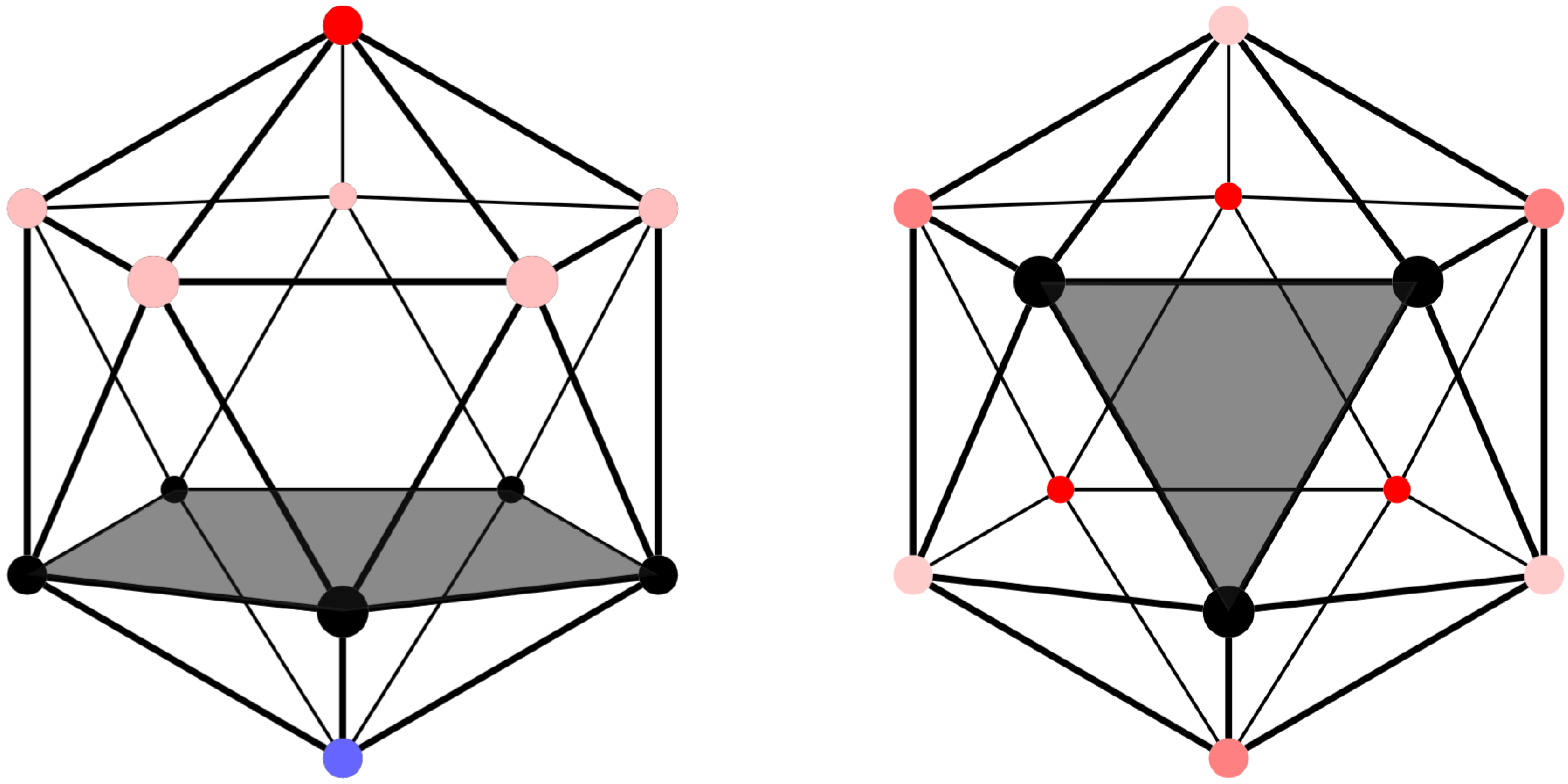
The Polyhedral Structure of Graphical Designs



Catherine Babecki

University of Washington, Seattle

Featuring Rekha Thomas, Stefan Steinerberger, and David Shiroma

The Polyhedral Structure of Graphical Designs

Catherine Babecki

University of Washington, Seattle

Featuring Rekha Thomas, Stefan Steinerberger, and David Shiroma

Spherical Designs

Definition: A spherical quadrature rule is a set of points $\{x_1, \ldots, x_n\} \subset \mathbb{S}^d$ and weights $a_i \in \mathbb{R}$ chosen so that

$$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x)\ dx \approx \sum_{i=1}^{n} a_i f(x_i)$$

whenever f is smooth in some suitable way.

Spherical Designs

Definition: A spherical quadrature rule is a set of points $\{x_1, \ldots, x_n\} \subset \mathbb{S}^d$ and weights $a_i \in \mathbb{R}$ chosen so that

$$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x)\ dx \approx \sum_{i=1}^{n} a_i f(x_i)$$

whenever f is smooth in some suitable way.

