Approximating Weighted Connectivity Augmentation Below Factor 2

Vera Traub

University of Bonn

Rico Zenklusen

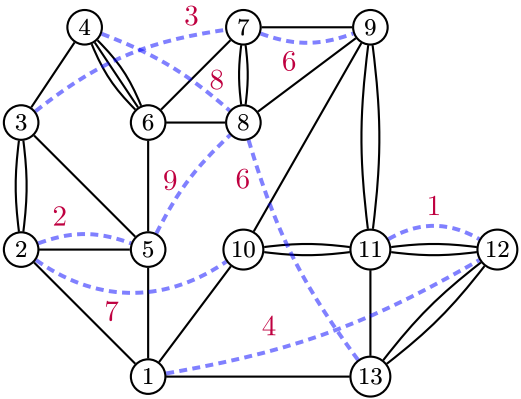
ETH Zurich

erc

Swiss National Science Foundation

Introduction to
Weighted Connectivity Augmentation

Weighted Connectivity Augmentation Problem (WCAP)



$G = (V, E)$

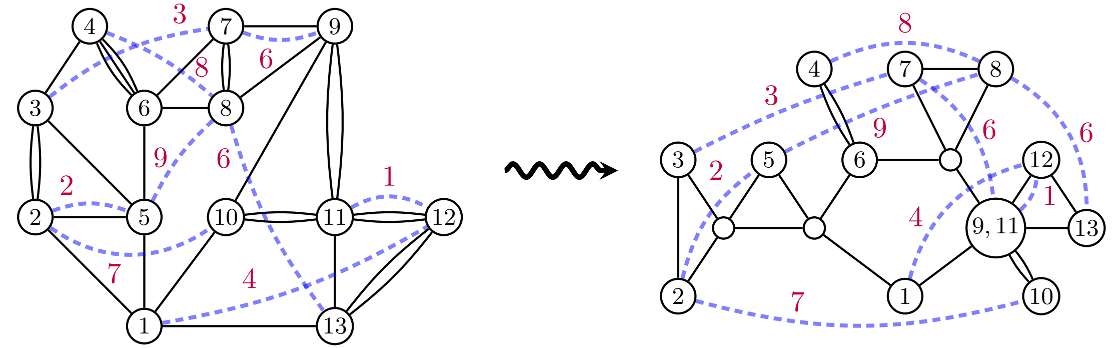
$L \subseteq \binom{V}{2}$

$c\colon L \to \mathbb{R}_{\geq 0}$

$\min\{c(U)\colon U \subseteq L$ with $(V, E \cup U)$ is $(k + 1)$-edge-connected$\}$

edge-connectivity of G

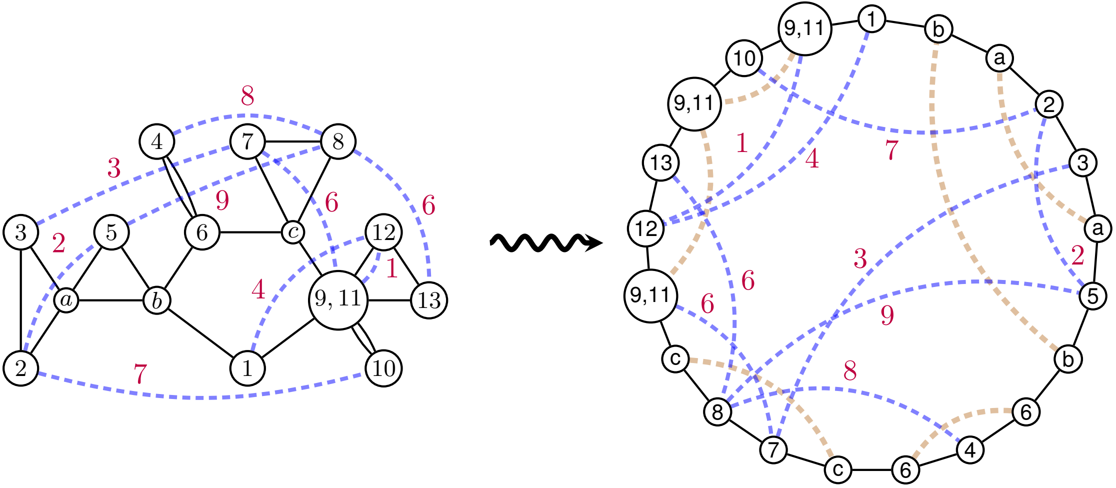
From WCAP to Weighted Cactus Augmentation . . .



