Understanding graphs locally

Annie Raymond
(different parts joint with Greg Blekherman,
James Saunderson, Mohit Singh,
Rekha Thomas and Fan Wei)

University of Massachusetts, Amherst

March 30, 2023

Large graphs are everywhere!

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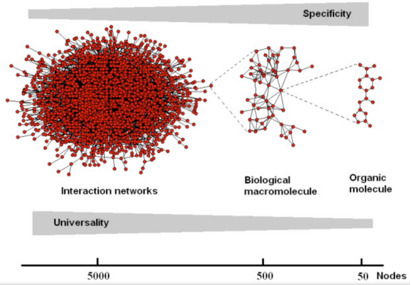

Biology

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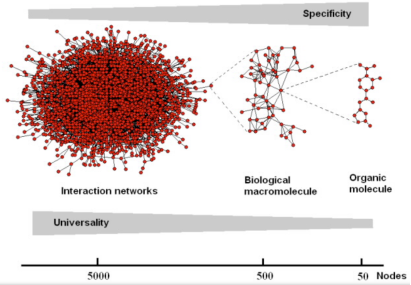

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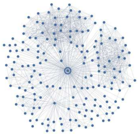
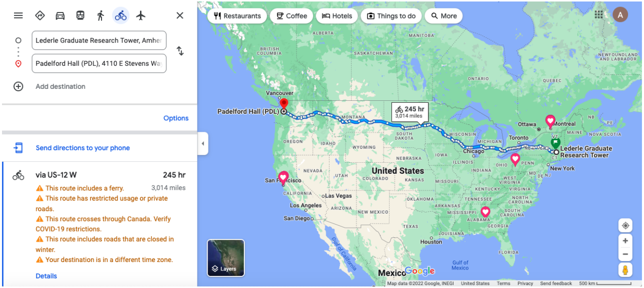
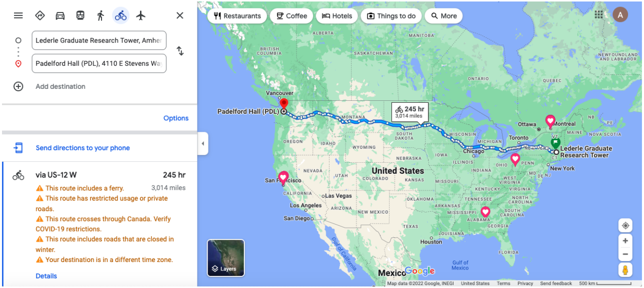


Facebook graph


Lederle Graduate Research Tower, Amher
Padelford Hall (PDL), 4110 E Stevens Wa
Add destination
Options
Send directions to your phone
via US-12 W
245 hr
3,014 miles
This route includes a ferry.
This route has restricted usage or private roads.
This route crosses through Canada. Verify COVID-19 restrictions.
This route includes roads that are closed in winter.
Your destination is in a different time zone.
Details
Restaurants
Coffee
Hotels
Things to do
More
Padelford Hall (PDL)
Lederle Graduate Research Tower
United States
Mexico

Google Maps



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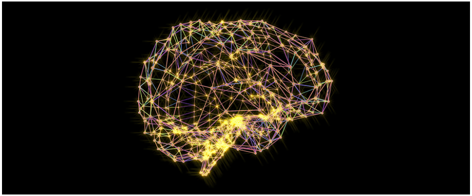

Alfred Pasieka/Science Photo Library/Getty Images

Problem

How can we understand such large graphs?

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This raises immediately two questions:

1. How do global and local properties relate?
2. What is even possible locally? $\rightarrow$ graph profiles!

How shall we count today?

The number of homomorphisms from a graph H to a graph G is $\hom(H;G) =$number of maps from $V(H)$ to $V(G)$ that yield a graph homomorphism, i.e., that map every edge of H to an edge of G.