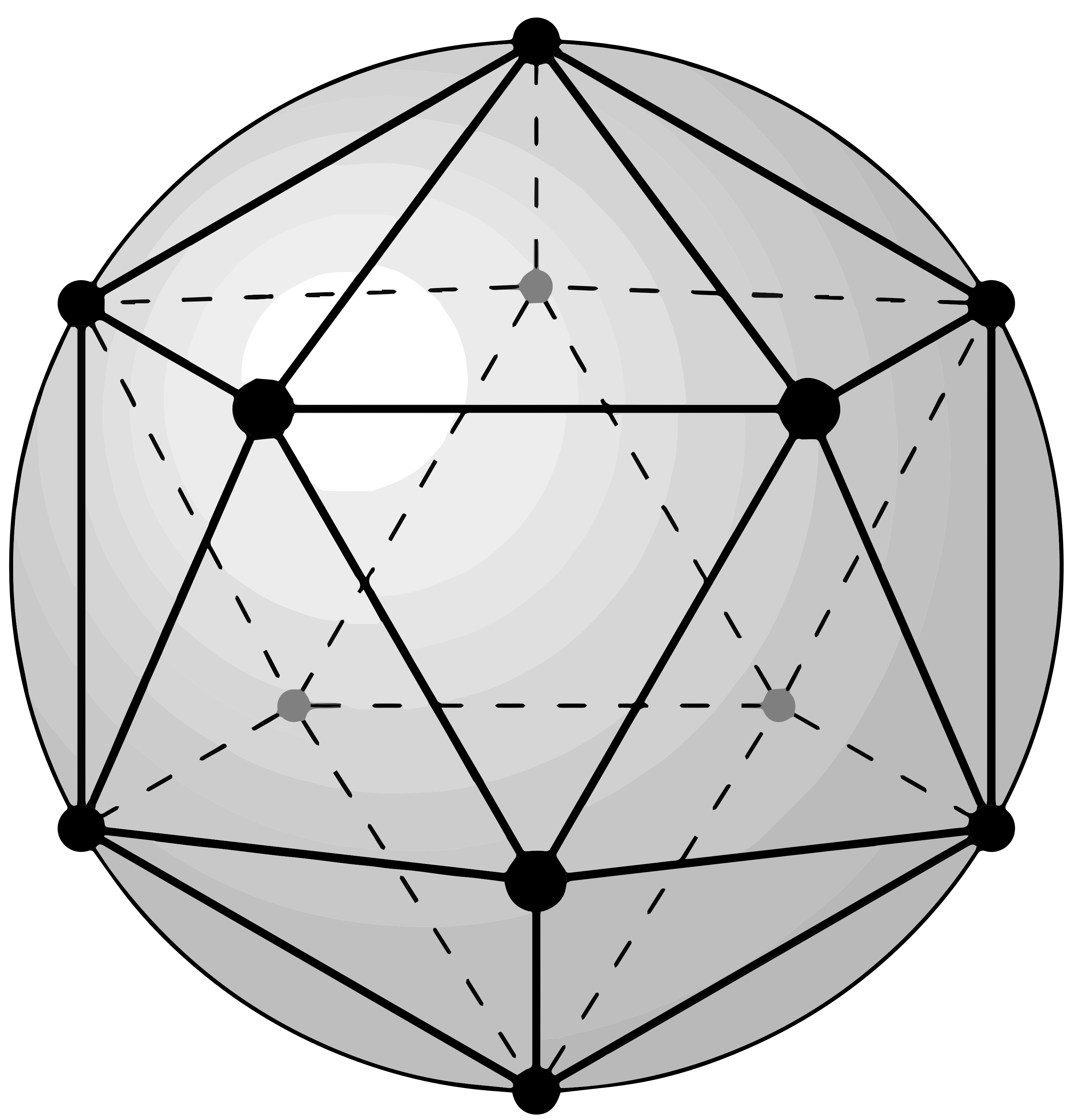
A spherical *t*-design is a quadrature rule which integrates all polynomials up to degree *t* exactly.

Spherical Designs

Definition: A spherical quadrature rule is a set of points $\{x_1, \ldots, x_n\} \subset \mathbb{S}^d$ and weights $a_i \in \mathbb{R}$ chosen so that

$$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x)\ dx \approx \sum_{i=1}^{n} a_i f(x_i)$$

whenever f is smooth in some suitable way.



A spherical *t*-design is a quadrature rule which integrates all polynomials up to degree *t* exactly.

This is a spherical 5-design!

Extension to Graphs

Definition: $G = ([n], E, w)$ a finite, connected, weighted graph. A subset $S \subset [n]$ with weights a_s averages a function $\varphi : [n] \to \mathbb{R}$ if