Cactus representation of min cuts allows for assuming that G is cactus.

([Dinitz,Karzanov,Lomonosov;1976])

. . . and finally to the Weighted Ring Augmentation Problem (WRAP)



Prior work

2-approximation obtainable through various techniques, including:

- specialized techniques [Frederickson, Jájá; 1981], [Khuller, Thurimella; 1993], [Khuller, Vishkin; 1994],
- primal-dual algorithms [Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson; 1994], and
- iterative rounding [Jain; 2001].

Prior work

2-approximation obtainable through various techniques, including:

- specialized techniques [Frederickson, Jájá; 1981], [Khuller, Thurimella; 1993], [Khuller, Vishkin; 1994],
- primal-dual algorithms [Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson; 1994], and
- iterative rounding [Jain; 2001].

Prior progress on beating factor 2 only for special cases, including:

- Unweighted Tree Augmentation. (1.393-approximation)

  [Nagamochi; 2003], [Cheriyan, Karloff, Khandekar, Könemann; 2008], [Even, Feldman, Kortsarz, Nutov; 2009], [Cohen, Nutov; 2013], [Kortsarz, Nutov; 2016], [Nutov; 2017], [Cheriyan, Gao; 2018a], [Cheriyan, Gao; 2018b], [Adjiashvili; 2018], [Fiorini, Groß, Könemann, Sanità; 2018], [Grandoni, Kalaitzis, Z. ;2018], [Kortsarz, Nutov; 2018], [Cecchetto, T., Z.; 2021]

Prior work

2-approximation obtainable through various techniques, including:

- specialized techniques [Frederickson, Jájá; 1981], [Khuller, Thurimella; 1993], [Khuller, Vishkin; 1994],
- primal-dual algorithms [Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson; 1994], and
- iterative rounding [Jain; 2001].

Prior progress on beating factor 2 only for special cases, including:

- Unweighted Tree Augmentation. (1.393-approximation)

  [Nagamochi; 2003], [Cheriyan, Karloff, Khandekar, Könemann; 2008], [Even, Feldman, Kortsarz, Nutov; 2009], [Cohen, Nutov; 2013], [Kortsarz, Nutov; 2016], [Nutov; 2017], [Cheriyan, Gao; 2018a], [Cheriyan, Gao; 2018b], [Adjiashvili; 2018], [Fiorini, Groß, Könemann, Sanità; 2018], [Grandoni, Kalaitzis, Z. ;2018], [Kortsarz, Nutov; 2018], [Cecchetto, T., Z.; 2021]
- Weighted Tree Augmentation. ($(1.5+\varepsilon)$-approximation)

  [T., Z. ; 2021], [T., Z. ; 2022]

Prior work

2-approximation obtainable through various techniques, including:

- specialized techniques [Frederickson, Jájá; 1981], [Khuller, Thurimella; 1993], [Khuller, Vishkin; 1994],
- primal-dual algorithms [Goemans, Goldberg, Plotkin, Shmoys, Tardos, Williamson; 1994], and
- iterative rounding [Jain; 2001].

Prior progress on beating factor 2 only for special cases, including:

- Unweighted Tree Augmentation. (1.393-approximation)

  [Nagamochi; 2003], [Cheriyan, Karloff, Khandekar, Könemann; 2008], [Even, Feldman, Kortsarz, Nutov; 2009], [Cohen, Nutov; 2013], [Kortsarz, Nutov; 2016], [Nutov; 2017], [Cheriyan, Gao; 2018a], [Cheriyan, Gao; 2018b], [Adjiashvili; 2018], [Fiorini, Groß, Könemann, Sanità; 2018], [Grandoni, Kalaitzis, Z. ;2018], [Kortsarz, Nutov; 2018], [Cecchetto, T., Z.; 2021]
- Weighted Tree Augmentation. ($(1.5+\varepsilon)$-approximation)

  [T., Z. ; 2021], [T., Z. ; 2022]
- Unweighted Connectivity Augmentation. (1.393-approximation)

