Solving Mixed-Integer Semidefinite Programs

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based on joint work with
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ICERM Workshop “Trends in Computational Discrete Optimization”

MISDP

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We consider solving Mixed-Integer Semidefinite Programs (MISDPs):

$$\begin{aligned} \inf \quad & b^\top y \\ \text{s.t.} \quad & \sum_{k=1}^{m} A^k y_k - A^0 \succeq 0, \\ & \ell_i \le y_i \le u_i && \forall\, i \in [m], \\ & y_i \in \mathbb{Z} && \forall\, i \in I, \end{aligned}$$

- symmetric matrices $A^k \in \mathbb{R}^{n \times n}$ for $k \in [m]_0 := \{0, \dots, m\}$, $b \in \mathbb{R}^m$
- bounds: $\ell_i \in \mathbb{R} \cup \{-\infty\}$, $u_i \in \mathbb{R} \cup \{\infty\}$ for all $i \in [m] := \{1, \dots, m\}$.
- integer variables: $I \subseteq [m]$

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Linear constraints can be expressed as SDP blocks with diagonal entries only. Thus, Mixed Integer Programs (MIPs) are a special case.

Goals of This Talk

The goals of this talk are:

- Explain how MISDPs can be solved.
- Present several improvement techniques:
  - Dual Fixing
  - Presolving
  - Conflict Analysis
  - Symmetry Handling
- Evaluate performance.
- Discuss similarities and differences to mixed-integer programming.

Following the title of the workshop, this talk will focus on computational aspects.

Overview

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1. Applications
2. Solution Methods
3. Dual Fixing
4. Presolving MISDPs
   - Bound Tightening
   - Computational Results
5. Conflict Analysis
6. Symmetry
7. Conclusions

Applications

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MISDPs have many applications:

- Quadratic TSP [Renata Sotirov $\rightarrow$ earlier ICERM workshop]
- Cardinality Constrained Least Squares
- Minimum k-Partitioning
- Computing restricted isometry constants in compressed sensing
- Optimal transmission switching problem in AC power flow
- Robustification of physical parameters in gas networks
- Subset selection for eliminating multicollinearity
- ...

Application Example: Robust Truss Topology Design

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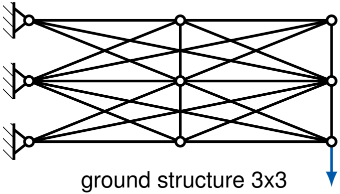
- n nodes $V \subset \mathbb{R}^d$
- n_f free nodes $V_f \subset V$
- m possible trusses E
- forces $f \in \mathbb{R}^{d_f}$ for $d_f = d \cdot n_f$



ground structure 3x3

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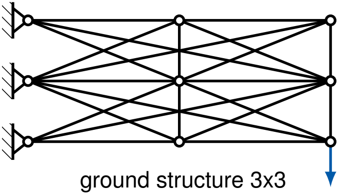
- Choose cross-sectional areas for trusses minimizing volume while creating a “stable” truss.
- Stability is measured by the compliance $\frac{1}{2} f^T u$ with node displacements u.



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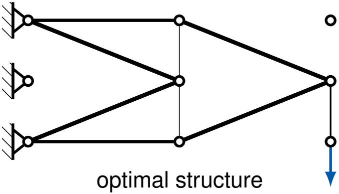

optimal structure

Application Example: Robust Truss Topology Design

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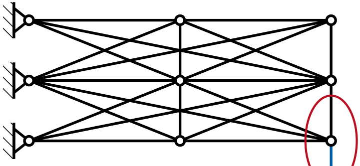

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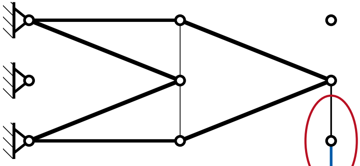

optimal structure

- Use uncertainty set $\{f \in \mathbb{R}^{d_f} \; : \; f = Qg : \|g\|_2 \leq 1\}$ instead of single force f.
- Restrict cross-sectional areas $x \in \mathbb{R}^m_+$ to a discrete set $\mathcal{A}$.

