Measuring the Importance of Facets of Cyclic Group Polyhedra Using Solid Angles

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Joint work with Allison Fitisone

Overview

1. Cyclic group relaxation of IP
2. Measure relative sizes of the facets
3. Compute solid angle measure in $\mathbb{R}^n$

General purpose cutting planes

Many widely used general-purpose cuts are derived from a **single row** of the optimal simplex tableau:

$$\mathbf{x}_B = A_B^{-1}\mathbf{b} + (-A_B^{-1}A_N)\mathbf{x}_N, \quad \mathbf{x}_B \in \mathbb{Z}_+^B,\ \mathbf{x}_N \in \mathbb{Z}_+^N.$$

Gomory Mixed Integer Cut Example:

- Tableau row:

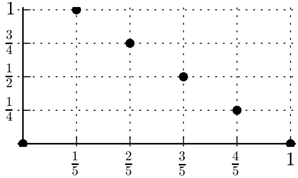
$$x = -(-\tfrac{4}{5}) + \tfrac{2}{5}x_1 - \tfrac{1}{5}x_2 + \tfrac{11}{5}x_3,$$

where $x, x_1, x_2, x_3 \in \mathbb{Z}_+$.

- Determine the cut:

$$?x_1 + ?x_2 + ?x_3 \geq 1.$$

- Cyclic group order $q = 5$
- $(-\frac{4}{5}) \equiv \frac{1}{5} \pmod 1$
- GMIC function π_{mic}



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- Determine the cut:

$$\tfrac{3}{4}x_1 + \tfrac{1}{4}x_2 + 1x_3 \geq 1.$$

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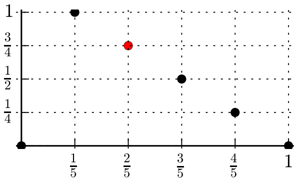

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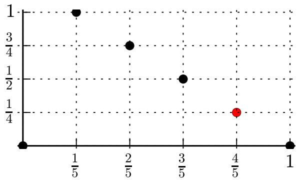

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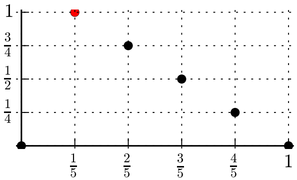


Cyclic group relaxation

- Tableau row: $x = -(-\frac{4}{5}) + \frac{2}{5}x_1 - \frac{1}{5}x_2 + \frac{11}{5}x_3, \quad x, x_1, x_2, x_3 \in \mathbb{Z}_+$.
  - Relax x from $\mathbb{Z}_+$ to $\mathbb{Z}$:

$$0 \equiv -\frac{1}{5} + \frac{2}{5}x_1 + \frac{4}{5}x_2 + \frac{1}{5}x_3 \pmod 1$$

  - Introduce $\mathbf{z} \in \mathbb{Z}_+^{q-1}$:

$$\frac{1}{5} \equiv \frac{1}{5}z_1 + \frac{2}{5}z_2 + \frac{3}{5}z_3 + \frac{4}{5}z_4 \pmod 1$$

  - $q = 5$ and $f = 1$
- **Master cyclic group polyhedron**

$$P(q, f) := \operatorname{conv}\left\{\mathbf{z} \in \mathbb{Z}_+^{q-1} \mid \sum_{i=1}^{q-1} \left(\frac{i}{q}\right) z_i \equiv \frac{f}{q} \pmod 1\right\}$$

- We are interested in $\pi \in \mathbb{R}_+^{q-1}$ s.t. $\pi \cdot \mathbf{z} \geq 1$ for all $\mathbf{z} \in P(q, f)$, and in particular, the facet-defining ones.

Master cyclic group polyhedron and Group-facet polytope

- Blocker $\mathfrak{B}\left(P(q,f)\right) = \left\{\pi \in \mathbb{R}_+^{q-1} \mid \pi \cdot \mathbf{z} \geq 1 \; \forall\, \mathbf{z} \in P(q,f)\right\}$
- $P(q,f) \subseteq \mathbb{R}_+^{q-1}$ is full-dimensional, and has $\mathbb{R}_+^{q-1}$ as recession cone.

