Approximating Weighted Connectivity Augmentation Below Factor 2

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Introduction to
Weighted Connectivity Augmentation
Weighted Connectivity Augmentation Problem (WCAP)

\[ G = (V, E) \]
\[ L \subseteq \binom{V}{2} \]
\[ c: L \to \mathbb{R}_{\geq 0} \]

\[ \min\{c(U): U \subseteq L \text{ with } (V, E \cup U) \text{ is } (k + 1)-\text{edge-connected}\} \]
Cactus representation of min cuts allows for assuming that $G$ is cactus.

([Dinitz,Karzanov,Lomonosov;1976])
...and finally to the Weighted Ring Augmentation Problem (WRAP)
2-approximation obtainable through various techniques, including:

- **specialized techniques** [Frederickson, Jájá; 1981], [Khuller, Thurimella; 1993], [Khuller, Vishkin; 1994],
- **primal-dual algorithms** [Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson; 1994], and
- **iterative rounding** [Jain; 2001].
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Prior progress on beating factor 2 only for special cases, including:

- Unweighted Tree Augmentation. (1.393-approximation)
  [Nagamochi; 2003], [Cheriyan, Karloff, Khandekar, Könemann; 2008], [Even, Feldman, Kortsarz, Nutov; 2009],
  [Cohen, Nutov; 2013], [Kortsarz, Nutov; 2016], [Nutov; 2017], [Cheriyan, Gao; 2018a], [Cheriyan, Gao; 2018b],
  [Adjishvili; 2018], [Fiorini, Groß, Könemann, Sanità; 2018], [Grandoni, Kalaitzis, Z.; 2018], [Kortsarz, Nutov; 2018],
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- **Weighted Tree Augmentation. (1.5 + \(\varepsilon\)-approximation)**
  [T., Z.; 2021], [T., Z.; 2022]
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- **Weighted Tree Augmentation.** ((1.5 + ε)-approximation)
  [T., Z.; 2021], [T., Z.; 2022]

- **Unweighted Connectivity Augmentation.** (1.393-approximation)
  [Byrka, Grandoni, Jaba Ameli; 2020], [Nutov; 2021], [Cecchetto, T., Z.; 2021]
Our contribution

Theorem [T.,Z.; 2023]

There is a $(1.5 + \varepsilon)$-approximation algorithm for Weighted Connectivity Augmentation (WCAP).
A bird’s-eye view on our approach

Directed WRAP: A weaker but highly structured version of WRAP.
Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.
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One way to improve on factor 2: relative greedy

Start with 2-approximate directed solution and improve it iteratively.
Directed WRAP

(a Highly Structured Simplification of WRAP)
Differences between directed WRAP and WRAP:

- Links are directed.
- To cover a 2-cut $C \subseteq V \setminus \{r\}$, a directed link needs to enter it.

Directed WRAP can be solved efficiently.
A simple 2-approximation for WRAP via directed WRAP

(similar to [Khuller, Vishkin, 1994] and [Cecchetto, T., Z., 2021])
Shortenings: toward structured solutions

shorten links
A non-shortenable solution ... 
- is an $r$-arborescence;
- has a planar canonical straight-line embedding;
- has at most one left-outgoing and one right-outgoing link per vertex, and out-degree 1 at $r$. 

Properties of non-shortenable solutions
Improving on directed WRAP solutions
Definition: mixed solution

Every 2-cut \( C \subseteq V \setminus \{r\} \) has either
- an entering directed link, or
- a crossing undirected link.
Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.
A possible (improving) step

Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

Which directed links should we drop?
Cut responsibility is defined wrt non-shortenable sol. \((V, \vec{F})\).