Application Example: Robust Truss Topology Design

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Elliptic Robust Discrete TTD [Ben-Tal/Nemirovski 1997; Mars 2013]

$$
\begin{aligned}
\inf \quad & \sum_{e \in E} \ell_e \sum_{a \in \mathcal{A}} a\, x_e^a \\
\text{s.t.} \quad & \begin{pmatrix} 2C_{\max} I & Q^T \\ Q & S(x) \end{pmatrix} \succeq 0, \\
& \sum_{a \in \mathcal{A}} x_e^a \leq 1 && \forall e \in E, \\
& x_e^a \in \{0, 1\} && \forall e \in E, a \in \mathcal{A},
\end{aligned}
$$

with truss lengths ℓ_e, upper bound $C_{\max}$ on compliance and stiffness matrix

$$S(x) = \sum_{e \in E} \sum_{a \in \mathcal{A}} S_e\, a\, x_e^a$$

for positive semidefinite, rank-one single truss stiffness matrices S_e.

Overview

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Solving Methods for MISDPs

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1. SDP-based branch-and-bound: Solve SDP-relaxations (special case of NLP-based B&B [Dakin 1965])
2. LP-based branch-and-bound: Cutting plane method based on LP-relaxations [Sherali and Fraticelli 2002]; [Krishnan and Mitchell 2006]
3. Outer approximation: Solve MIP-relaxations [Duran and Grossmann 1986].

Implementations:

1. YALMIP [Löfberg 2004] and SCIP-SDP
2. YALMIP and SCIP-SDP
3. Pajarito [Coey, Lubin, and Vielma 2020]

SDP-based Branch-and-Bound

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- Relax integrality.
- Branch on integral y-variables.
- Need to solve a continuous SDP in each branch-and-bound node.
- Relaxations can be solved by problem-specific approaches (e.g. conic bundle or low-rank methods) or interior-point solvers.
- Convergence assumptions of SDP-solvers should be satisfied.
- Usually much slower than solving LPs and no warmstart.

LP-based Approach

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For LP-based approach and outer approximation:

- ▷ Usual approach for convex MINLP: gradient cuts

$$g_j(\overline{x}) + \nabla g_j(\overline{x})^\top (x - \overline{x}) \leq 0.$$

- ▷ But function of smallest eigenvalue is not differentiable everywhere.
- ▷ Instead use characterization $X \succeq 0 \quad \Leftrightarrow \quad u^\top X u \geq 0$ for all $u \in \mathbb{R}^n$.
- ▷ If $Z := \sum_{k=1}^m A^k y_k^* - A^0 \not\succeq 0$, compute eigenvector v to smallest eigenvalue. Then

$$v^\top Z v = \sum_{k=1}^m v^\top A^k v\, y_k - v^\top A^0 v \geq 0$$

is valid and cuts off $y^* \quad \rightarrow$ Eigenvector cut.

Cutting Planes: MISOCP vs. MISDP

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- Cutting planes are often used by solvers for mixed-integer second-order cone problems (MISOCPs).
- Approximation for SOCPs possible with polynomial number of cuts [Ben-Tal/Nemirovski 2001].
- Approximation for SDPs needs exponential number of cuts:

Theorem ([Braun, Fiorini, Pokutta, Steurer 2015])

There are SDPs of dimension $n \times n$ for which any polyhedral approximation is of size $2^{\Omega(n)}$.

SCIP-SDP

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Our solver: SCIP-SDP

- ▷ Based on SCIP (`www.scipopt.org`)
- ▷ Supports both SDP-based B&B and LP-based branch-and-cut.
- ▷ Introduced by [Mars 2013], continued by [Gally 2019] and Matter [2022], ...
- ▷ Apache 2.0 license.
- ▷ Current version: 4.1: `wwwopt.mathematik.tu-darmstadt.de/scipsdp`
- ▷ Approximately 50 000 lines of C-code
- ▷ SDP-solvers: interfaces to Mosek, DSDP, SDPA
- ▷ Matlab-Interface: `github.com/scipopt/MatlabSCIPInterface`

Computations:

- ▷ Use SCIP developer version (8.0.3).
- ▷ Use Mosek 9.2.40 for solving SDP-relaxations.
- ▷ Linux cluster with 3.5 GHz Intel Xeon E5-1620 Quad-Core CPUs.
- ▷ Nodes and times are shifted geometric means.
- ▷ Time limit 1 h.