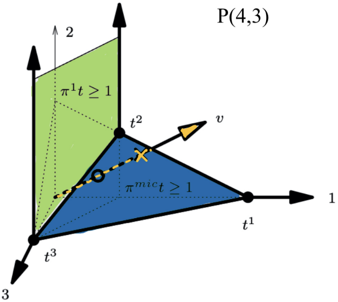
$$\Longrightarrow \mathfrak{B}\left(P(q,f)\right) = \Pi(q,f) + \mathbb{R}_+^{q-1}$$

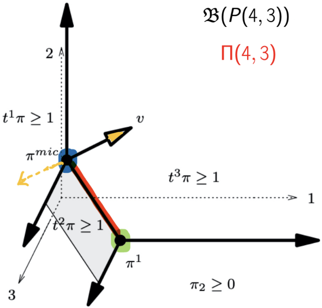
- **Group-facet polytope** $\Pi(q,f)$ consists of $\pi = \left(\pi_1, \ldots, \pi_{q-1}\right)$ with $\pi_0 = 0, \pi_f = 1$ and satisfies
  - (nonnegativity) $\pi_i \geq 0$ for $i = 1, \ldots, q-1$
  - (symmetry) $\pi_i + \pi_j = 1$, $i + j \equiv f \pmod q$
  - (subadditivity) $\pi_i + \pi_j \geq \pi_k$, $i + j \equiv k \pmod q$
- Non-trivial facet of $P(q,f)$ $\Leftrightarrow$ vertex of $\Pi(q,f)$
- Number of facets / vertices π grows exponentially with q.
- Which π's are more important?

Relative sizes of facets — Shooting experiment

[Kuhn, 1991], [Evans, 2002], [Gomory–Johnson–Evans, 2003], [Hunsaker, 2003], [Dash–Günlük, 2006], [Shim, 2009], [Shim–Johnson, 2013], [Chopra–Shim–Steffy, 2015]

- Choose random direction $\mathbf{v} \in \mathbb{R}_+^{q-1}$.
- Move from the origin along the ray $\lambda\mathbf{v}$ $(\lambda > 0)$ until hitting $P(q, f)$.
- Record which facet π^* is hit: $\pi^* = \arg\min_\pi \{\pi \cdot \mathbf{v} \mid \pi \in \Pi(q, f)\}$.
- Facets that receive more hits are deemed more important.





Relative sizes of facets — Shooting experiment

Proportion of hits received by facets of $P(7,6)$ according to [Hunsaker, 2003] (out of 1000 shots) and [Shim, 2009] (geometry)

Facets of $P(7,6)$						Hunsaker	Shim
1	2	3	4	5	*6*	0.332	0.3238
9	4	6	8	3	*12*	0.253	0.2500
4	8	5	2	6	*10*	0.245	0.2500
6	5	4	3	2	*8*	0.170	0.1762

Discrepancies?

Relative sizes of facets — Shooting experiment

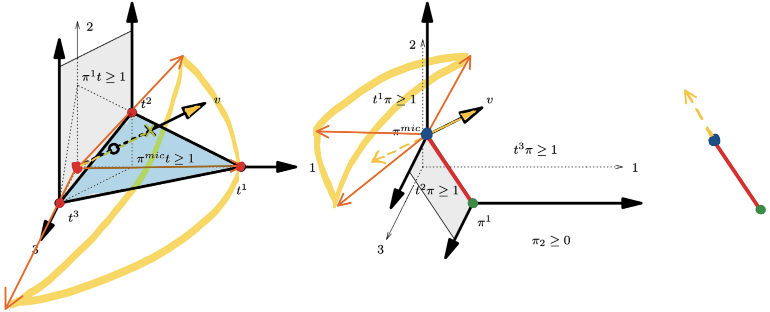
Proportion of hits received by facets of $P(11,10)$ according to [Hunsaker, 2003] (out of 1000 shots) and [Shim, 2009] (out of 10000 shots)