$$\sum_{s \in S} a_s \varphi(s) = \frac{1}{n} \sum_{v \in [n]} \varphi(v).$$

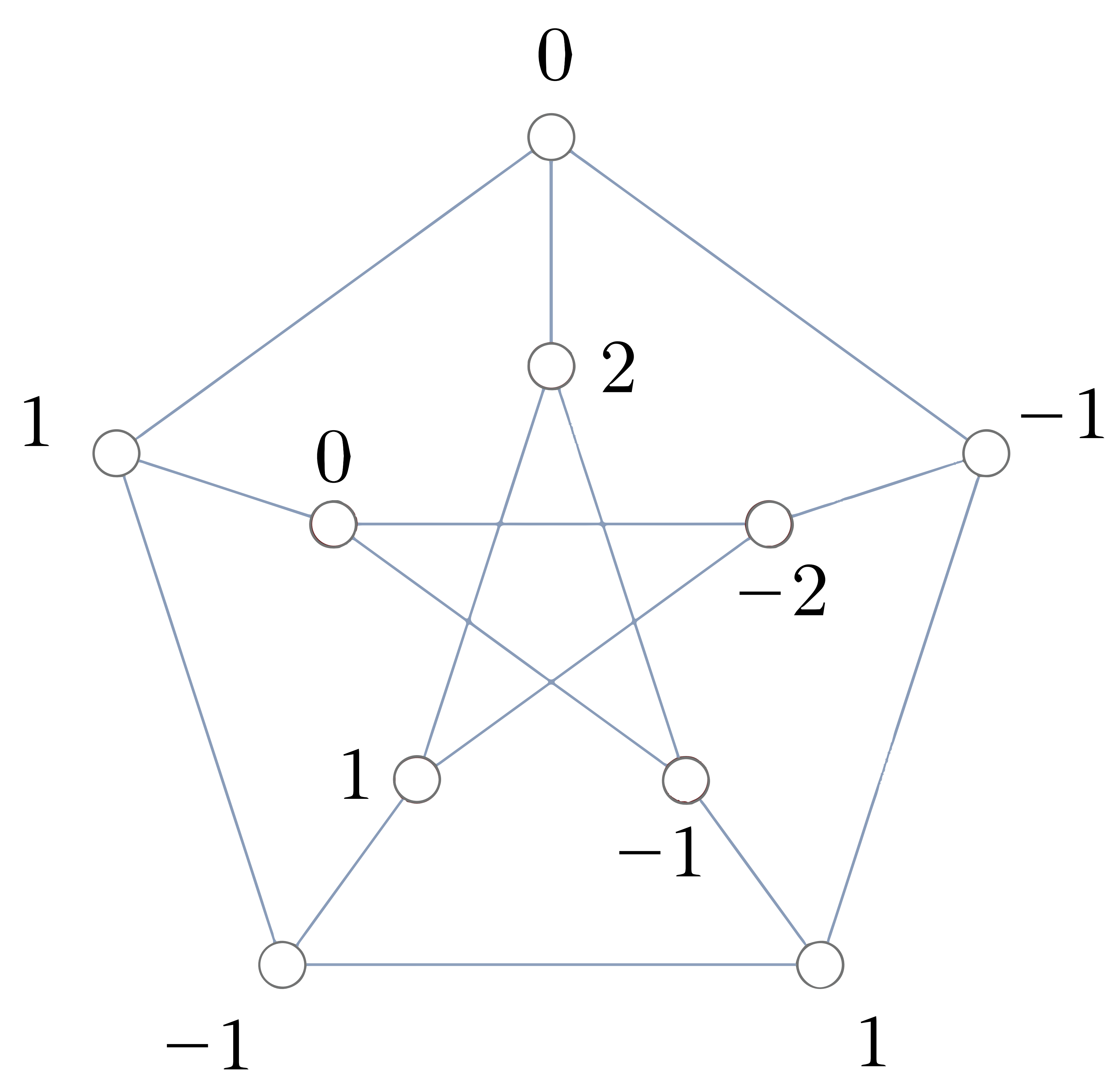
$w : E \to \mathbb{R}_{>0}$

Extension to Graphs

Definition: $G = ([n], E, w)$ a finite, connected, weighted graph. A subset $S \subset [n]$ with weights a_s averages a function $\varphi : [n] \to \mathbb{R}$ if