Comparison of SDP and LP-based Approach

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Testset: 185 instances from different sources.

type	# solved	# nodes	time
SDP	167	1066.1	132.2
LP	109	419.2	336.5
all optimal (106):			
SDP		605.0	93.2
LP		507.0	63.2

Conclusions:

- LP-based approach solves significantly less instances.
- On the instances solved by both, it is faster by 32 % and uses less nodes.
- Open question: Predict which method is faster and explain why.

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Dual Fixing

- Extension of reduced-cost fixing to general MINLPs by [Ryoo and Sahinidis 1996] and primal MISDPs by [Helmberg 2000].
- Our approach uses conic duality and only requires feasibility.

Theorem [Gally, P., Ulbrich 2018]

- (X, W, V): Primal feasible solution, where W, V are primal variables corresponding to variable bounds ℓ, u in the dual,
- f: Corresponding primal objective value,
- L: Lower bound on the optimal objective value of the MISDP.

Then for every optimal solution $y^\star$ of the MISDP

$$y_j^\star \leq \ell_j + \frac{f - L}{W_{jj}} \text{ if } \ell_j > -\infty \quad \text{and} \quad y_j^\star \geq u_j - \frac{f - L}{V_{jj}} \text{ if } u_j < \infty.$$

- If $f - L < W_{jj}$ for binary y_j, then $y_j^\star = 0$, if $f - L < V_{jj}$, then $y_j^\star = 1$.
- 9% reduction of B&B-nodes, 23% speedup [Gally et al. 2018].

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Presolving

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- Motivated by presolving for mixed-integer (linear) programs (MIPs): [Bixby and Rothberg 2007]: slow-down of 10.8 when turning off presolving.
- Previous work for MISDPs: [Mars 2013], [Gally 2019], [Gally et al. 2017].
- We introduced several new methods and generalizations from MIP methods.
- Speed-up of presolving depends very strongly on instances.
  One reason: Instances are usually "hand-crafted".
  $\rightsquigarrow$ Cannot expect similar speed-ups as for MIPs.
- We also apply most methods in each node (node presolving).
- Standard presolving is applied as well: linear constraints, aggregations, ...
- Concentrate here on one particular method: bound tightening.

Bound Tightening I

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For an index $k \in [m]$, define

$$P_k := \{i \in [m] \setminus \{k\} \;:\; A^i \succeq 0\}, \qquad N_k := \{i \in [m] \setminus \{k\} \;:\; A^i \preceq 0\},$$

as well as

$$\underline{\mu}_k := \inf \Big\{\mu \;:\; A^k \mu + \sum_{i \in P_k} A^i u_i + \sum_{j \in N_k} A^j \ell_j - A^0 \succeq 0\Big\},$$

$$\overline{\mu}_k := \sup \Big\{\mu \;:\; A^k \mu + \sum_{i \in P_k} A^i u_i + \sum_{j \in N_k} A^j \ell_j - A^0 \succeq 0\Big\}$$

or $\pm\infty$ if $\pm\infty$ occurs in bounds (ℓ, u).

Bound Tightening II

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Lemma (Tighten Bounds (TB))

Let all matrices be (positive or negative) semidefinite. Then, $\underline{\mu}_k \leq y_k \leq \overline{\mu}_k$ *is valid for all* $k \in [m]$*. We can round bounds for integral variables.*

Proof.

Suppose that $y_k < \underline{\mu}_k$ or $y_k > \overline{\mu}_k$. Then there exists $x \in \mathbb{R}^n$ with

$$\begin{aligned}
0 &> x^\top \Big(A^k\, y_k + \sum_{i \in P_k} A^i\, u_i + \sum_{i \in N_k} A^i\, \ell_i - A^0 \Big) x \\
&= x^\top A^k x\, y_k + \sum_{i \in P_k} \underbrace{x^\top A^i x}_{\geq 0}\, u_i + \sum_{i \in N_k} \underbrace{x^\top A^i x}_{\leq 0}\, \ell_i - x^\top A^0 x \\
&\geq x^\top A^k x\, y_k + \sum_{i \in P_k} x^\top A^i x\, y_i + \sum_{i \in N_k} x^\top A^i x\, y_i - x^\top A^0 x. \ ↯
\end{aligned}$$

□

One-Variable SDPs

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- For computing bound tightenings, need to solve one-variable SDPs.