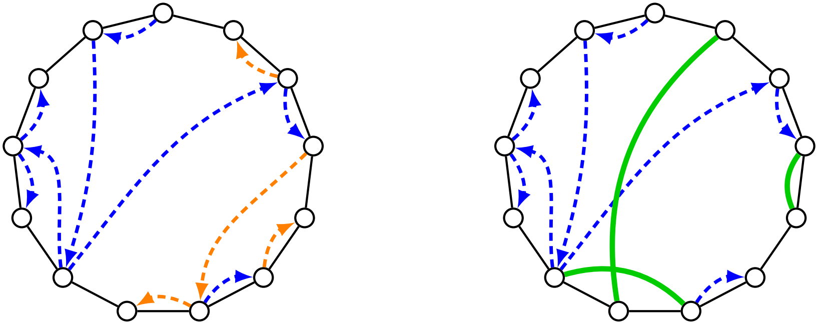
  [Byrka, Grandoni, Jabal Ameli; 2020], [Nutov; 2021], [Cecchetto, T., Z.; 2021]

Our contribution

Theorem [T.,Z.; 2023]

There is a $(1.5+\varepsilon)$-approximation algorithm for Weighted Connectivity Augmentation (WCAP).

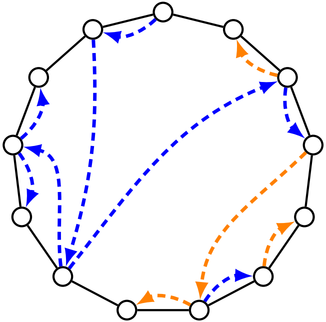
A bird's-eye view on our approach

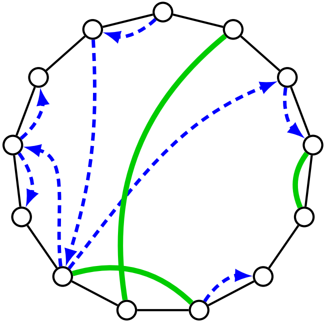


Directed WRAP: A weaker but highly structured version of WRAP.

Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

A bird's-eye view on our approach





Directed WRAP: A weaker but highly structured version of WRAP.

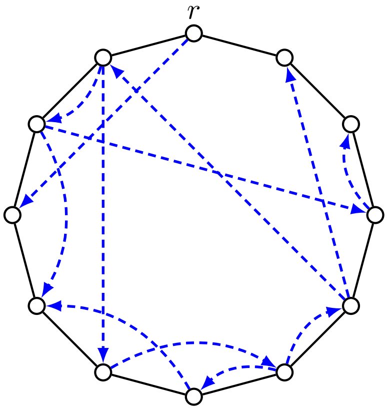
Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

One way to improve on factor 2: relative greedy

Start with 2-approximate directed solution and improve it iteratively.

Directed WRAP

(a Highly Structured Simplification of WRAP)



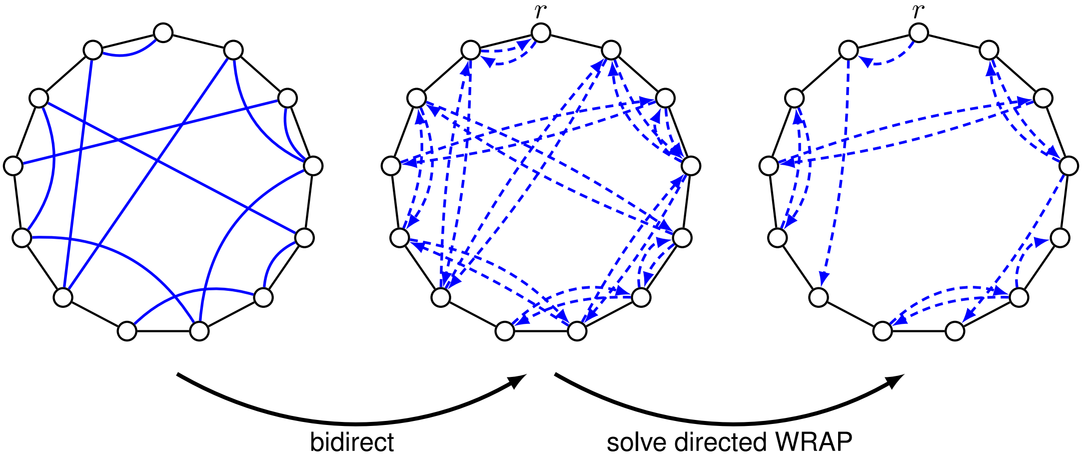
Differences between directed WRAP and WRAP:

- Links are directed.
- To cover a 2-cut $C \subseteq V \setminus \{r\}$, a directed link needs to enter it.

