Unit and distinct distances in typical norms

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Joint work with Noga Alon and Matija Bucić.

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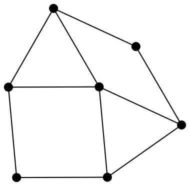

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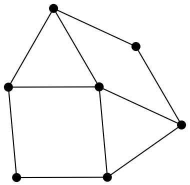

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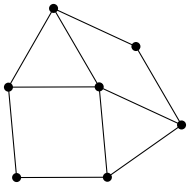

Clearly, the number of such pairs is at most $\binom{n}{2}$.

The problem of estimating the answer to this problem for large n is still widely open
(despite of a lot of attention for 70 years).

Denoting the maximum possible number of pairs of points with unit distance as $U_{\|.\|_2}(n)$, the best known bounds for large n are

$$n^{1+\Omega(1/\log\log n)} \leq U_{\|.\|_2}(n) \leq O(n^{4/3}).$$

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If the boundary of the unit ball of $\|.\|$ contains a straight line segment, then it is not hard to show that $U_{\|.\|}(n) \geq \Omega(n^2)$.

On the other hand, if the norm $\|.\|$ is strictly convex (i.e. the unit ball of $\|.\|$ is strictly convex), then like in the Euclidean case one has

$$U_{\|.\|}(n) \leq O(n^{4/3}).$$

Definition

$U_{\|.\|}(n)$ is the maximum possible number of unit distances according to $\|.\|$ among a set of n points in $\mathbb{R}^2$ (where $\|.\|$ is a given norm on $\mathbb{R}^2$) .

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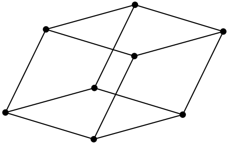
This can be shown by considering affine-linear images of hypercubes in $\mathbb{R}^2$, where every edge is mapped to a unit vector according to $\|.\|$.



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How big is $U_{\|.\|}(n)$ for most norms $\|.\|$ on $\mathbb{R}^2$ (for large n)?

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We improve Matoušek's upper bound for $U_{\|.\|}(n)$ for most norms $\|.\|$ on $\mathbb{R}^2$ by removing the $\log_2 \log_2 n$ factor.

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This shows that the general lower bound

$$U_{\|.\|}(n) \geq (1/2 - o(1)) \cdot n \cdot \log_2 n$$

is tight up to constant factors for most norms on $\mathbb{R}^2$.

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We disprove this conjecture in every dimension $d \geq 3$ (which also answers the second question).

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Unfortunately, the first theorem does not give explicit examples of norms $\|.\|$ on $\mathbb{R}^d$ satisfying the upper bound for $U_{\|.\|}(n)$.

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Erdős' Distinct Distance Problem (also from 1946) is another famous problems in discrete geometry.

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For n points in $\mathbb{R}^2$, what is the minimum possible number of distinct distances occurring among the pairs of points? What about for n points in $\mathbb{R}^d$?

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Problem

What happens for other norms $||.||$ on $\mathbb{R}^d$?

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Theorem (Alon, Bucić, S., 2023+)

For any fixed $d \geq 2$, for most norms $\|.\|$ on $\mathbb{R}^d$, we have (for large n)

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Definition

$D_{\|.\|}(n)$ is the minimum possible number of distinct distances according to $\|.\|$ among a set of n points in $\mathbb{R}^d$ (where $\|.\|$ is a given norm on $\mathbb{R}^d$) .

It is easy to see that $D_{\|.\|}(n) \leq n-1$ for every $d \geq 1$ and every norm $\|.\|$ on $\mathbb{R}^d$.

Brass conjectured that this can be improved to $D_{\|.\|}(n) \leq o(n)$ for every $d \geq 2$ and every norm $\|.\|$ on $\mathbb{R}^d$.

We disprove this conjecture in a strong form, showing that for most norms $\|.\|$ on $\mathbb{R}^d$, the function $D_{\|.\|}(n)$ is not only linear in n but in fact asymptotically equals n.

Theorem (Alon, Bucić, S., 2023+)

For any fixed $d \geq 2$, for most norms $\|.\|$ on $\mathbb{R}^d$, we have (for large n)

$$D_{\|.\|}(n) \geq (1-o(1)) \cdot n.$$

Again, we unfortunately do not get explicit examples of such norms.