Facets of $P(11,10)$										Hunsaker	Shim
1	2	3	4	5	6	7	8	9	*10*	0.134	0.1354
4	8	12	16	9	2	6	10	14	*18*	0.140	0.1291
9	18	16	3	12	21	8	6	15	*24*	0.089	0.0935
10	9	8	7	6	5	4	3	2	*12*	0.092	0.0872
16	21	4	20	14	8	24	7	12	*28*	0.080	0.0816
25	6	20	12	15	18	10	24	5	*30*	0.083	0.0813
8	5	2	10	7	4	12	9	6	*14*	0.070	0.0579
6	12	7	2	8	14	9	4	10	*16*	0.069	0.0646
20	7	16	14	12	10	8	17	4	*24*	0.033	0.0403
15	8	12	5	9	13	6	10	3	*18*	0.030	0.0404
13	4	6	8	10	12	14	16	7	*20*	0.037	0.0402
6	12	7	13	8	3	9	4	10	*16*	0.038	0.0297
4	8	12	5	9	13	6	10	14	*18*	0.022	0.0276
9	18	5	14	12	10	19	6	15	*24*	0.026	0.0250
14	6	20	12	15	18	10	24	16	*30*	0.013	0.0187
18	14	10	6	13	20	16	12	8	*26*	0.015	0.0173
13	15	6	8	10	12	14	5	7	*20*	0.012	0.0162
9	18	16	14	12	10	8	6	15	*24*	0.017	0.0140

Shooting experiment size estimates solid angle

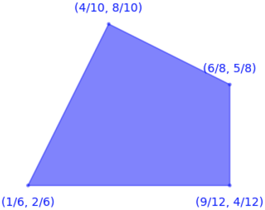
Proposition

Shooting experiment size of a facet $\pi \cdot \mathbf{z} \geq 1$ *of* $P(q, f)$
$\simeq$ *solid angle subtended by the facet of* $P(q, f)$ *at the origin*
$=$ *solid angle of the normal cone at the vertex* π *of* $\mathfrak{B}\,(P(q, f))$
$=$ *(normalized) solid angle of the normal cone at the vertex* π *of* $\Pi(q, f)$



Relative sizes of facets — Solid angle

Facets of $P(7,6)$						Solid angle	Hunsaker	Shim
1	**2**	3	4	5	*6*	0.3238	0.332	0.3238
9	**4**	6	8	3	*12*	0.2500	0.253	0.2500
4	**8**	5	2	6	*10*	0.2500	0.245	0.2500
6	**5**	4	3	2	*8*	0.1762	0.170	0.1762



Relative sizes of facets — Solid angle

Facets of $P(11,10)$										Solid angle	Hunsaker	Shim
1	**2**	**3**	**4**	5	6	7	8	9	*10*	0.1331	0.134	0.1354
4	**8**	**12**	**16**	9	2	6	10	14	*18*	0.1272	0.140	0.1291
9	**18**	**16**	**3**	12	21	8	6	15	*24*	0.0908	0.089	0.0935
10	**9**	**8**	**7**	6	5	4	3	2	*12*	0.0895	0.092	0.0872
16	**21**	**4**	**20**	14	8	24	7	12	*28*	0.0831	0.080	0.0816
25	**6**	**20**	**12**	15	18	10	24	5	*30*	0.0809	0.083	0.0813
8	**5**	**2**	**10**	7	4	12	9	6	*14*	0.0627	0.070	0.0579
6	**12**	**7**	**2**	8	14	9	4	10	*16*	0.0605	0.069	0.0646
20	**7**	**16**	**14**	12	10	8	17	4	*24*	0.0411	0.033	0.0403
15	**8**	**12**	**5**	9	13	6	10	3	*18*	0.0400	0.030	0.0404
13	**4**	**6**	**8**	10	12	14	16	7	*20*	0.0370	0.037	0.0402
6	**12**	**7**	**13**	8	3	9	4	10	*16*	0.0353	0.038	0.0297
4	**8**	**12**	**5**	9	13	6	10	14	*18*	0.0286	0.022	0.0276
9	**18**	**5**	**14**	12	10	19	6	15	*24*	0.0236	0.026	0.0250
14	**6**	**20**	**12**	15	18	10	24	16	*30*	0.0190	0.013	0.0187
18	**14**	**10**	**6**	13	20	16	12	8	*26*	0.0175	0.015	0.0173
13	**15**	**6**	**8**	10	12	14	5	7	*20*	0.0158	0.012	0.0162
9	**18**	**16**	**14**	12	10	8	6	15	*24*	0.0130	0.017	0.0140

Motivations for studying solid angles

- Draw more accurate conclusions regarding the importance of facets of master cyclic group polyhedra
- Estimate the number of simplices in a triangulation of an n-cube
- Solid angle sum over rational polytope:

  Ehrhart polynomial and lattice counting
- Approximate the relative volume in many applications:
  - pixel purity index (PPI) scores
  - feasibility domain of an ecological community
  - ......

