Degree Sequence Optimization and Sparse Integer Programming

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Given a graph H and a function $f_i : \{0,1, . . ., d_i(H)\} \text{ ---> } Z$
for each vertex i, find a subgraph G of H which minimizes

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Example: with $f_i(x)=(x-1)^2$ for all i can decide if H has a perfect matching

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NP-hard: H has a nonempty cubic subgraph if and only if for some i the optimal value of DSO with $f_i(x)=(x-3)^2$ and $f_j(x)=x(x-3)^2$ for all $j \neq i$ is zero

Some Special Cases

Complete Graphs

When $H = K_n$ is the complete graph, the problem is:

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This answers a 30 year long open question (Colbourn et al. 1986)

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Proof: Involved dynamic programming with states being 7-tuples

Convex Functions and General Factors

The general Factor problem is:

Given a graph H and a subset B_i of $\{0,1, . . ., d_i(H)\}$ for each i, decide if H has a subgraph G with $d_i(G)$ in B_i for each i

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Reduces to DSO with $f_i(x):=0$ for x in B_i and $f_i(x):=1$ for x not in B_i

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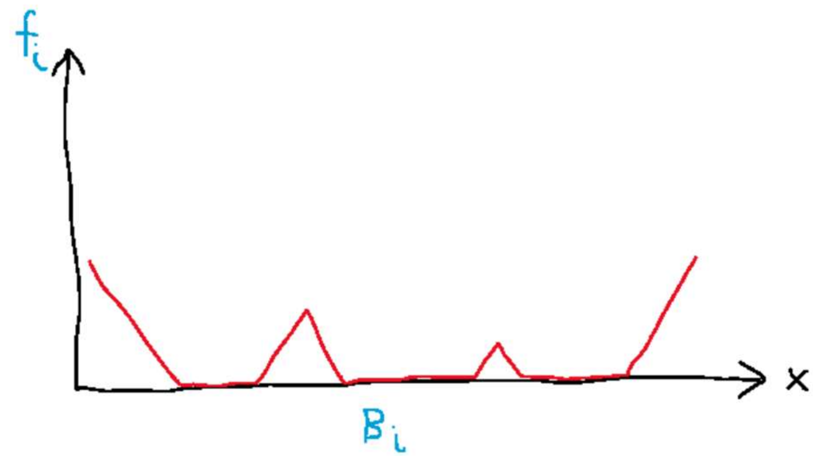

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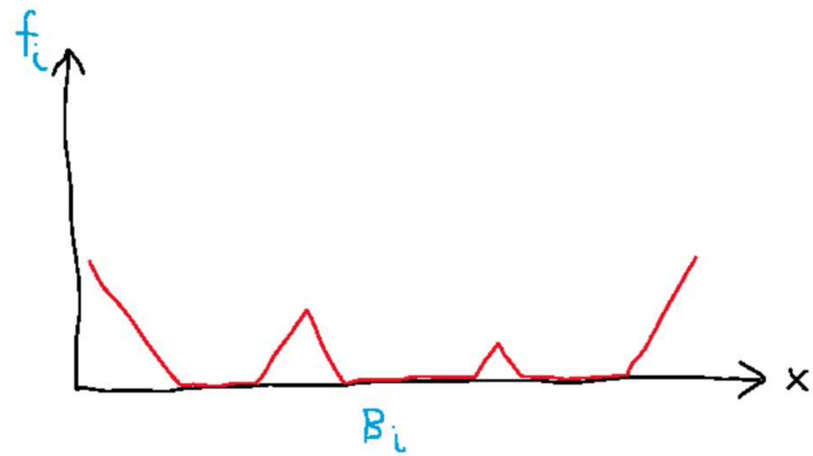

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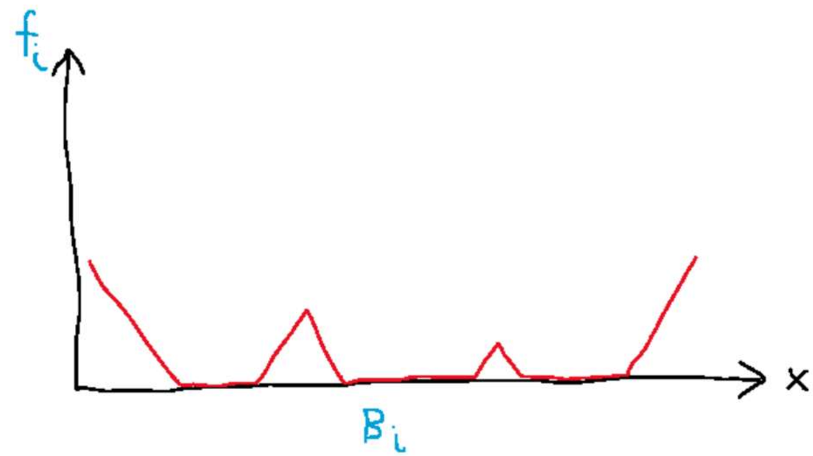