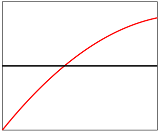
$$\inf \{\mu \,:\, \mu A - B \succeq 0,\ \ell \leq \mu \leq u\}.$$

  for symmetric $A, B \in \mathbb{R}^{n\times n}$.
- Can easily see: $\mu \mapsto \lambda_{\min}(\mu A - B)$ is concave.
- Let $\hat{v}$ be a unit eigenvector for $\lambda_{\min}(\hat{\mu} A - B)$ for $\hat{\mu} \in \mathbb{R}$. Then $\hat{v}^\top A \hat{v}$ is a supergradient, i.e.,

$$\lambda_{\min}(\mu A - B) \leq \lambda_{\min}(\hat{\mu} A - B) + (\mu - \hat{\mu})\, \hat{v}^\top A \hat{v}$$

  for all $\mu \in \mathbb{R}$.

- Goal: Want increase μ from ℓ until $\lambda_{\min}(\mu A - B) = 0$.
- Yields semismooth Newton algorithm …



One-Variable SDPs

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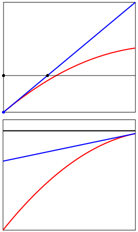
v_k = eigenvector for $\lambda_k := \lambda_{\min}(A\mu_k - B)$

$\mu_{k+1} = \mu_k - \frac{\lambda_k}{(v^k)^\top A v^k}$



Handle easy cases, e.g., infeasible if $\lambda_{\min}(A\,u - B) < 0$, supergradient positive.

- Always converges.
- Converges Q-superlinearly to a zero $\mu^\star$ of $f(\mu) = \lambda_{\min}(\mu\,A - B)$, given that $\partial f(\mu^\star)$ is nonsingular and the starting point lies near $\mu^\star$ [Qi and Sun, 1993].
- Very fast in practice; bottleneck: eigenvector computation …

Condensed Computational Results

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Testset with 185 instances, results from [Matter and P. 2023]:

Setting	# solved	# nodes	time
nopresol	168	1405.3	180.23
bound tightening	167	1297.6	152.43
MIX	167	1085.2	139.52

- Bound tightening applied in every node produces a speed-up of about 7 %.
- MIX includes bound tightening and several other methods.
  It produces a speed-up of about 22 %.

- Some techniques do not do anything on some instances.
- The methods are effective if they can be applied and induce a small time overhead.

Overview

Conflict Analysis I

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- The original idea is to learn from infeasible nodes in a branch-and-bound-tree.
- Idea transferred from SAT-solving to MIPs by [Achterberg 2007].
- More generally, can be seen as a way to learn cuts from solutions of the duals $\rightarrow$ similar to “dual ray/solution analysis” [Witzig et al. 2017, Witzig 2021]

Conflict Analysis II

Consider

$$\inf\left\{b^\top y \;:\; A(y) := \sum_{k=1}^{m} A^k y_k - A^0 \succeq 0,\; Dy \geq d,\; \ell \leq y \leq u\right\}$$

and $\hat{X} \succeq 0$, $\hat{z}$, $\hat{r}^\ell$, $\hat{r}^u \geq 0$. Aggregation yields:

$$\langle A(y), \hat{X}\rangle + \hat{z}^\top Dy + (\hat{r}^\ell)^\top y - (\hat{r}^u)^\top y \;\geq\; \hat{z}^\top d + (\hat{r}^\ell)^\top \ell - (\hat{r}^u)^\top u.$$

Idea: Do not add this (redundant) inequality, but perform bound propagation, taking integrality conditions into account.

Could also use CMIR on generated inequality as alternative to generalization of Gomory cuts [Sotirov and de Meijer 2022]. This seems not to be effective.

Conflict Analysis III

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The dual can provide $(\hat{X}, \hat{z}, \hat{r}^\ell, \hat{r}^u)$:

$$\begin{aligned} \sup \quad & \langle A^0, X\rangle + z^\top d + \ell^\top r^\ell - u^\top r^u \\ \text{s.t.} \quad & \langle A^j, X\rangle + (D^\top z)_j + r^\ell_j - r^u_j = b_j \quad \forall\, j \in [m], \\ & X \succeq 0,\ z,\ r^\ell,\ r^u \geq 0. \end{aligned}$$

Similarly for a primal ray satisfying:

$$\begin{aligned} & \langle A^j, X\rangle + (D^\top z)_j + r^\ell_j - r^u_j = 0 \qquad \forall\, j \in [m], \\ & \langle A^0, X\rangle + d^\top z + \ell^\top r^\ell - u^\top r^u > 0, \\ & X \succeq 0,\ z,\ r^\ell,\ r^u \geq 0. \end{aligned}$$

Lemma

Let $(\hat{X}, \hat{z}, \hat{r}^\ell, \hat{r}^u)$ be a primal ray. Then the aggregated inequality is infeasible with respect to the local bounds ℓ and u.

