beta^{\mathbf{b}},$$

where for any multiexponent $\mathbf{b} = (b_1, \ldots, b_{n-1})$ in $\mathbb{N}^{n-1}$,

$$A_{\mathbf{b}} := \frac{(-2)^{\sum b_i}}{\prod_{i=1}^{n-1} b_i!} \Gamma \left( \frac{1 + b_1}{2} \right) \Gamma \left( \frac{1 + b_1 + b_2}{2} \right) \cdots \Gamma \left( \frac{1 + b_{n-2} + b_{n-1}}{2} \right) \Gamma \left( \frac{1 + b_{n-1}}{2} \right).$$
Eigenvalue and domain of convergence

Suppose $V^T V$ is tridiagonal.
Suppose the $v_i$'s are not all pairwise orthogonal (otherwise $\tilde{\Omega}_n(C) = \frac{1}{2^n}$).

**Proposition**

- $M_n(C)$ has the same eigenvalues as $V^T V$.
- The smallest eigenvalue $\lambda_{\text{min}}$ satisfies $0 < \lambda_{\text{min}} < 1$.

When does the hypergeometric series converge / diverge?

$$ T(x) = \sum_{b \in \mathbb{N}^{n-1}} A_b \, x_1^{b_1} x_2^{b_2} \cdots x_{n-1}^{b_{n-1}} $$

**Proposition**

- $(\beta_1, \ldots, \beta_{n-1})$ lies in the domain of convergence of $T(x)$.
- $\left( \frac{\beta_1}{1-\lambda_{\text{min}}}, \ldots, \frac{\beta_{n-1}}{1-\lambda_{\text{min}}} \right)$ lies on the boundary of convergence domain.
Asymptotic truncation error of the series

Truncating the series in partial degrees \((N_1, \ldots, N_{n-1})\),
the error term is bounded by

\[
E(N_1, \ldots, N_{n-1}) = \sum_{b \in B} \left| A_b \beta^b \right|,
\]

where \(B = \{ b \in \mathbb{N}^{n-1} \mid b_i \geq N_i \text{ for at least one } i \}\).

We show the asymptotic decay of \(E(N_1, \ldots, N_{n-1})\) in relation to \(1 - \lambda_{\text{min}}\).

**Theorem (Fitisone–Zhou, 2023)**

For any \(\rho\) such that \(1 - \lambda_{\text{min}} < \rho < 1\), there exist partial degrees \(N_1, \ldots, N_{n-1}\) such that for any integer \(\ell \geq 1\), we have

\[
E(N_1 + \ell, \ldots, N_{n-1} + \ell) \leq \rho^\ell E(N_1, \ldots, N_{n-1}).
\]
Asymptotic truncation error proof

Proof idea (for case $n = 3$).

- Since $\left(\frac{\beta_1}{1-\lambda_{\min}}, \frac{\beta_2}{1-\lambda_{\min}}\right)$ lies on the boundary of convergence domain, there exist $x_1, x_2 \in \mathbb{R}^+$ such that $\frac{|\beta_i|}{1-\lambda_{\min}} = \frac{1}{|\Psi_i(x_1, x_2)|}$ for $i = 1, 2$.

- $\Psi_i(b) = \lim_{t \to \infty} \frac{A_{tb} + e_i}{A_{tb}}$ rational and homogeneous of degree zero.

- $\frac{|A_{b_1+1, b_2} \beta_1^{b_1+1} \beta_2^{b_2}|}{|A_{b_1, b_2} \beta_1^{b_1} \beta_2^{b_2}|}, \frac{|A_{b_1+1, b_2+1} \beta_1^{b_1} \beta_2^{b_2+1}|}{|A_{b_1, b_2} \beta_1^{b_1} \beta_2^{b_2}|} \leq (1 - \lambda_{\min})(1 + \epsilon) =: \mu$

- $E(N_1 + \ell, N_2 + \ell) \leq S_1' + \cdots + S_4' \leq \mu^\ell (1 + \epsilon)^\ell (S_1 + S_2 + S_3) \leq \rho^\ell E(N_1, N_2)$
Thank you!

- **Manuscript:**
  A. Fitisone and Y. Zhou.
  Solid angle measure of polyhedral cones
  eprint arXiv:2304.11102 [math.MG], 2023

- **SageMath code:**
  https://github.com/yuan-zhou/solid-angle-code