Normalized solid angle measure

Definition (Property)

The **normalized solid angle measure** of a cone $C \subseteq \mathbb{R}^n$ with respect to $\mathbb{R}^n$ is defined as

$$\tilde{\Omega}_n(C) = \frac{\mathrm{vol}_{n-1}(C \cap S_{n-1})}{\mathrm{vol}_{n-1}(S_{n-1})} \left(= \frac{\mathrm{vol}_n(C \cap B_n)}{\mathrm{vol}_n(B_n)} = \frac{\int_C f(\mathbf{x})d\mathbf{x}}{\int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x}} \right),$$

where

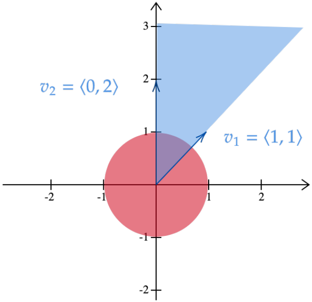
- vol_n is the usual volume form in $\mathbb{R}^n$,
- S_{n-1} is the unit $(n-1)$-sphere (residing in $\mathbb{R}^n$) centered at $\mathbf{0}$,
- B_n is the unit ball in $\mathbb{R}^n$ centered at $\mathbf{0}$
- $f\colon \mathbb{R}^n \to \mathbb{R}$ is any function that is invariant under rotations around $\mathbf{0}$, e.g. $f = e^{-\|\mathbf{x}\|^2}$

In the following, we consider **simplicial** cones $C = \mathfrak{c}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

Solid angles in $\mathbb{R}^2$

In $\mathbb{R}^2$, the normalized solid (or plane) angle measure can be computed via the standard inner product.

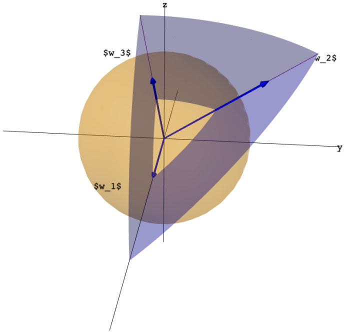
$$\tilde{\Omega}_2(\mathfrak{c}(v_1, v_2)) = \frac{1}{2\pi} \cos^{-1}\left(\frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|}\right)$$



$$\tilde{\Omega}_2(\mathfrak{c}(v_1, v_2)) = \frac{1}{2\pi} \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{8}$$

Solid angles in $\mathbb{R}^3$

The normalized solid angle measure is the ratio of the area of the spherical triangle formed by the unit vectors to the surface area of the unit sphere.



$$\tilde{\Omega}_3(\mathfrak{c}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)) = \frac{1}{4\pi}\left(2\tan^{-1}\left(\frac{|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|}{1 + \mathbf{v}_2 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_3}\right)\right).$$

Solid angles in $\mathbb{R}^n$

$$\tilde{\Omega}_n(C) = \frac{\mathsf{vol}_{n-1}(C \cap S_{n-1})}{\mathsf{vol}_{n-1}(S_{n-1})} = \frac{\mathsf{vol}_n(C \cap B_n)}{\mathsf{vol}_n(B_n)} = \frac{\int_C f(\mathbf{x})d\mathbf{x}}{\int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x}}$$

- Computing $\tilde{\Omega}_n(\mathfrak{c}(\mathbf{v}_1, \ldots, \mathbf{v}_n))$ amounts to computing the volume of a "spherically faced simplex"
- Closed formula not known
- Probabilistic methods — require large samples
- Numerical integration methods — high in computational cost
- We consider the multivariate power series method independently discovered by [Aomoto, 1977] and [Ribando, 2006]

Solid angle formula — Ribando's hypergeometric series

Theorem (Ribando, 2006)