$$\sum_{s \in S} a_s \varphi(s) = \frac{1}{n} \sum_{v \in [n]} \varphi(v).$$

$$\varphi = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

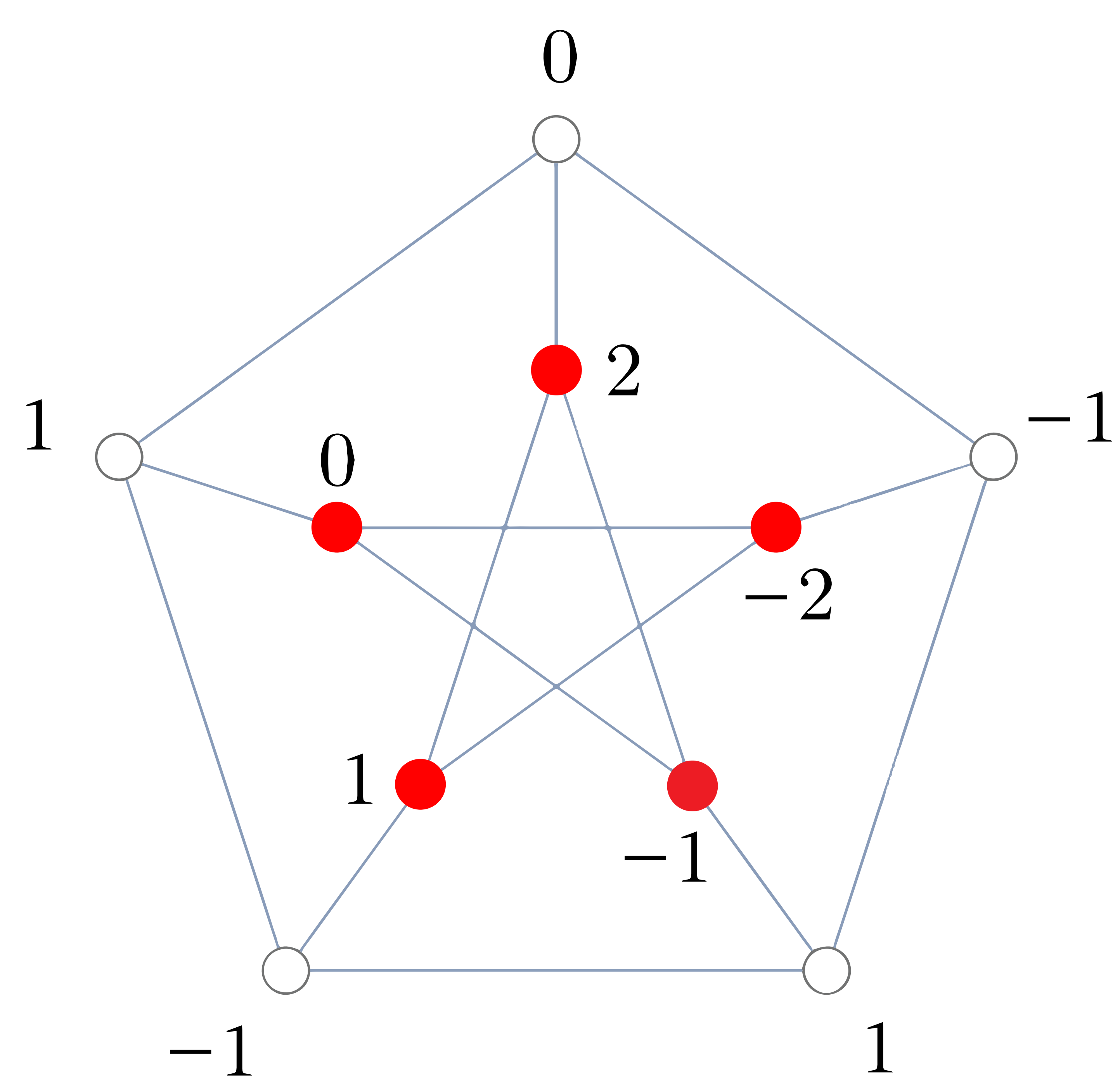


Extension to Graphs

Definition: $G = ([n], E, w)$ a finite, connected, weighted graph. A subset $S \subset [n]$ with weights a_s averages a function $\varphi : [n] \to \mathbb{R}$ if

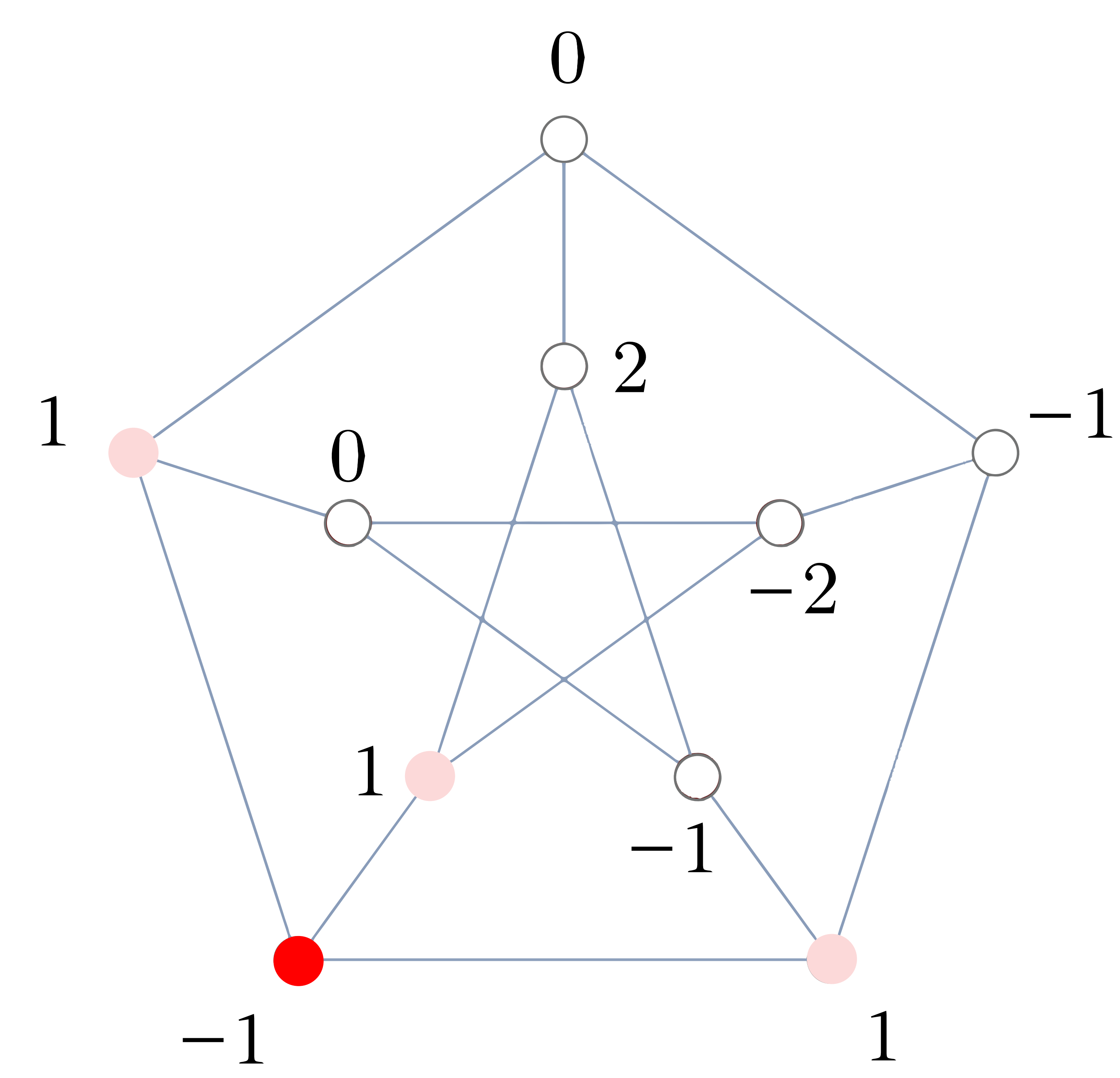
$$\sum_{s \in S} a_s \varphi(s) = \frac{1}{n} \sum_{v \in [n]} \varphi(v).$$