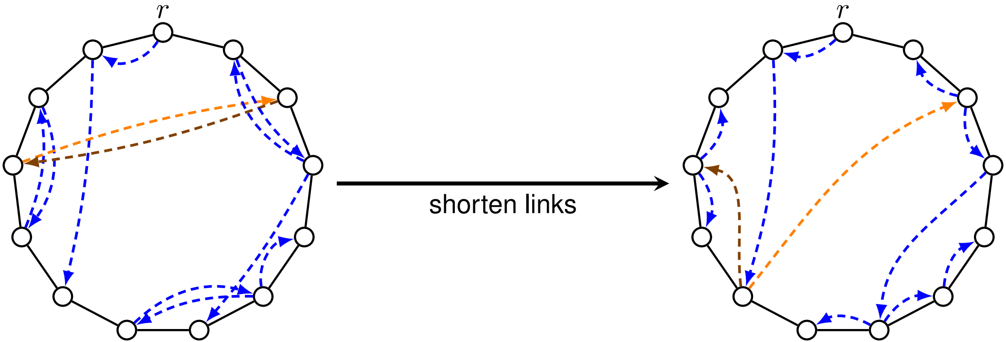
Directed WRAP can be solved efficiently.

A simple 2-approximation for WRAP via directed WRAP

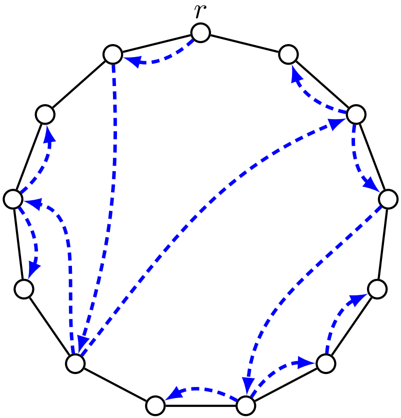
(similar to [Khuller, Vishkin, 1994] and [Cecchetto, T., Z., 2021])



Shortenings: toward structured solutions



Non-shortenable solutions



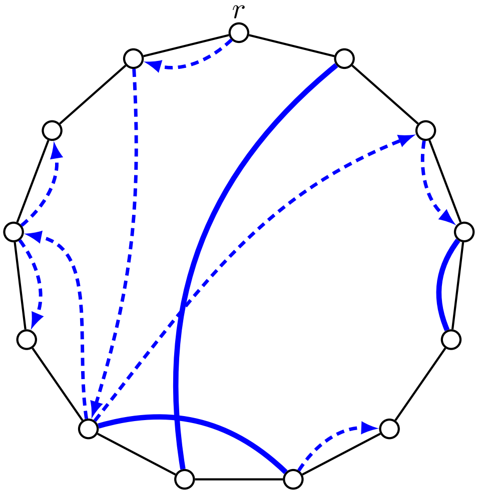
Properties of non-shortenable solutions

A non-shortenable solution ...

- is an r-arborescence;
- has a planar canonical straight-line embedding;
- has at most one left-outgoing and one right-outgoing link per vertex, and out-degree 1 at r.