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Proof: Reduces to weighted matching on a suitable larger graph

Bounded Tree Width or Depth

Theorem (Onn 2022) For any fixed k, can solve DSO in polynomial time for any functions f_i and any graph H with tw(H) ≤ k

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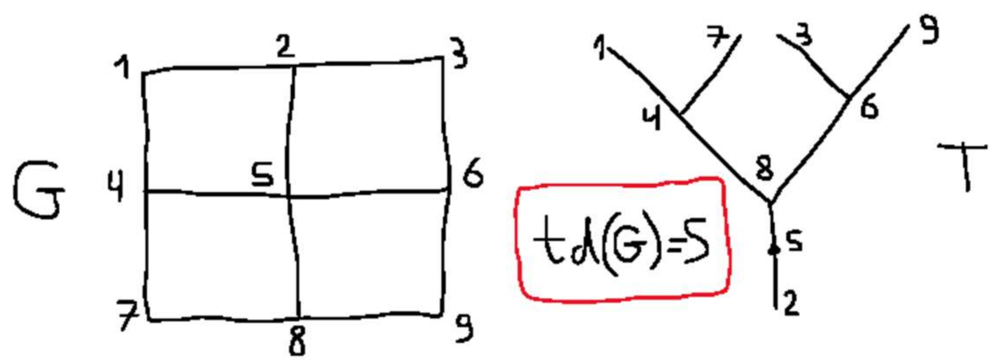

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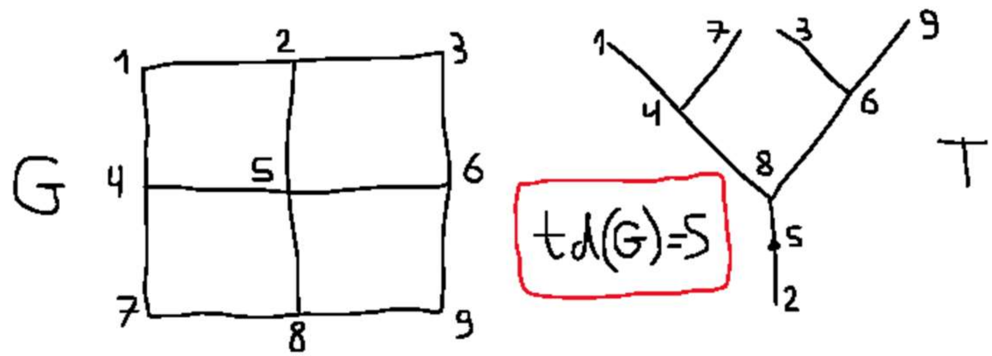

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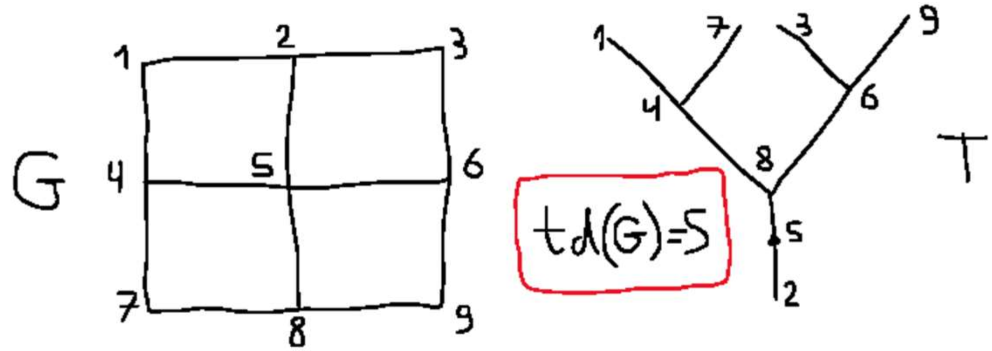
Given a graph G on [n], a rooted tree on [n] is valid for G if for each edge ij of G, one of i and j is on the path from the root to the other.

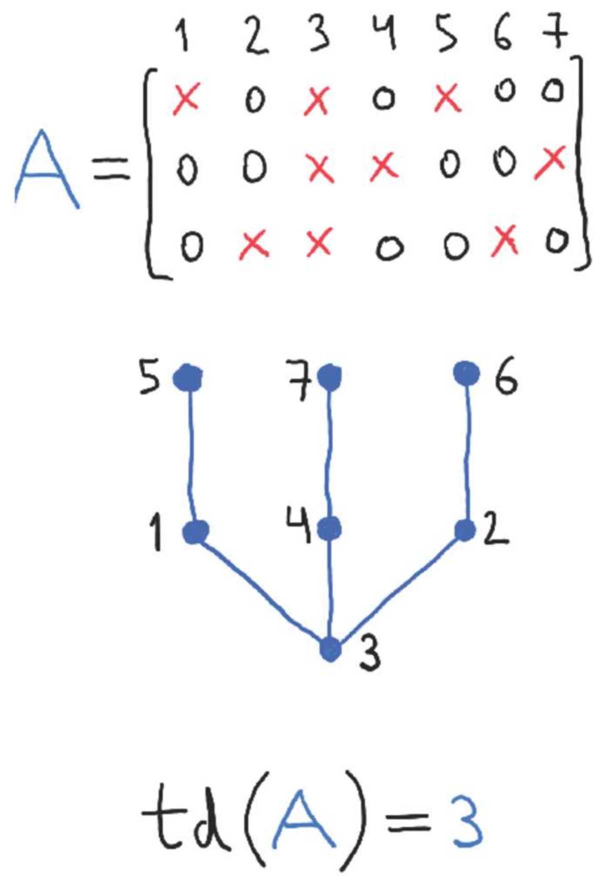
The tree-depth of G is the smallest height of a rooted tree valid for G