Conflict Analysis – Computations

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Generate a conflict constraint for each feasible or infeasible node. Store them as constraints and perform bound propagation.

type	# solved	# nodes	time
default	167	1066.1	132.2
conflicts	168	989.6	122.2
all optimal (167):			
default		788.7	94.2
conflicts		726.3	86.4

- Using conflicts provides a speed-up and node-reduction of about 8 %.
- On average 12792.0 conflict constraints are generated per instance.
- Average number of conflict constraints per node: 1.25.

Overview

Symmetry Detection

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Goal: apply known symmetry handling methods.

For a permutation σ of $[n]$:

$$\sigma(A)_{ij} = A_{\sigma^{-1}(i),\sigma^{-1}(j)} \quad \forall i, j \in [n].$$

Definition

A permutation $\pi \in \mathcal{S}_m$ of variables is a formulation symmetry if there exists a permutation $\sigma \in \mathcal{S}_n$ such that

1. $\pi(I) = I$, $\pi(\ell) = \ell$, $\pi(u) = u$, and $\pi(b) = b$
   (π leaves integer variables, variable bounds, and the objective coefficients invariant),
2. $\sigma(A^0) = A^0$ and, for all $i \in [m]$, $\sigma(A^i) = A^{\pi^{-1}(i)}$.

Such symmetries can be detected by using graph automorphism algorithms.

Symmetry: Computed Symmetries

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instance	symmetry group
0+-115305C_MISDPld000010	$\mathcal{S}_2$
0+-115305C_MISDPrd000010	$\mathcal{S}_2$
band60605D_MISDPld000010	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_{10} \times \mathcal{S}_3 \times \mathcal{S}_4$
band60605D_MISDPrd000010	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_{10} \times \mathcal{S}_3 \times \mathcal{S}_4$
band70704A_MISDPld000010	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_3 \times \mathcal{S}_3$
band70704A_MISDPrd000010	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_3 \times \mathcal{S}_3$
clique_60_k10_6_6, clique_60_k15_4_4, clique_60_k20_3_3, clique_60_k4_15_15, clique_60_k5_12_12, clique_60_k6_10_10, clique_60_k7_8_9, clique_60_k8_7_8, clique_60_k9_6_7, clique_70_k3_23_24	$\mathcal{S}_2$
diw_34	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{D}_4 \times \mathcal{S}_4 \times \mathcal{S}_4$
diw_37	$\mathcal{S}_2 \times \mathcal{S}_4 \times \mathcal{S}_3 \times \mathcal{S}_4$
diw_38	$\mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_3$
diw_43	$\mathcal{S}_3$
diw_44	$\mathcal{S}_3$

$\mathcal{S}_k$ = full symmetric group on k elements; $\mathcal{D}_k$ = dihedral group.

Symmetry: Computational Results

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Results from [Hojny and P. 2023]:

	all (184)			all optimal (168)		only symmetric (21)
	time (s)	symtime (s)	# gens	time (s)	#nodes	time (s)
without	130.6	–	–	95.0	778.3	45.07
with	125.3	0.44	99	90.8	760.6	29.84

- Speed-up of about 4 % for all instances;
- Speed-up of about 34 % for the 21 instances that contain symmetry.
- Number of generators is quite small.
- Note that we do not exploit symmetries in the solutions of the SDPs (yet).

Overview

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Conclusions and Outlook

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Take away messages:

- Several methods help to improve performance.
- Most of the methods are effective only on certain instances, but do not incur a significant overhead.
- Solving LP-relaxations is only helpful for certain instance types.
- Solving SDPs is still one bottleneck, but often yields strong bounds.
- Generic MISDP-solver SCIP-SDP is available.

Future work:

- More methods are likely to be helpful – probably motivated by particular structures.
- Investigate SOCP relaxations for propagation.
- We are always interested in new instances.

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Thank you for your attention!

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