Let $C \subseteq \mathbb{R}^n$ be the simplicial cone generated by the unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let $V \in \mathbb{R}^{n \times n}$ be the matrix whose i^{th} column is $\mathbf{v}_i$. Let $\alpha_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ for $1 \leq i, j \leq n$. Define

$$T_\alpha = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

*Then, T_α **converges absolutely to** $\tilde{\Omega}_n(C)$, if and only if its **associated matrix** $M_n(C)$ **is positive definite**.*

$$M_n(C) = \begin{bmatrix} 1 & -|\mathbf{v}_1 \cdot \mathbf{v}_2| & \cdots & -|\mathbf{v}_1 \cdot \mathbf{v}_n| \\ -|\mathbf{v}_2 \cdot \mathbf{v}_1| & 1 & \cdots & -|\mathbf{v}_2 \cdot \mathbf{v}_n| \\ \vdots & \vdots & \ddots & \vdots \\ -|\mathbf{v}_n \cdot \mathbf{v}_1| & -|\mathbf{v}_n \cdot \mathbf{v}_2| & \cdots & 1 \end{bmatrix}$$

Ribando's hypergeometric series

$$T_\alpha = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{\mathbf{a}\in\mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma\left(\frac{1+\sum_{m\neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

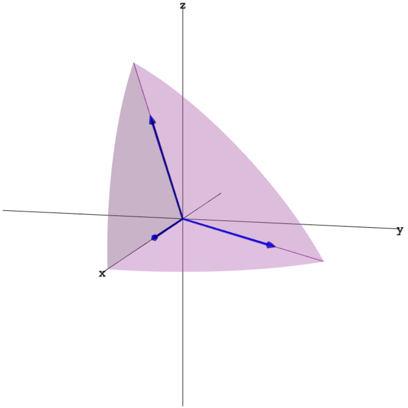
- $\alpha = (\alpha_{12}, \alpha_{13}, \ldots \alpha_{n-1,n})$ is a multivariable in $\binom{n}{2}$ variables;
- $\mathbf{a} = (a_{12}, a_{13}, \ldots, a_{n-1,n})$ is a multiexponent;
- $\alpha^{\mathbf{a}} := \prod_{i<j}^{n} \alpha_{ij}^{a_{ij}}$;
- $\sum_{m\neq i} a_{im} = \sum_{j<i} a_{ji} + \sum_{j>i} a_{ij}$;
- Γ is the Euler-Gamma function.
- T_α is a hypergeometric series as the ratio of the neighboring coefficients is a rational function of the index.

Issue with T_α: convergence condition not met

Let $v_1 = [1,1,0]$, $v_2 = [1,0,1]$, $v_3 = [1,0,0]$, and $C = \mathfrak{c}(v_1, v_2, v_3)$.

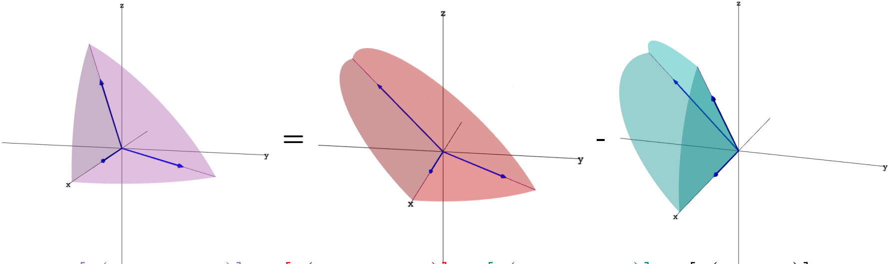
- Scale to unit vectors;
- $V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$
- $\implies V^t V = \begin{bmatrix} 1 & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \end{bmatrix}$
- $M_3(C) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \end{bmatrix}$

  is NOT positive-definite



Remedy: Cone decomposition

Let $\mathbf{v}_1 = [1,1,0]$, $\mathbf{v}_2 = [1,0,1]$, $\mathbf{v}_3 = [1,0,0]$, $\mathbf{v}_4 = [0,-1,1]$.



$$[\mathfrak{c}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)] = [\mathfrak{c}(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4)] - [\mathfrak{c}(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)] + [\mathfrak{c}(\mathbf{v}_3,\mathbf{v}_4)]$$

$$\tilde{\Omega}_3\left(\mathfrak{c}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\right) = \tilde{\Omega}_3\left(\mathfrak{c}(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4)\right) - \tilde{\Omega}_3\left(\mathfrak{c}(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)\right) + 0$$

where $[C]$ denotes the indicator function of the cone C,
and $\mathfrak{c}(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4)$, $\mathfrak{c}(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)$ have positive-definite associated matrices.