Improving on directed WRAP solutions

Mixed solutions

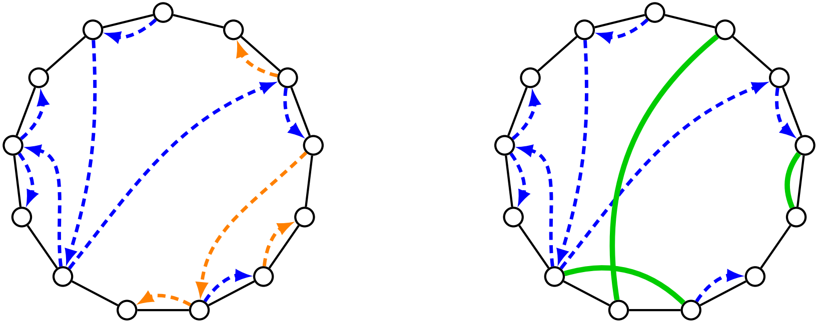


Definition: mixed solution

Every 2-cut $C \subseteq V \setminus \{r\}$ has either

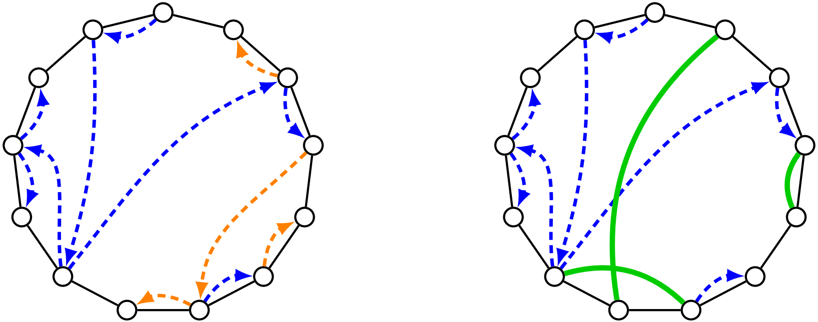
- an entering directed link, or
- a crossing undirected link.

A possible (improving) step



Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

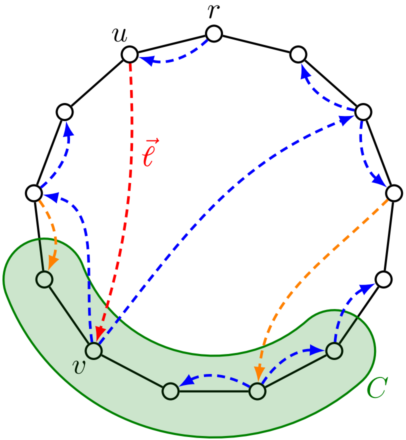
A possible (improving) step



Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

Which directed links should we drop?

Cut responsibility



Cut responsibility is defined wrt non-shortenable sol. $(V, \vec{F})$.

Definition: responsibility of arc for a 2-cut

$(u, v) \in \vec{F}$ is responsible for a 2-cut $C \subseteq V \setminus \{r\}$ if

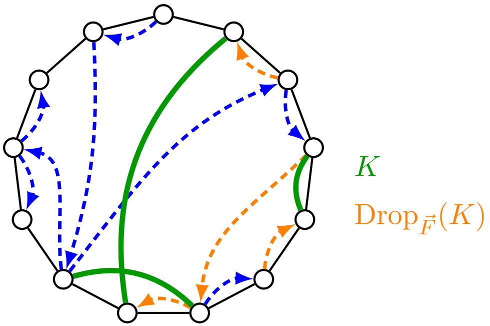
(i) (u, v) enters C, and

(ii) no directed link on r-u path of $(V, \vec{F})$ enters C.

$$\mathcal{R}_{\vec{F}}(\vec{\ell}) := \Big\{\text{all 2-cuts } \vec{\ell} \text{ is responsible for}\Big\}$$

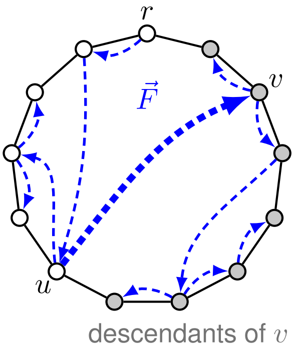
Drop of an undirected link set

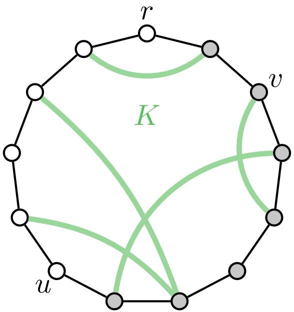
$$\text{Drop}_{\vec{F}}(K) := \left\{ \vec{\ell} \in \vec{F} \colon \text{each } C \in \mathcal{R}_{\vec{F}}(\vec{\ell}) \text{ is crossed by some link in } K \right\}$$



Then $\left(\vec{F} \setminus \text{Drop}_{\vec{F}}(K)\right) \cup K$ is a mixed solution.

Characterizing when all 2-cuts a link is responsible are covered?