$$\varphi = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$



$$a = (0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$$

$$\varphi = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

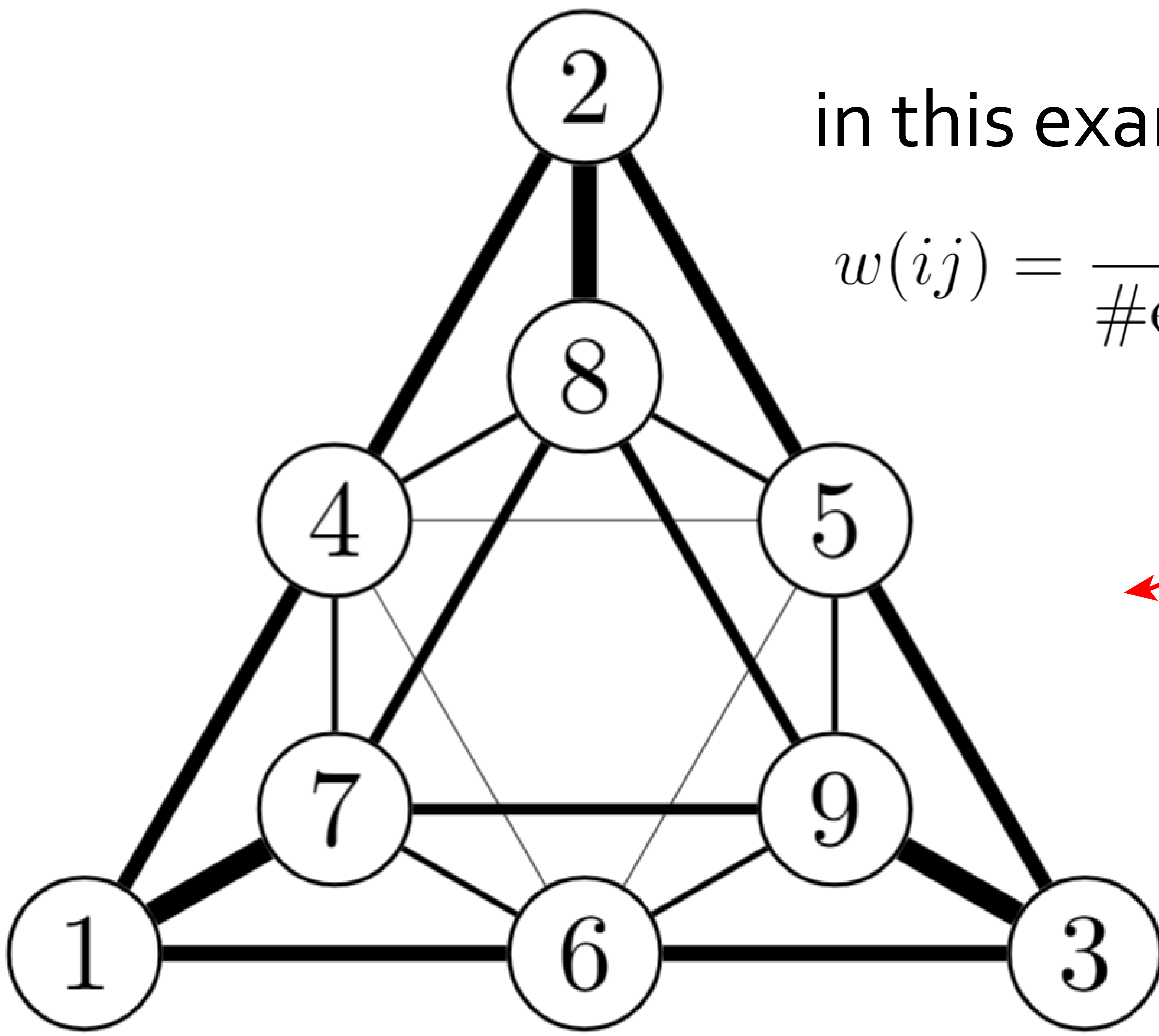


$$a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, 1, 0, 0, 0, 0, \frac{1}{3}\right)$$

What Functions Should We Average?

Definition: $G = ([n], E, w)$. Eigenvectors of the operator $D - A$.

- $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$
- $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$



in this example,

$$w(ij) = \frac{1}{\#\text{edges adjacent to } i} + \frac{1}{\#\text{edges adjacent to } j}$$

What Functions Should We Average?

Definition: $G = ([n], E, w)$. Eigenvectors of the operator $D - A$.

- $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$
- $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$

$$D - A$$

$$\begin{bmatrix}
23/15 & 0 & 0 & -1/2 & 0 & -1/2 & -8/15 & 0 & 0 \\
0 & 23/15 & 0 & -1/2 & -1/2 & 0 & 0 & -8/15 & 0 \\
0 & 0 & 23/15 & 0 & -1/2 & -1/2 & 0 & 0 & -8/15 \\
-1/2 & -1/2 & 0 & 12/5 & -1/3 & -1/3 & -11/30 & -11/30 & 0 \\
0 & -1/2 & -1/2 & -1/3 & 12/5 & -1/3 & 0 & -11/30 & -11/30 \\
-1/2 & 0 & -1/2 & -1/3 & -1/3 & 12/5 & -11/30 & 0 & -11/30 \\
-8/15 & 0 & 0 & -11/30 & 0 & -11/30 & 31/15 & -2/5 & -2/5 \\
0 & -8/15 & 0 & -11/30 & -11/30 & 0 & -2/5 & 31/15 & -2/5 \\
0 & 0 & -8/15 & 0 & -11/30 & -11/30 & -2/5 & -2/5 & 31/15
\end{bmatrix}$$

What Functions Should We Average?