Decomposition Theorem

Theorem (Fitisone–Zhou, 2023)

Given linearly independent unit vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in \mathbb{R}^n$*, the cone* $\mathfrak{c}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ *can be decomposed into a finite family of cones, each of which is either:*

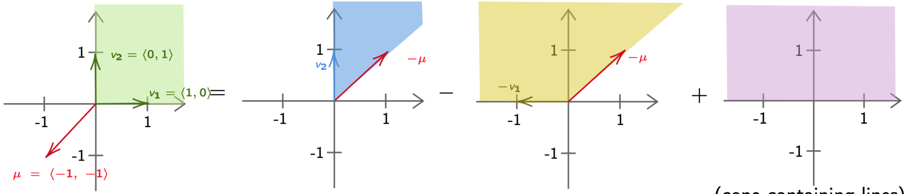
I. *a cone of affine dimension less than* n*, or*
II. *a full-dimensional cone* $\mathfrak{c}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ *such that*
    a. $\|\mathbf{v}_i\| = 1$ *for* $i = 1, 2, \ldots, n$
    b. $\mathbf{v}_n = \mathbf{w}_n$
    c. $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1} \rangle = \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1} \rangle$
    d. $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ *for all* $1 \leq i, j \leq n$ *such that* $j \neq i, j \neq i \pm 1$.

Remarks:

- A cone in I has solid angle 0.
- Properties b and c ensure induction works.
- Property d ensures $V^T V$ is tridiagonal, so $M_n(\mathfrak{c})$ is positive-definite.

Decomposition Theorem

Proof idea. Use induction on dimension. Apply Brion–Vergne decomposition with respect to a line carefully chosen, to the dual cone. Then take the dual again.



(cone containing lines)

Corollary (Fitisone–Zhou, 2023)

The decomposition theorem gives explicitly $N \le (n-1)!$ cones C_i whose $\tilde{\Omega}_n(C_i)$ can be computed via the power series formula T_α and the signs $s_i \in \{\pm 1\}$, such that

$$\tilde{\Omega}_n(C) = \sum_{i=1}^{N} s_i \tilde{\Omega}_n(C_i).$$

We can make the cones C_i to have tridiagonal associated matrices.

Issue with T_α: too many terms $\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}$

$$T_\alpha = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{\mathbf{a} \in \mathbb{N}^{\binom{n}{2}}} \left[\frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j} a_{ij}!} \prod_{i=1}^{n} \Gamma\left(\frac{1 + \sum_{m \neq i} a_{im}}{2} \right) \right] \alpha^{\mathbf{a}}.$$

Recall property (d) from the theorem: tridiagonal $V^T V$!

$$V^T V = \begin{bmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \beta_{n-2} & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & \beta_{n-1} & 1 \end{bmatrix}$$

Remedy: series T_α in $\binom{n}{2}$ variables simplifies to series in $(n-1)$ variables

$$T_\beta = \frac{|\det V|}{(4\pi)^{\frac{n}{2}}} \sum_{\mathbf{b} \in \mathbb{N}^{n-1}} A_{\mathbf{b}} \beta^{\mathbf{b}},$$

where for any multiexponent $\mathbf{b} = (b_1, \ldots, b_{n-1})$ in $\mathbb{N}^{n-1}$,

$$A_{\mathbf{b}} := \frac{(-2)^{\sum b_i}}{\prod_{i=1}^{n-1} b_i!} \Gamma\left(\frac{1+b_1}{2}\right) \Gamma\left(\frac{1+b_1+b_2}{2}\right) \cdots \Gamma\left(\frac{1+b_{n-2}+b_{n-1}}{2}\right) \Gamma\left(\frac{1+b_{n-1}}{2}\right).$$

Eigenvalue and domain of convergence

Suppose $V^T V$ is tridiagonal.
Suppose the $\mathbf{v}_i$'s are **not** all pairwise orthogonal (otherwise $\tilde{\Omega}_n(C) = \frac{1}{2^n}$).