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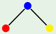
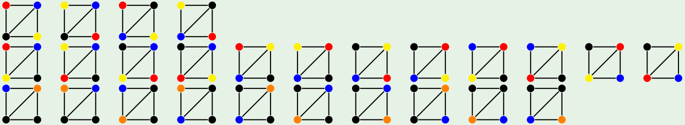
- the number of homomorphisms hom( ; G) $= 26$,



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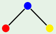
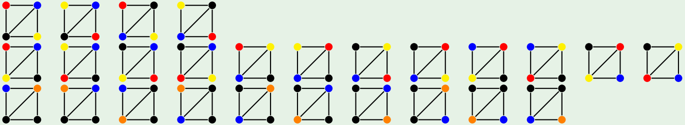
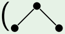
- the number of homomorphisms hom( ; $G) = 26$,



- the homomorphism density $t($  $; G) = \frac{26}{64}$

Graph profiles

Let $\mathcal{U} = \{H_1, H_2, \ldots, H_l\}$. The (homomorphism) density profile of $\mathcal{U}$ is

$$\mathcal{D}_{\mathcal{U}} := \text{cl}(\{(t(H_1; G), \ldots, t(H_l; G)) | G \text{ is a graph}\})$$

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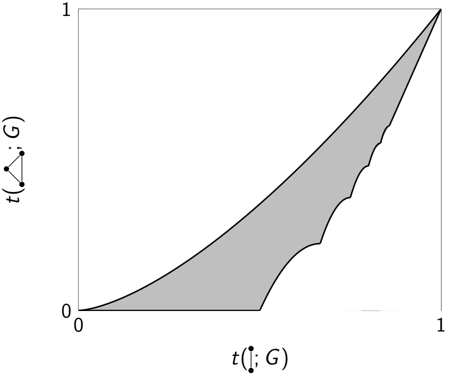

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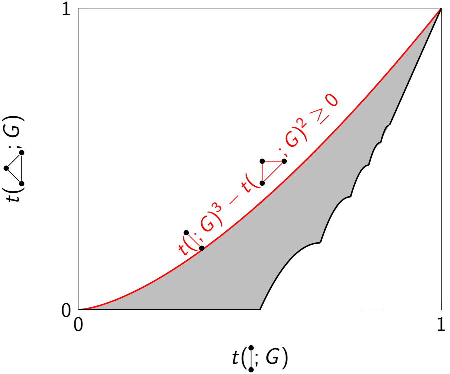

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Different techniques to certify graph inequalities

- Integer programming
- Hölder's inequality
- Entropy method
- Lagrangian method
- Cauchy-Schwarz/sums of squares
- ...

Integer programming example

Theorem (R., 2014)

The Turán inequality $t($$; G) \leq 1 - \frac{1}{r-1}$ *for all graphs* G *such that* $t(K_r; G) = 0$ *can be proven by studying*

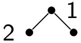
$$\max \sum_{e \in E(K_n)} x_e$$

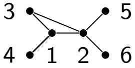
$$\text{such that} \sum_{e \in E(Q)} x_e \leq \binom{r}{2} - 1 \text{ for all cliques } Q \text{ of size } r \text{ in } K_n,$$

$$x_e \in \{0, 1\} \text{ for all } e \in E(K_n).$$

Tools to certify graph inequalities with sos

- Restrict to maps that send labelled vertices to specific vertices of G

$t\Big($ 

; 

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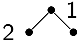
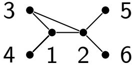
$t\Big($ 

$;$ 

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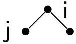
- Symmetrization=unlabeling

$[[t($ 

$; G)]] := \frac{1}{|V(G)|^2} \sum_{1 \leq i,j \leq |V(G)|} t($ 

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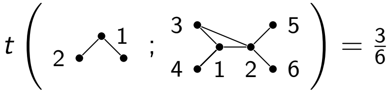

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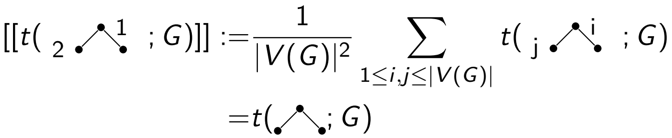
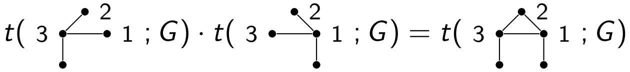