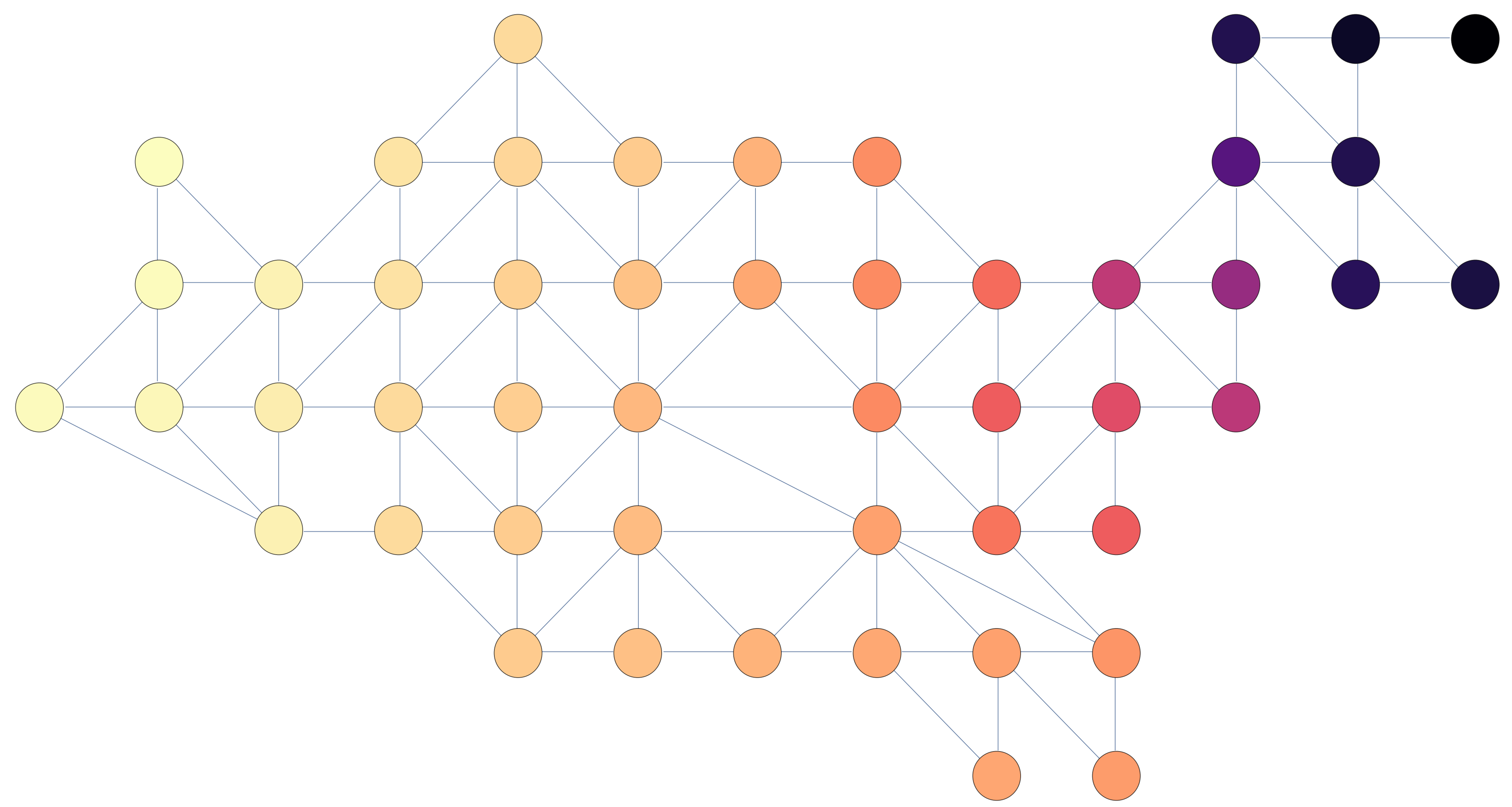
Definition: $G = ([n], E, w)$. Eigenvectors of the operator $D - A$.

- $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix $A_{ij} = w(ij)$ if $ij \in E$
- $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $D_{ii} = \deg i = \sum_{ij\in E} w(ij)$