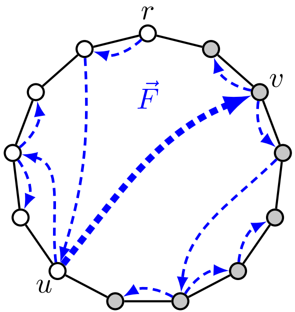




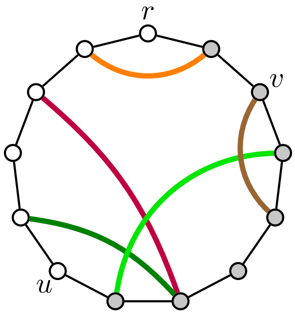
Lemma

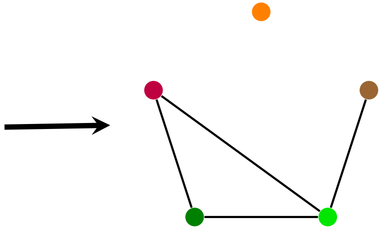
$(u, v) \in \mathrm{Drop}_{\vec{F}}(K)$ $\Leftrightarrow$ There is a path in $H[K]$ connecting a link incident to v to one incident to a vertex that is not a descendant of v (in $\vec{F}$).

Characterizing when all 2-cuts a link is responsible are covered?



descendants of v



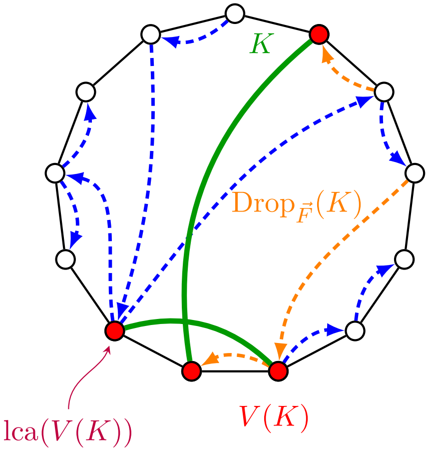


link intersection graph $H[K]$

Lemma

$(u, v) \in \mathrm{Drop}_{\vec{F}}(K) \quad \Leftrightarrow$ There is a path in $H[K]$ connecting a link incident to v to one incident to a vertex that is not a descendant of v (in $\vec{F}$).

More structure: a nice coverage property



Lemma

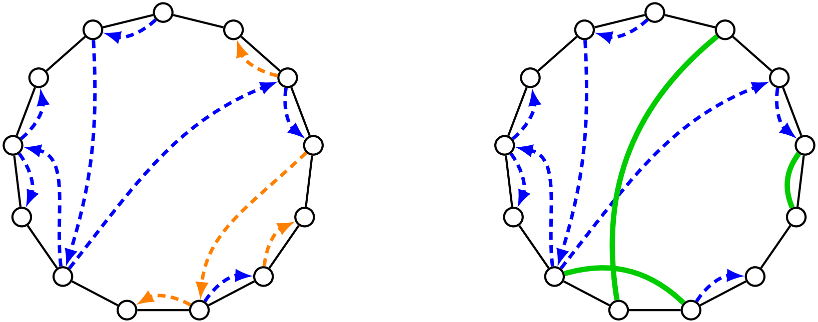
If K is a link set connected in link intersection graph:

least common ancestor

$$\mathrm{Drop}_{\vec{F}}(K) = \left(\bigcup_{v \in V(K)} \delta^-_{\vec{F}}(v) \right) \setminus \delta^-_{\vec{F}}(\mathrm{lca}(V(K))).$$

Improvements through thin components

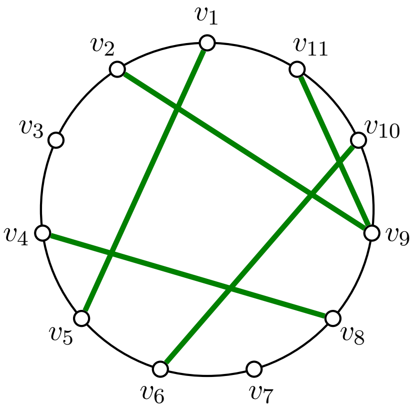
Finding improving steps



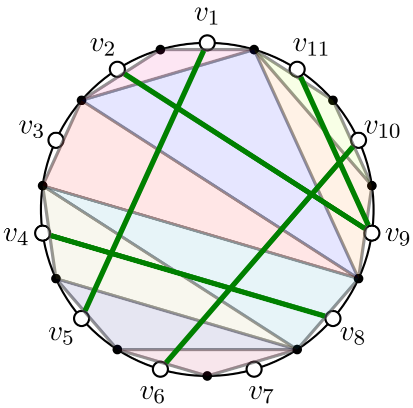
Improvement step: Exchange set of directed links by (cheaper set of) undirected ones.