- Multiplication=gluing along vertices that have the same labels



Sums of square example

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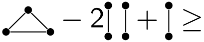
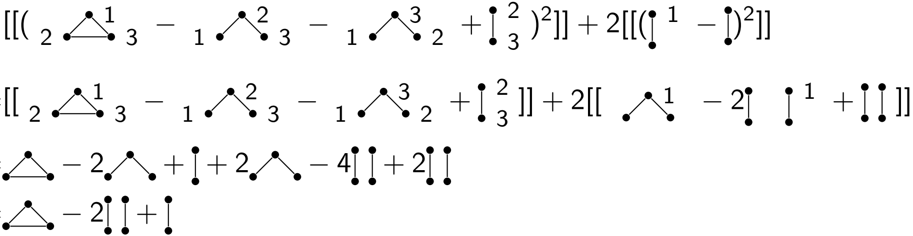
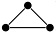
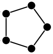
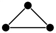
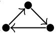
Claim:  (Goodman bound)

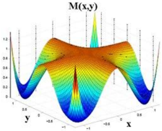
Proof:



Some sos successes

- Density profile for $\mathcal{U} = \{$, $\}$ (Razborov, 2008)
- The best bound for the Turán $K_4^{(3)}$ problem (Razborov, 2014) and a solution to the problem in the ℓ_2-norm (Balogh, Clemen, Lidický 2021)
- Every n-vertex triangle-free graph has at most $\left(\frac{n}{5}\right)^5$ cycles of length 5, i.e., ind(; $G) \leq \left(\frac{n}{5}\right)^5$ whenever ind(; $G) = 0$ (Grzesik 2012; Hatami, Hladký, Král', Norine, Razborov 2013)
- Ramsey multiplicity bounds and results (e.g., Parczyk, Pokutta, Spiegel, Szabó 2022)
- $t_{\text{ind}}($$; G) + t_{\text{ind}}($$; G) \geq \frac{1}{9} - o_n(1)$ (Gilboa, Glebov, Hefetz, Linial, Morgenstern 2022)
- The smallest eigenvalue of the signless Laplacian matrix of an n-vertex graph G is at most $\frac{15n}{94}$ (Balogh, Clemen, Lidický, Norin, Volec, 2022)

Certifying polynomial inequalities with sos



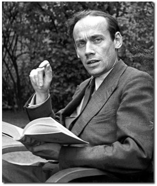
A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$
is nonnegative if $p(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$

p sum of squares (sos), i.e., $p = \sum_{i=1}^{l} f_i^2$ where $f_i \in \mathbb{R}[\mathbf{x}] \;\Rightarrow\; p \geq 0$

Hilbert (1888): Not all nonnegative polynomials are sos.

Artin (1927): Every nonnegative polynomial can be written as a (finite) sum of squares of rational functions.

Motzkin (1967, with Taussky-Todd): $M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is a nonnegative polynomial but is not a sos.

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RSST (2017): Graph sums of squares can be understood by doing symmetry-reduction for real polynomials.

Undecidability results

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Theorem (BRW, 2022)

Determining the validity of a polynomial inequality between weighted homomorphism densities or between weighted homomorphism numbers is undecidable.

Undecidability results - part 2

The number of homomorphisms from a graph H to a graph G with edge weights $\mathbf{w} : E(G) \to \mathbb{R}$, denoted as $G_{\mathbf{w}}$, is

$$\hom(H; G_{\mathbf{w}}) = \sum_{\substack{\varphi: V(H) \to V(G_{\mathbf{w}}): \\ \varphi \text{ is a homomorphism}}} \prod_{\{i,j\} \in E(H)} w_{\varphi(i),\varphi(j)}.$$

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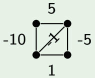
Let $G_{\mathbf{w}} =$ 

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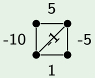
Let $G_{\mathbf{w}} =$ 

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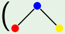
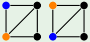
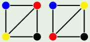
- hom( ; G) $= 142$

  $2(100+25+25+1+1)+2(-10\cdot5-5\cdot5-5\cdot1-10\cdot1+1\cdot10+1\cdot5-1\cdot5-1\cdot1) = 142$

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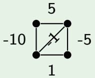
Let $G_\mathbf{w} =$ 

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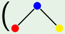
- hom( ; G) $= 142$

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  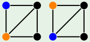
  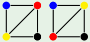

- $t($  $; G_\mathbf{w}) = \frac{142}{64}$ (the expected weight of a random map)

Undecidability results - part 3

All these results build on the following solution to Hilbert's 10th problem:

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

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Theorem (Matiyasevich 1970 (Davis, Putnam, Robinson), Tarski 1948)

Given $k \in \mathbb{N}$ *and* $p \in \mathbb{Z}[x_1, \ldots, x_k]$*, the problem of determining whether* $p(x_1, \ldots, x_k) \geq 0$ *for every* $x_1, \ldots, x_k \in \mathbb{Z}$ *is undecidable.*

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In contrast, given $k \in \mathbb{N}$ and $p \in \mathbb{Z}[x_1, \ldots, x_k]$, the problem of determining whether $p(x_1, \ldots, x_k) \geq 0$ for all $x_1, \ldots, x_k \in \mathbb{R}$ is decidable.

Simplifying graph profiles

Definition

The tropicalization of $\mathcal{S} \subseteq \mathbb{R}^l_{\geq 0}$ is

$$\text{trop}(\mathcal{S}) := \lim_{k \to 0} \log_{\frac{1}{k}}(\mathcal{S}).$$

Theorem (BRST20)

$\text{trop}(\mathcal{D}_{\mathcal{U}}) = \text{cl}(\text{conv}(\log(\mathcal{D}_{\mathcal{U}})))$ *which is a closed convex cone in* $\mathbb{R}^l_{\leq 0}$ *determined by linear inequalities corresponding to the pure binomial inequalities valid on* $\mathcal{D}_{\mathcal{U}}$*. Similarly for* $\mathcal{N}_{\mathcal{U}}$ *(except* $\mathbb{R}^l_{\geq 0}$*).*

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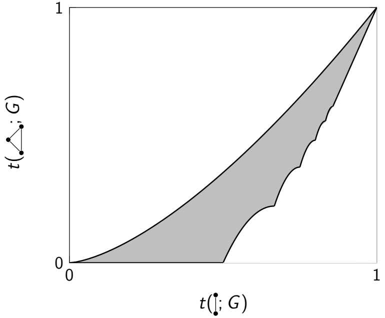
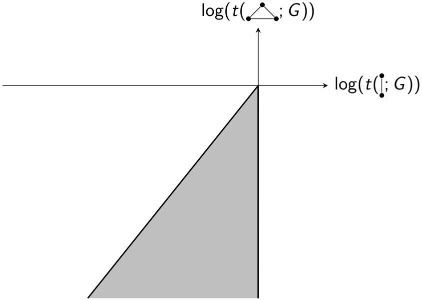
$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_l^{\alpha_l} - x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \geq 0$
$\to \ \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n - \beta_1 y_1 - \beta_2 y_2 - \ldots - \beta_n y_n \geq 0$
where $y_i = \log x_i$.

Tropicalization example





$$t(K_2; G)^3 - t(K_3; G)^2 \geq 0$$

becomes

$$3\log(t(K_2; G)) - 2\log(t(K_3; G)) \geq 0$$

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Conjecture (BRST2020): trop($\mathcal{N}_{\mathcal{U}}$) is a rational polyhedral cone.

BR22: If $\mathcal{U}$ contains only chordal series-parallel graphs, then trop($\mathcal{N}_{\mathcal{U}}$) and trop($\mathcal{D}_{\mathcal{U}}$) is a rational polyhedral cone.

Some consequences of the tropicalization of paths (BR22)

- $\max\{c \in \mathbb{R} : \hom(P_m; G) \geq \hom(P_n; G)^c \text{ for all graphs } G\}$

$$= \begin{cases} \frac{m}{n+1} & \text{when } m \text{ is even and } n \text{ is odd and } m \leq n \\ \frac{km-(m-1)}{k(n-1)+2k-n} & \text{when } m \text{ and } n \text{ are both even and } m \leq n \\ \frac{m}{n} & \text{when } m \text{ is odd and } m \leq n \\ 1 & \text{when } m \geq n \end{cases}$$

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$$\hom(P_a; G)^{c-b} \hom(P_c; G)^{b-a} \geq \hom(P_b; G)^{c-a}$$

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- C_5 is strongly common (Behague, Morrison, Noel 2022)

Future directions

- What are the strengths and limitations of different techniques to prove graph inequalities?
- Computing graph profiles in more than two dimensions (and even computing more in two dimensions!)
- Is the tropicalization of $\mathcal{N}_{\mathcal{U}}$ always a polyhedral rational cone?
- Is determining the validity of a **pure binomial** inequality between homomorphism numbers decidable?