Sparse Integer Programming

Tree-Depth of a Matrix

The **graph** of **m×n** matrix A is the graph $G(A)$ on [n] with j,k an edge iff $A_{i,j}$, $A_{i,k}$ are nonzero for some i. The **tree-depth** of A is $td(A) := td(G(A))$



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Sparse Integer Programming

IP: $\min\{\, cx \;:\; Ax = b,\;\; l \le x \le u,\;\; x \in Z^n \,\}$

A integer $m\times n$ matrix, l,u,c in Z^n, b in Z^m

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Theorem (Koutecky-Levin-Onn) IP parameterized by a and d is FPT.
Specifically, there are functions $g(a,d)$, $h(d)$ so IP is solvable in time:

$g(a,d)\mathrm{poly}(n)$ when $d = td(A)$

$(a+1)^{h(d)}\mathrm{poly}(n)$ when $d = td(A^T)$

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Koutecky-Levin-Onn 2018, Eisenbrand-Hunkenschroder-Klein-Koutecky-Levin-Onn 2019

Koutecky-Onn 2021, "Sparse Integer Programming is FPT", Bulletin EATCS

One Application: N-Fold and Block Shaped IP

$$A = \begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{2,6} & A_{2,7} & A_{2,8} \\
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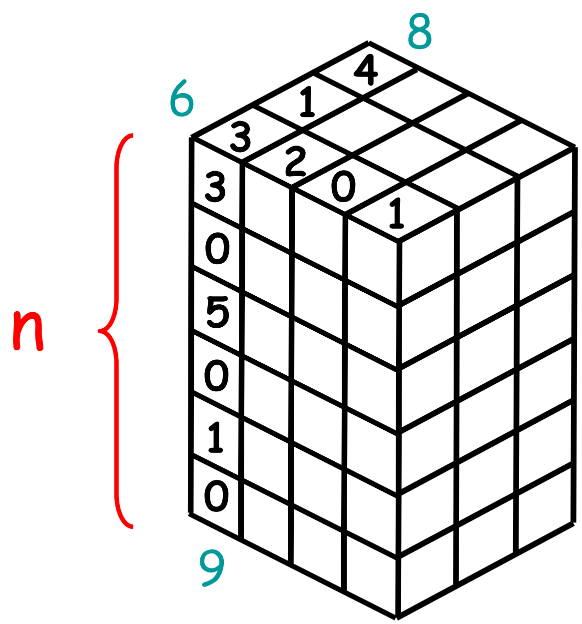
\end{pmatrix}$$

Corollary: Block shaped IP and N-fold IP are solvable in FPT time

In Particular: Multiway Tables

Optimization in FPT time over $m_1 \times \cdots \times m_k \times n$ tables with given margins (and multicommodity flows), where this line started by De Loera – Onn:



Proof Sketch

Theorem (Koutecky-Levin-Onn) IP is FPT parameterized by a,d, where:

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2. Using these bounds it can be shown that suitable auxiliary integer programs can be used to efficiently find Graver-best steps recursively on a small height tree validating small tree-depth $d=td(A)$ or $d=td(A^T)$

3. It can be shown that few Graver-best steps suffice to reach optimum

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Proof: The matrix A of the following IP has parameters a=n-1 and d=td(A^T)=td(H)+1 so solvable in polynomial time $n^{h(k+1)}$poly(n):

$$\min \sum_{i=1}^{n} \sum_{j=0}^{d_i(H)} f_i(j)\, y_{i,j}$$

$$\sum_{e \in \delta_i(H)} x_e - \sum_{j=0}^{d_i(H)} j\, y_{i,j} = 0$$

$$\sum_{j=0}^{d_i(H)} y_{i,j} = 1$$

$$0 \leq x_e,\ y_{i,j} \leq 1, \quad \text{integer}$$

Colored Degree Sequence Optimization

Adding suitable constraints to this IP, we can even solve the colored version of DSO, where the edges are colored by p colors and we need to find a subgraph having prescribed number of edges of each color:

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A special case of this problem is the notorious exact matching problem for which a randomized algorithm is known but not a deterministic one