Which undirected links should we add?

Thin components



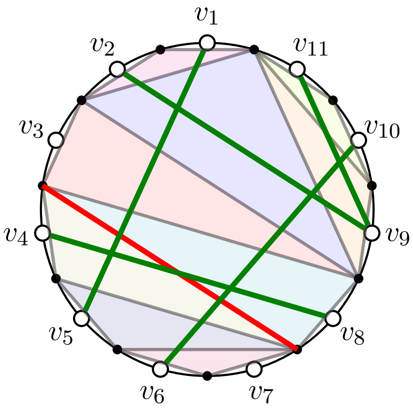
Thin components

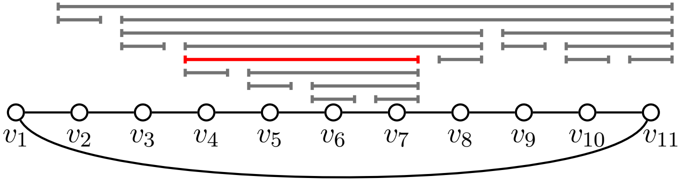


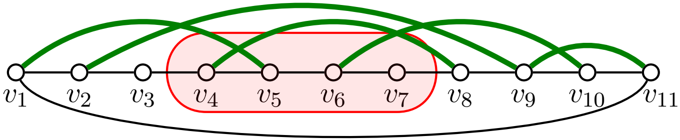
Definition: α-thin component $K \subseteq L$

$K \subseteq L$ is α-thin if $\exists$ triangulation s.t. the side of any triangle is crossed by $\leq \alpha$ links in K.

Thin components



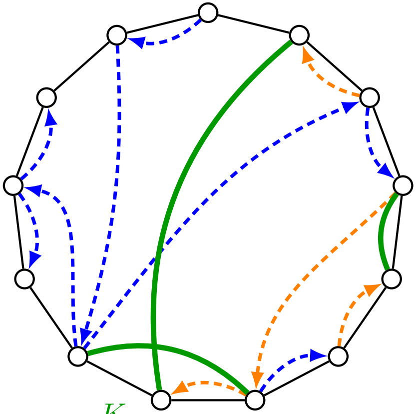




Definition: α-thin component $K \subseteq L$

$K \subseteq L$ is α-thin if $\exists$ triangulation s.t. the side of any triangle is crossed by $\leq \alpha$ links in K.

Optimization theorem

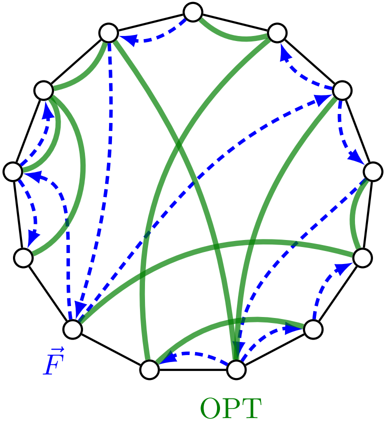


Optimization Theorem

Given a non-shortenable directed WRAP solution $\vec{F}$ and $\alpha = O(1)$, one can efficiently find an α-thin $K \subseteq L$ minimizing

$$\frac{c(K)}{c(\text{Drop}_{\vec{F}}(K))}.$$

Decomposition Theorem



Fix $\varepsilon > 0$.