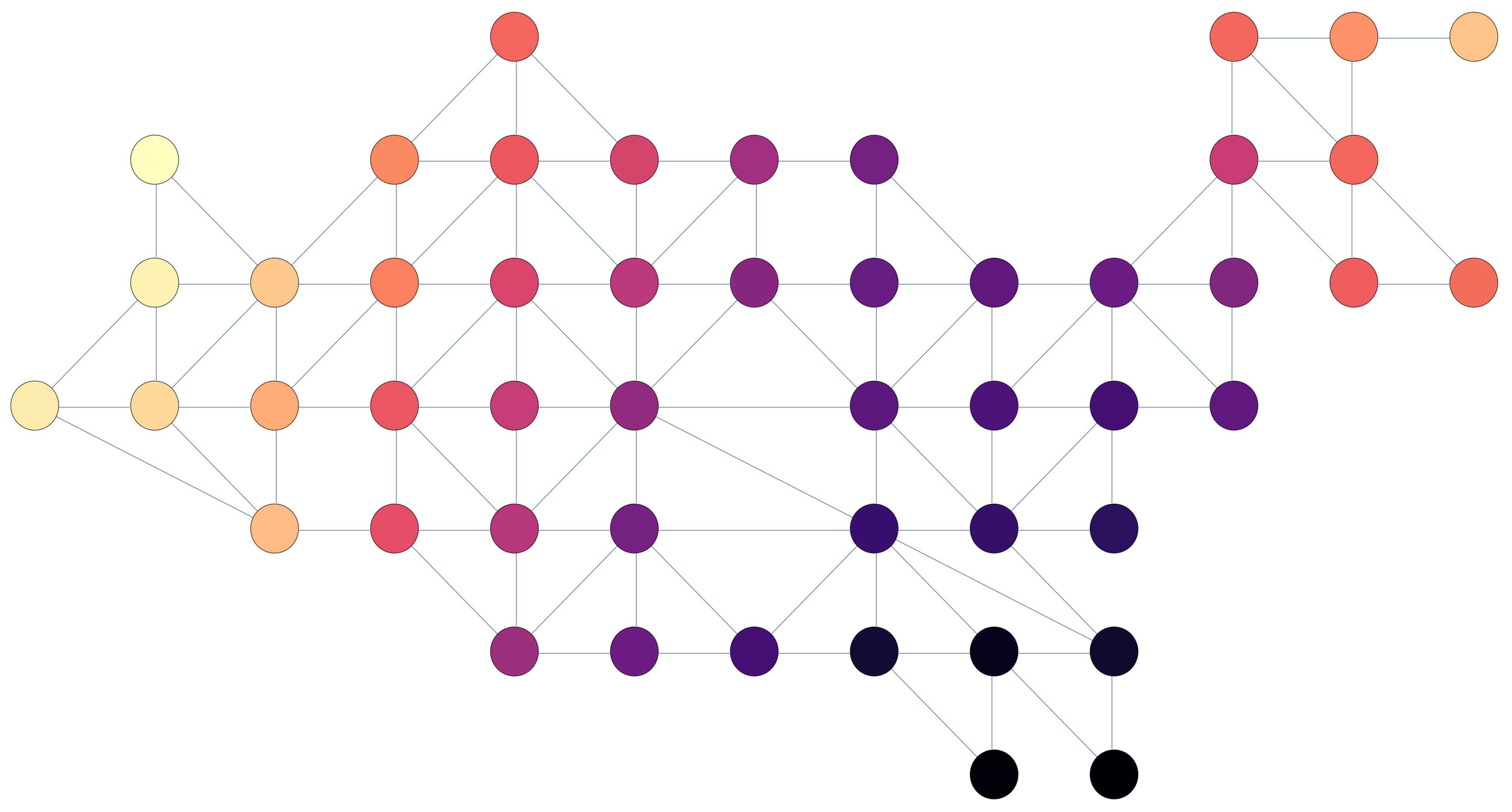
Eigenvectors by eigenspace

$\lambda_1 = 0$	1	1	1	1	1	1	1	1	1
$\lambda_2 = 1.069$	$-.6262$	$.5616$	$.0646$	$-.0264$	$.256$	$-.2291$	$-.3059$	$.2744$	$.0316$
	$-.2870$	$-.3988$	$.6858$	$-.2798$	$.1171$	$.1627$	$-.1402$	$-.1948$	$.3350$
$\lambda_3 = 1.861$	$.3193$	$.3193$	$.3193$	$.1406$	$.1406$	$.1406$	$-.4600$	$-.4600$	$-.4600$
$\lambda_4 = 2.661$	$.3248$	$-.4014$	$.0766$	$.0507$	$.2150$	$-.2658$	$-.4852$	$.5996$	$-.1144$
	$.2760$	$.1433$	$-.4193$	$-.2776$	$.1827$	$.0949$	$-.4123$	$-.2141$	$.6263$
$\lambda_5 = 2.672$	$.3468$	$.3458$	$.3468$	$-.4499$	$-.4499$	$-.4499$	$.1032$	$.1032$	$.1032$
$\lambda_6 = 3.003$	$.0724$	$-.0995$	$.0271$	$.1880$	$.5015$	$-.6894$	$.2707$	$-.3721$	$.1015$
	$.0731$	$.0261$	$-.0992$	$-.6876$	$.5066$	$.1810$	$.2734$	$.0977$	$-.3711$

What Eigenspaces?