Proposition

- *$M_n(C)$ has the same eigenvalues as $V^T V$.*
- *The smallest eigenvalue $\lambda_{\min}$ satisfies $0 < \lambda_{\min} < 1$.*

When does the hypergeometric series converge / diverge?

$$T(\mathbf{x}) = \sum_{\mathbf{b} \in \mathbb{N}^{n-1}} A_{\mathbf{b}} \, x_1^{b_1} x_2^{b_2} \cdots x_{n-1}^{b_{n-1}}$$

Proposition

- *$(\beta_1, \ldots, \beta_{n-1})$ lies in the domain of convergence of $T(\mathbf{x})$.*
- *$\left(\frac{\beta_1}{1-\lambda_{\min}}, \ldots, \frac{\beta_{n-1}}{1-\lambda_{\min}}\right)$ lies on the boundary of convergence domain.*

Asymptotic truncation error of the series

Truncating the series in partial degrees $(N_1, \ldots, N_{n-1})$,
the error term is bounded by

$$E(N_1, \ldots, N_{n-1}) = \sum_{\mathbf{b} \in B} \left| A_{\mathbf{b}} \beta^{\mathbf{b}} \right|,$$

where $B = \{\mathbf{b} \in \mathbb{N}^{n-1} \mid b_i \geq N_i \text{ for at least one } i\}$.

We show the asymptotic decay of $E(N_1, \ldots, N_{n-1})$ in relation to $1 - \lambda_{\min}$.

Theorem (Fitisone–Zhou, 2023)

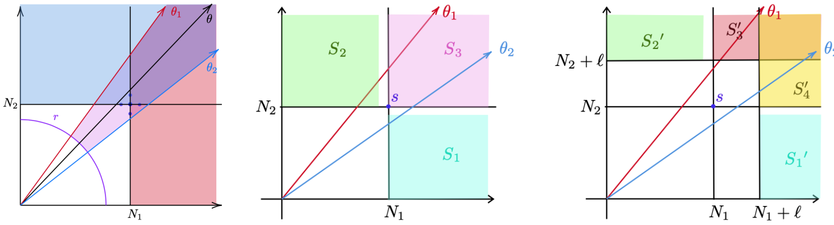
For any ρ such that $1 - \lambda_{\min} < \rho < 1$, there exist partial degrees $N_1, \ldots, N_{n-1}$ such that for any integer $\ell \geq 1$, we have

$$E(N_1 + \ell, \ldots, N_{n-1} + \ell) \leq \rho^{\ell}\, E(N_1, \ldots, N_{n-1}).$$

Asymptotic truncation error proof

Proof idea (for case $n = 3$).

- Since $\left(\frac{\beta_1}{1-\lambda_{\min}}, \frac{\beta_2}{1-\lambda_{\min}}\right)$ lies on the boundary of convergence domain, there exist $x_1, x_2 \in \mathbb{R}_+$ such that $\frac{|\beta_i|}{1-\lambda_{\min}} = \frac{1}{|\Psi_i(x_1,x_2)|}$ for $i = 1, 2$, $\Psi_i(\mathbf{b}) = \lim_{t\to\infty} \frac{A_{t\mathbf{b}+\mathbf{e}_i}}{A_{t\mathbf{b}}}$ rational and homogeneous of degree zero.
- $\frac{\left|A_{b_1+1,b_2}\, \beta_1^{b_1+1} \beta_2^{b_2}\right|}{\left|A_{b_1,b_2}\, \beta_1^{b_1} \beta_2^{b_2}\right|}, \frac{\left|A_{b_1,b_2+1}\, \beta_1^{b_1} \beta_2^{b_2+1}\right|}{\left|A_{b_1,b_2}\, \beta_1^{b_1} \beta_2^{b_2}\right|} \leq (1-\lambda_{\min})(1+\epsilon) =: \mu$
- $E(N_1+\ell, N_2+\ell) \leq S_1' + \cdots + S_4' \leq \mu^\ell (1+\epsilon)^\ell (S_1+S_2+S_3) \leq \rho^\ell E(N_1, N_2)$



Thank you!

- Manuscript:
  A. Fitisone and Y. Zhou.
  Solid angle measure of polyhedral cones
  eprint arXiv:2304.11102 [math.MG], 2023
- SageMath code:
  https://github.com/yuan-zhou/solid-angle-code