Thank you!

Tropicalization of homomorphism numbers of paths

Theorem (BR, 2021)

The following inequalities hold for homomorphism numbers of paths into any graph G with no isolated vertices:

- *log-convexity between odd paths:*
  $\hom(P_a; G)^{c-b} \hom(P_c; G)^{b-a} \geq \hom(P_b; G)^{c-a}$ *for* $a \leq b \leq c$*,* a, c *odd*
- *log-convexity for odd and even paths, even middle:*
  $\hom(P_a; G)^{c-b} \hom(P_c; G)^{b-a} \geq \hom(P_b; G)^{c-a}$ *for* $a \leq b \leq c$*,* a *odd,* b, c *even*
- *"weak convexity" for odd and even path, odd middle:*
  $\hom(P_a; G)^{\frac{c}{2}} \hom(P_c; G) \geq \hom(P_b; G)^{\frac{c}{2}}$ *for* $a \leq b \leq c$*,* a, b *odd,* c *even*
- *non-decreasing:*
  $\hom(P_a; G) \leq \hom(P_b; G)$ *for* $a \leq b$
- *log-subadditivity:*
  $\hom(P_a; G) \hom(P_b; G) \leq \hom(P_{a+b}; G)$

Moreover, any pure binomial inequality in paths can be deduced in a finite way from the above inequalities. In particular, for a binomial inequality where the largest path has v vertices, only inequalities involving paths on at most $2v$ vertices need to be considered.

Example of how to deduce an inequality

Suppose we want to recover $\hom(P_3; G)^3 \geq \hom(P_4; G)^2$. We know:

1. $\hom(P_3; G) \hom(P_5; G) \geq \hom(P_4; G)^2$ (log-convexity)
2. $\hom(P_3; G) \hom(P_7; G) \geq \hom(P_5; G)^2$ (log-convexity)
3. $\hom(P_4; G)^2 \geq \hom(P_8; G)$ (log-subadditivity)
4. $\hom(P_8; G) \geq \hom(P_7; G)$ (non-decreasing)

So we have

$$\begin{aligned}
&\hom(P_3; G)^3 \hom(P_5; G)^2 \hom(P_7; G) \\
&\quad = (\hom(P_3; G) \hom(P_5; G))^2 (\hom(P_3; G) \hom(P_7; G)) \\
&\quad \geq \hom(P_4; G)^4 \hom(P_5; G)^2 \\
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and so $\hom(P_3; G)^3 \geq \hom(P_4; G)^2$.

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Equivalently, we can recover $3y_3 - 2y_4 \geq 0$ where $y_i := \log(\hom(P_i; G))$.
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1. $y_3 - 2y_4 + y_5 \geq 0$ (log-convexity)
2. $y_3 - 2y_5 + y_7 \geq 0$ (log-convexity)
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4. $-y_7 + y_8 \geq 0$ (non-decreasing)

So we have

$$\begin{array}{rcccccl} 2\cdot(& y_3 & -2y_4 & +y_5 & & & \geq 0) \\ +(& y_3 & & -2y_5 & +y_7 & & \geq 0) \\ +(& & & & -y_7 & +y_8 & \geq 0) \\ +(& & 2y_4 & & & -y_8 & \geq 0) \\ \hline & 3y_3 & -2y_4 & & & & \geq 0 \end{array}$$

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Checking the validity of a pure binomial in paths is equivalent to checking if there exists a conical combination that yields it.

Some important properties of $\mathcal{N}_{\mathcal{U}}$ and $\text{trop}(\mathcal{N}_{\mathcal{U}})$

- $\mathcal{N}_{\mathcal{U}}$ has the Hadamard property since
  $\text{hom}(H; G_1) \cdot \text{hom}(H; G_2) = \text{hom}(H; G_1 \times G_2)$
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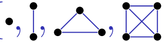
- Any extreme ray of the dual cone $\text{trop}(\mathcal{N}_{\mathcal{U}})^*$ is spanned by a vector with at most one negative coordinate (BR 2021). For example, this means that we know the following inequality is redundant:

$$\text{hom}(P_{2a+1}; G)\,\text{hom}(P_{2(a+b+c)+1}; G) \geq \text{hom}(P_{2a+c+1}; G)\,\text{hom}(P_{2(a+b)+c+1}; G).$$