**Definition: responsibility of arc for a 2-cut**

\((u, v) \in \vec{F}\) is responsible for a 2-cut \(C \subseteq V \setminus \{r\}\) if

(i) \((u, v)\) enters \(C\), and

(ii) no directed link on \(r\)-\(u\) path of \((V, \vec{F})\) enters \(C\).

\[
\mathcal{R}_{\vec{F}}(\vec{\ell}) := \left\{ \text{all 2-cuts } \vec{\ell} \text{ is responsible for} \right\}
\]
$$\text{Drop}_F(K) := \{ \vec{\ell} \in F : \text{each } C \in R_F(\vec{\ell}) \text{ is crossed by some link in } K \}$$

Then \((F \setminus \text{Drop}_F(K)) \cup K\) is a mixed solution.
Characterizing when all 2-cuts a link is responsible are covered?

Lemma

\[(u, v) \in \text{Drop}_F(K) \iff \text{There is a path in } H[K] \text{ connecting a link incident to } v \text{ to one incident to a vertex that is not a descendant of } v \text{ (in } F).\]
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More structure: a nice coverage property

**Lemma**

If \( K \) is a link set connected in link intersection graph:

\[
\text{Drop}_{\mathcal{F}}(K) = \left( \bigcup_{v \in V(K)} \delta_{\mathcal{F}}^-(v) \right) \setminus \delta_{\mathcal{F}}^-(\text{lca}(V(K))).
\]
Improvements through thin components
Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

Which undirected links should we add?
Definition: $\alpha$-thin component $K \subseteq L$ is $\alpha$-thin if $\exists$ triangulation s.t. the side of any triangle is crossed by $\leq \alpha$ links in $K$. 
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Optimization theorem

Given a non-shortenable directed WRAP solution $\vec{F}$ and $\alpha = O(1)$, one can efficiently find an $\alpha$-thin $K \subseteq L$ minimizing

$$\frac{c(K)}{c(Drop_{\vec{F}}(K))}.$$
Decomposition Theorem

Fix $\varepsilon > 0$.

$\vec{F} :=$ non-shortenable directed WRAP solution

There exists a partition $\mathcal{K}$ of $\text{OPT}$ into $4^{\lceil 1/\varepsilon \rceil}$-thin components s.t.:

$$\sum_{K \in \mathcal{K}} c(\text{Drop}_{\vec{F}}(K)) \geq (1 - \varepsilon) \cdot c(\vec{F}).$$
Decomposition Theorem

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$\mathcal{K} = \{K_1, K_2, K_3\}$
Approximation Algorithms

**Relative Greedy: \((1 + \ln 2 + \varepsilon)\)-approximation**

- Start with a non-shortenable directed WRAP solution \(\vec{F}_0\) of cost \(\leq 2 \cdot c(\text{OPT})\).
- Iteratively add components \(K_1, \ldots, K_m\) (chosen greedily using optimization theorem).
- Remove \(\text{Drop}_{\vec{F}_0}(K_1), \ldots, \text{Drop}_{\vec{F}_0}(K_m)\).
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**Local Search: \((1.5 + \epsilon)\)-approximation**

- Algorithm based on the approach from [T., Z.; 2022].
- We maintain an undirected WRAP solution \(F \subseteq L\).
- \(F\) can be turned into non-shortenable directed WRAP solution \(\vec{F}\).
- When looking for an exchange step, use \(\vec{F}\) to measure progress.
- Remove \(\ell \in F\) if (the at most 2) links in \(\vec{F}\) corresponding to \(\ell\) have been dropped.
Local search step

Drop $\vec{F}(K)$
Conclusions
Theorem [T. & Z., 2023]

There is a \((1.5 + \varepsilon)\)-approximation for Weighted Connectivity Augmentation.

- Directed WRAP is highly structured problem that can be leveraged to approach WRAP.
- Cut responsibility helps to make search for good link set algorithmically tractable.
- Careful definition of thin components $\rightarrow$ Optimization & Decomposition Theorem.
- Main technical parts (skipped here): Proof of Optimization & Decomposition Theorem.
- Derived structural results can be exploited in relative greedy or local search algorithms.