What does “most norms” mean?

A norm $\|.\|$ on $\mathbb{R}^d$ is characterized by its unit ball

$$B_{\|.\|} = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\},$$

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The set of such convex bodies can be endowed with the Hausdorff metric, where the distance of B and B' is given by

$$d_H(B, B') = \max\left\{\sup_{b\in B}\inf_{b'\in B'}||b-b'||_2,\ \sup_{b'\in B'}\inf_{b\in B}||b-b'||_2\right\}$$

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This gives a metric (and hence a topology) on the set of all norms on $\mathbb{R}^d$.

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With “most norms on $\mathbb{R}^d$” we mean “all norms outside some meagre set”.

Theorem (Alon, Bucić, S., 2023+)

For any fixed $d \geq 2$, for most norms $\|.\|$ on $\mathbb{R}^d$, we have

$$U_{\|.\|}(n) \leq \frac{d}{2} \cdot n \cdot \log_2 n.$$

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The proofs of our theorems use arguments from combinatorics, polyhedral and discrete geometry, topology and algebra.

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The overall approach for the proof is very similar for both of the theorems above, so we focus on the first theorem (whose proof is a bit simpler).

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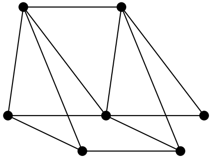

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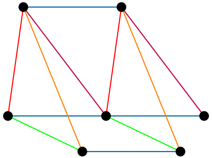

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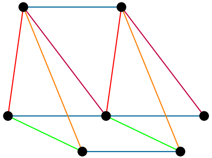

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We show that there must be some subset of the vectors $u_1, \ldots, u_k$ with many linear dependencies.

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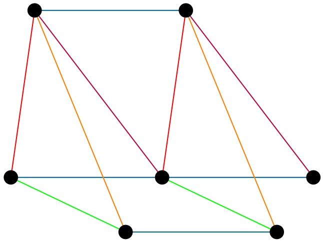

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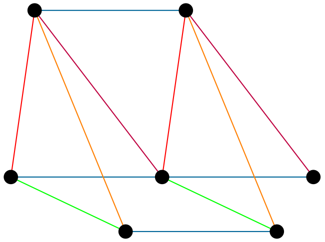

We show that there is a subset $I \subseteq \{1, \ldots, k\}$ such that we have $u_j \in \text{span}_{\mathbb{Q}}\{u_i \mid i \in I\}$ for at least $d \cdot |I| + 1$ indices $j \in \{1, \ldots, k\}$.

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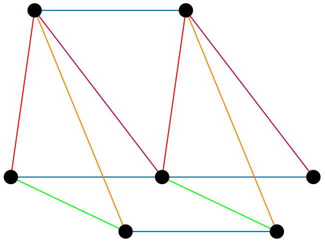

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This is a special property of the norm $\|.\|$, and we show that most norms cannot have this property (the norms with this property are a meagre set).

Lemma (Alon, Bucić, S., 2023+)

Let $u_1, \ldots, u_k \in \mathbb{R}^d$ be non-zero vectors and let $S \subseteq \mathbb{R}^d$ be a set of n points such that there are more than $(d/2) \cdot n \cdot \log_2 n$ pairs $\{x, y\} \subseteq S$ with $x - y \in \{\pm u_1, \ldots, \pm u_k\}$.

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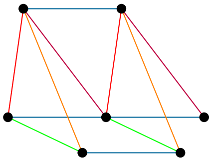

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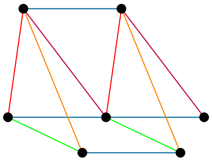

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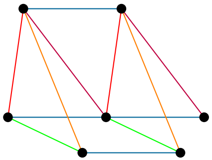

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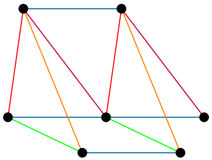

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Now consider one of these linearly independent subsets accounting for more than $(1/2) \cdot n \cdot \log_2 n$ pairs $\{x, y\} \subseteq S$ with $x - y \in \{\pm u_1, \ldots, \pm u_k\}$.

We now obtain a contradiction to the following fact.