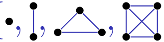
trop($\mathcal{N}_\mathcal{U}$) when $\mathcal{U} = \{$$, \ldots, K_l\}$

Theorem (BRST 2020)

Let $\mathcal{U} = \{K_1, \ldots, K_l\}$ *where* K_i *is a complete graph on* i *vertices. Let*

$$Q_\mathcal{U} = \left\{ \begin{array}{ll} \mathbf{y} \in \mathbb{R}^l | & i \cdot y_{i-1} - (i-1) \cdot y_i \geq 0 \quad 2 \leq i \leq l \\ & y_l \geq 0 \end{array} \right\}$$

where $y_i = \log(\hom(K_i; G))$. *Then* $\operatorname{trop}(\mathcal{N}_\mathcal{U}) = Q_\mathcal{U}$.

trop($\mathcal{N}_{\mathcal{U}}$) when $\mathcal{U} = \{$  $, \ldots, K_l\}$

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Proof.

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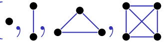
trop($\mathcal{N}_{\mathcal{U}}$) when $\mathcal{U} = \{$  $, \ldots, K_l\}$

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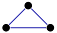
$\text{trop}(\mathcal{N}_{\mathcal{U}}) = \{$, , , $, \ldots, K_l\}$

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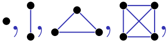
Proof.

Claim 2: The extreme rays of $Q_{\mathcal{U}}$ are $\mathbf{r}_i = (r_1, \ldots, r_l)$ for $1 \leq i \leq l$ where

$$r_j = \begin{cases} j & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

$\mathrm{trop}(\mathcal{N}_\mathcal{U}) = \{$  $, \ldots, K_l\}$

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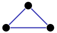
l constraints in l variables, and $\mathbf{r}_i = (1, 2, \ldots, i, 0, \ldots, 0)$ satisfies all but the ith constraint at equality (which it still satisfies).

$\text{trop}(\mathcal{N}_{\mathcal{U}}) = \{\bullet,$ , , $, \ldots, K_l\}$

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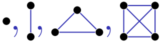
Claim 3: The extreme rays of $Q_{\mathcal{U}}$ are in $\text{trop}(\mathcal{N}_{\mathcal{U}})$, and hence $Q_{\mathcal{U}} = \text{trop}(\mathcal{N}_{\mathcal{U}})$.

$\text{trop}(\mathcal{N}_\mathcal{U}) = \{$  $, \ldots, K_l\}$

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Proof.

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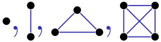
To realize $\mathbf{r}_i$, let G_n be an i-partite complete graph where each part has $\frac{n}{i}$ vertices (i.e., a Turán graph) with a disjoint copy of K_l.

$\text{trop}(\mathcal{N}_{\mathcal{U}}) = \{$  $, \ldots, K_l\}$

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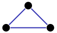
To realize $\mathbf{r}_i$, let G_n be an i-partite complete graph where each part has $\frac{n}{i}$ vertices (i.e., a Turán graph) with a disjoint copy of K_l.

Then as $n \to \infty$, $\frac{\log \text{hom}(K_j; G_n)}{\log n} \to j$ if $j \leq i$ and $\frac{\log \text{hom}(K_j; G_n)}{\log n} \to 0$ if $j > i$. □

$\text{trop}(\mathcal{N}_{\mathcal{U}}) = \{$, , , , $\dots, K_l\}$

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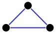
Consequence: Every pure binomial inequality involving complete graphs can be deduced in a finite way from the inequalities above.

trop($\mathcal{N}_\mathcal{U}$) = {•, , , , ..., K_l}

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Consequence: Every pure binomial inequality involving complete graphs can be deduced in a finite way from the inequalities above.

For example, the general Kruskal-Katona inequalities $\text{hom}(K_p; G)^q \geq \text{hom}(K_q; G)^p$ for any $2 \leq p < q$ can be recovered from the set of inequalities $\text{hom}(K_{i-1}; G)^i \geq \text{hom}(K_i; G)^{i-1}$