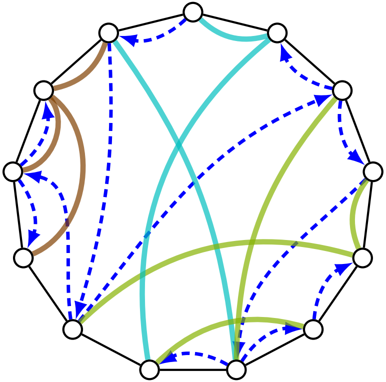
$\vec{F}$:= non-shortenable directed WRAP solution

Decomposition Theorem

There exists a partition $\mathcal{K}$ of OPT into $4\lceil {}^1/_\varepsilon \rceil$-thin components s.t.:

$$\sum_{K \in \mathcal{K}} c(\mathrm{Drop}_{\vec{F}}(K)) \;\geq\; (1-\varepsilon) \cdot c(\vec{F}).$$

Decomposition Theorem



$\mathcal{K} = \{K_1, K_2, K_3\}$

Fix $\varepsilon > 0$.

$\vec{F}$:= non-shortenable directed WRAP solution

Decomposition Theorem

There exists a partition $\mathcal{K}$ of OPT into $4\lceil 1/\varepsilon \rceil$-thin components s.t.:

$$\sum_{K \in \mathcal{K}} c(\mathrm{Drop}_{\vec{F}}(K)) \geq (1 - \varepsilon) \cdot c(\vec{F}).$$

Relative Greedy: $(1 + \ln 2 + \varepsilon)$-approximation

- Start with a non-shortenable directed WRAP solution $\vec{F}_0$ of cost $\leq 2 \cdot c(\mathrm{OPT})$.
- Iteratively add components $K_1, \ldots, K_m$ (chosen greedily using optimization theorem).
- Remove $\mathrm{Drop}_{\vec{F}_0}(K_1), \ldots, \mathrm{Drop}_{\vec{F}_0}(K_m)$.

Approximation Algorithms

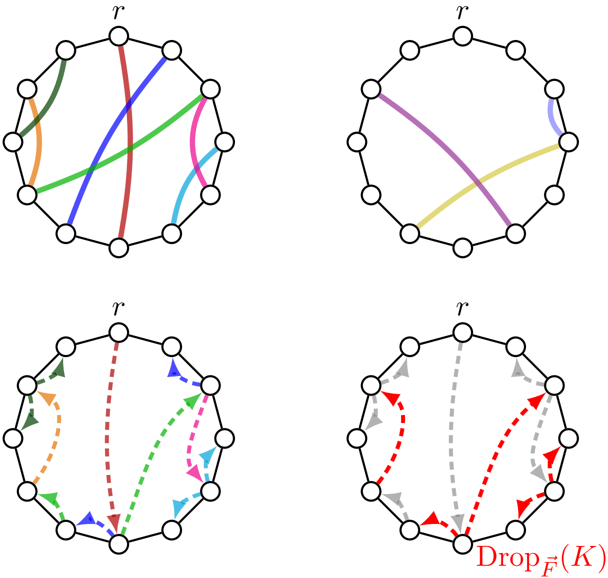
Relative Greedy: $(1 + \ln 2 + \varepsilon)$-approximation

- Start with a non-shortenable directed WRAP solution $\vec{F}_0$ of cost $\leq 2 \cdot c(\mathrm{OPT})$.
- Iteratively add components $K_1, \ldots, K_m$ (chosen greedily using optimization theorem).
- Remove $\mathrm{Drop}_{\vec{F}_0}(K_1), \ldots, \mathrm{Drop}_{\vec{F}_0}(K_m)$.

Local Search: $(1.5 + \varepsilon)$-approximation

- Algorithm based on the approach from [T.,Z.;2022].
- We maintain an undirected WRAP solution $F \subseteq L$.
- F can be turned into non-shortenable directed WRAP solution $\vec{F}$.
- When looking for an exchange step, use $\vec{F}$ to measure progress.
- Remove $\ell \in F$ if (the at most 2) links in $\vec{F}$ corresponding to ℓ have been dropped.

Local search step



Conclusions

Theorem [T. & Z., 2023]

There is a $(1.5 + \varepsilon)$-approximation for Weighted Connectivity Augmentation.

- Directed WRAP is highly structured problem that can be leveraged to approach WRAP.
- Cut responsibility helps to make search for good link set algorithmically tractable.
- Careful definition of thin components $\rightarrow$ Optimization & Decomposition Theorem.
- Main technical parts (skipped here): Proof of Optimization & Decomposition Theorem.
- Derived structural results can be exploited in relative greedy or local search algorithms.