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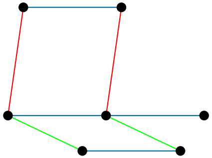
Let $u_1, \ldots, u_k \in \mathbb{R}^d$ be vectors that are linearly independent over $\mathbb{Q}$. Then for any subset $S \subseteq \mathbb{R}^d$ of n points there can be at most $(1/2) \cdot n \cdot \log_2 n$ pairs $\{x, y\} \subseteq S$ with $x - y \in \{\pm u_1, \ldots, \pm u_k\}$.



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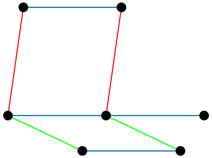

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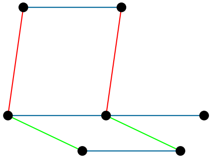

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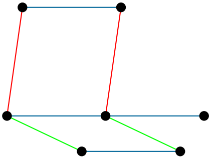

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Alternatively, one can also just prove this fact with an inductive argument.

How do the polyhedra appear?

It remains to prove the following:

Lemma (Alon, Bucić, S., 2023+)

For any fixed $d \geq 2$, for most norms $\|.\|$ on $\mathbb{R}^d$, there do not exist distinct vectors $u_1, \ldots, u_{d\ell+1} \in \mathbb{R}^d$ for any $\ell \geq 1$ with $\|u_1\| = \cdots = \|u_{d\ell+1}\| = 1$ such that $u_j \in \text{span}_{\mathbb{Q}}\{u_1, \ldots, u_\ell\}$ for $j = 1, \ldots, d\ell + 1$.

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We want to show that this set of norms is always nowhere dense.

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Let B_0 be a unit ball of some norm on $\mathbb{R}^d$, and consider an open ball around B_0 in the Hausdorff metric.

We are given $\ell \geq 1$, a rational $\eta > 0$, and coefficients $a_{ji} \in \mathbb{Q}$.

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We can approximate B_0 by a polytope B_1 with small facets. Let B be a polytope obtained from B_1 by shifting the facets slightly in a suitable way.

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Now, suppose $\|.\|$ is a norm for which there are vectors $u_1, \dots, u_{d\ell+1}$ as above such that the unit ball of $\|.\|$ is very close to B (in Hausdorff metric).

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We find points $u'_1, \dots, u'_{d\ell+1}$ on the boundary of B close to $u_1, \dots, u_{d\ell+1}$. Then $u'_1, \dots, u'_{d\ell+1}$ lie on distinct facets of B (since the facets are small).

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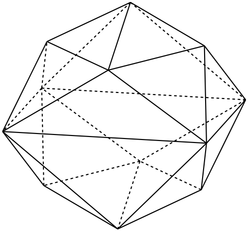
Since $u_j = \sum_{i=1}^{\ell} a_{ji} u_i$ for $j = 1, \dots, d\ell + 1$, we get a linear equation which is very close to satisfied for the constant terms for the constraints of B.

Let B_0 be a unit ball of some norm on $\mathbb{R}^d$, and take an open ball around B_0 in the Hausdorff metric. We approximated B_0 by a polytope B_1.

We then defined B to be a polytope obtained from B_1 by shifting the facets slightly in a suitable way.



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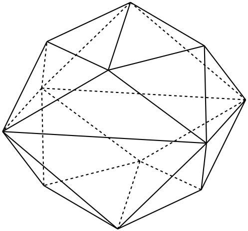

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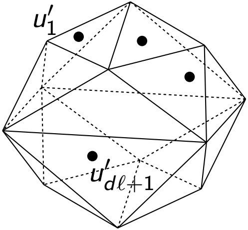

We considered a certain set of norms $\|.\|$ on $\mathbb{R}^d$ with the goal of showing that it is nowhere dense.

If $\|.\|$ is a norm in this set, and the unit ball of $\|.\|$ is very close to B (in Hausdorff metric), there must be points $u'_1, \ldots, u'_{d\ell+1}$ on distinct facets of B such that a certain linear equation is very close to being satisfied for the constant terms for the constraints of B.

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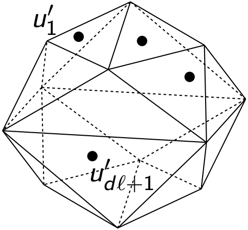

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But we can choose the polytope B so that this cannot happen.

Thank you very much for your attention!