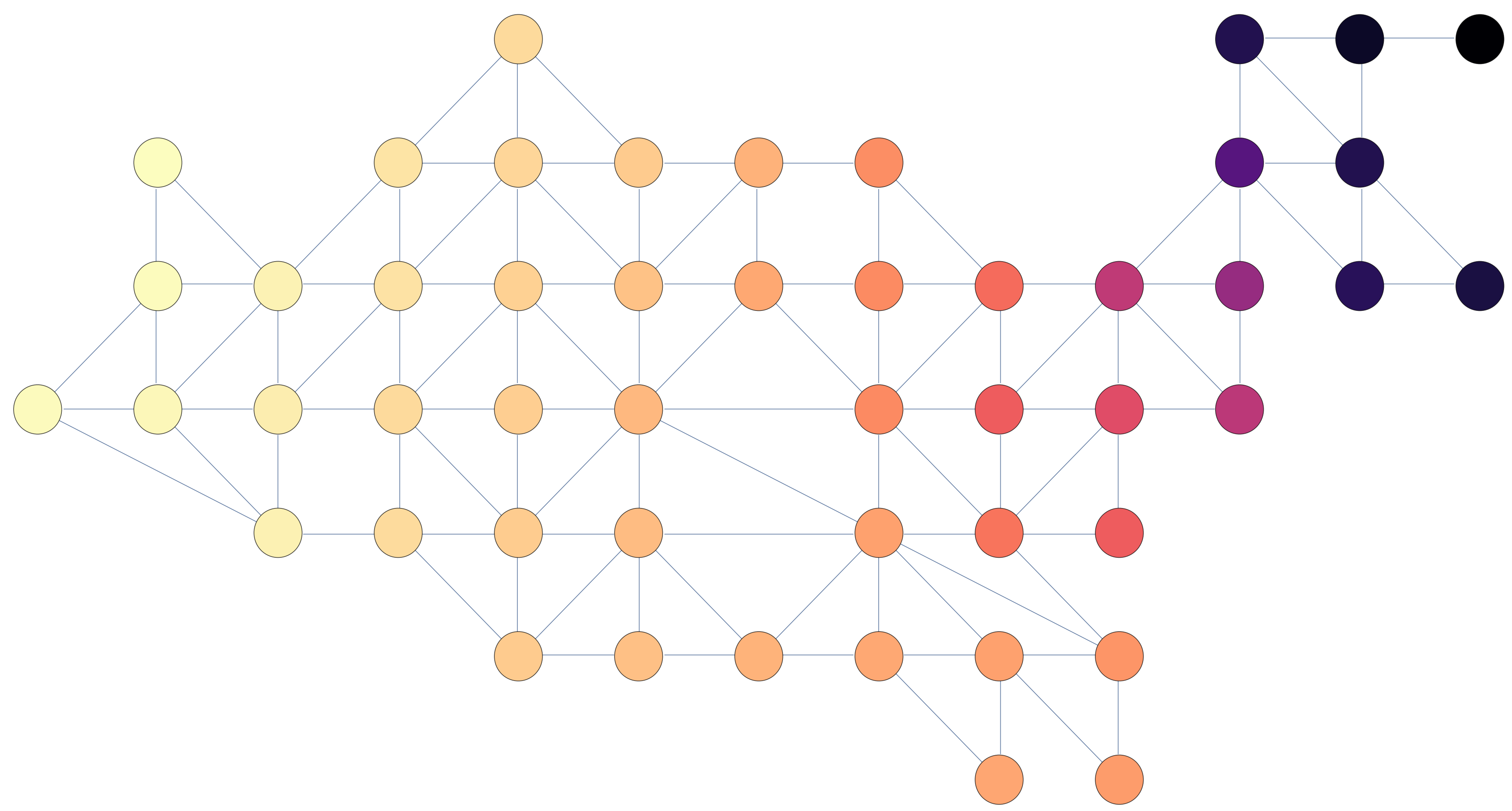


The first eigenvector by frequency

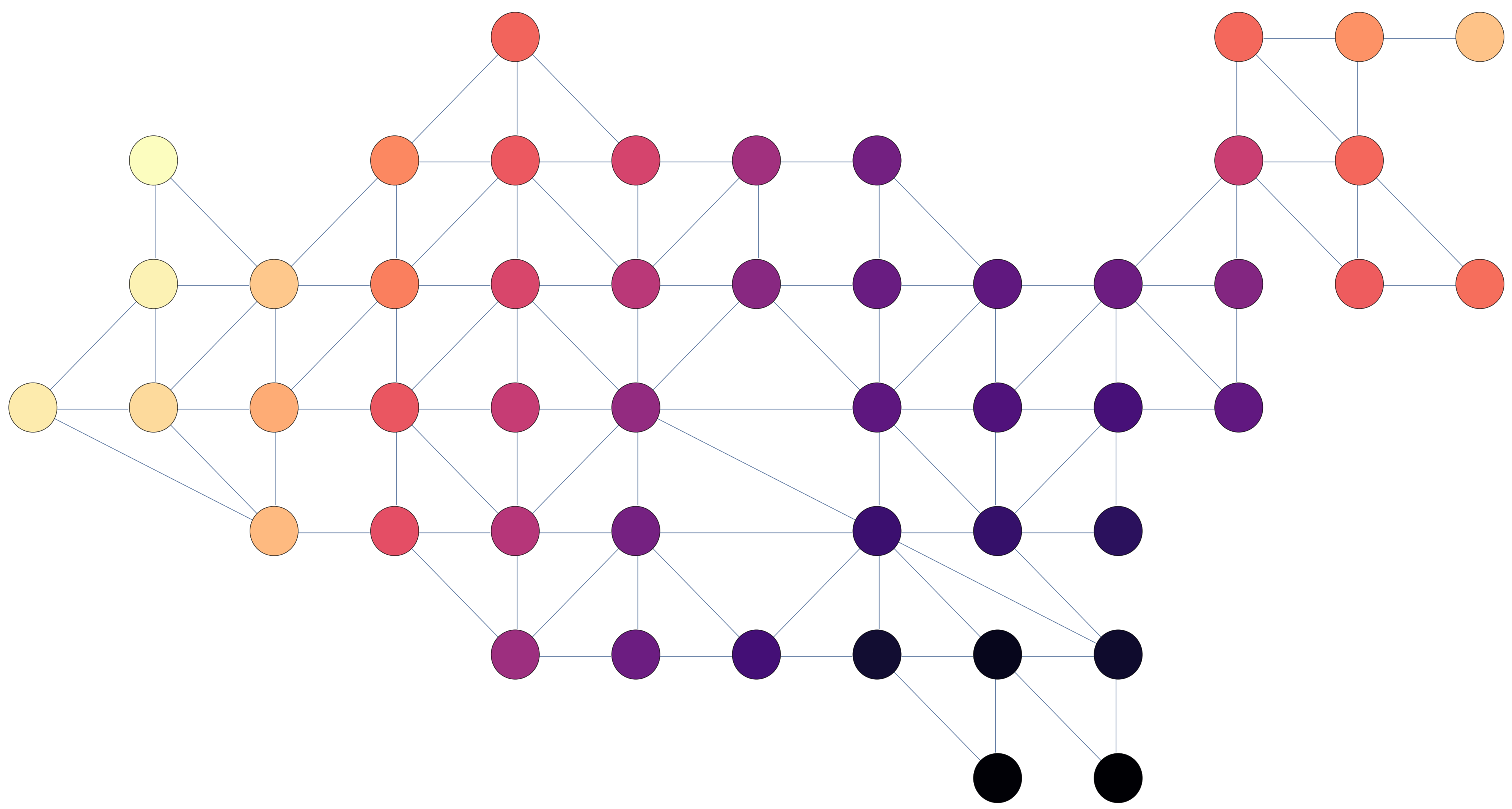


The second eigenvector by frequency

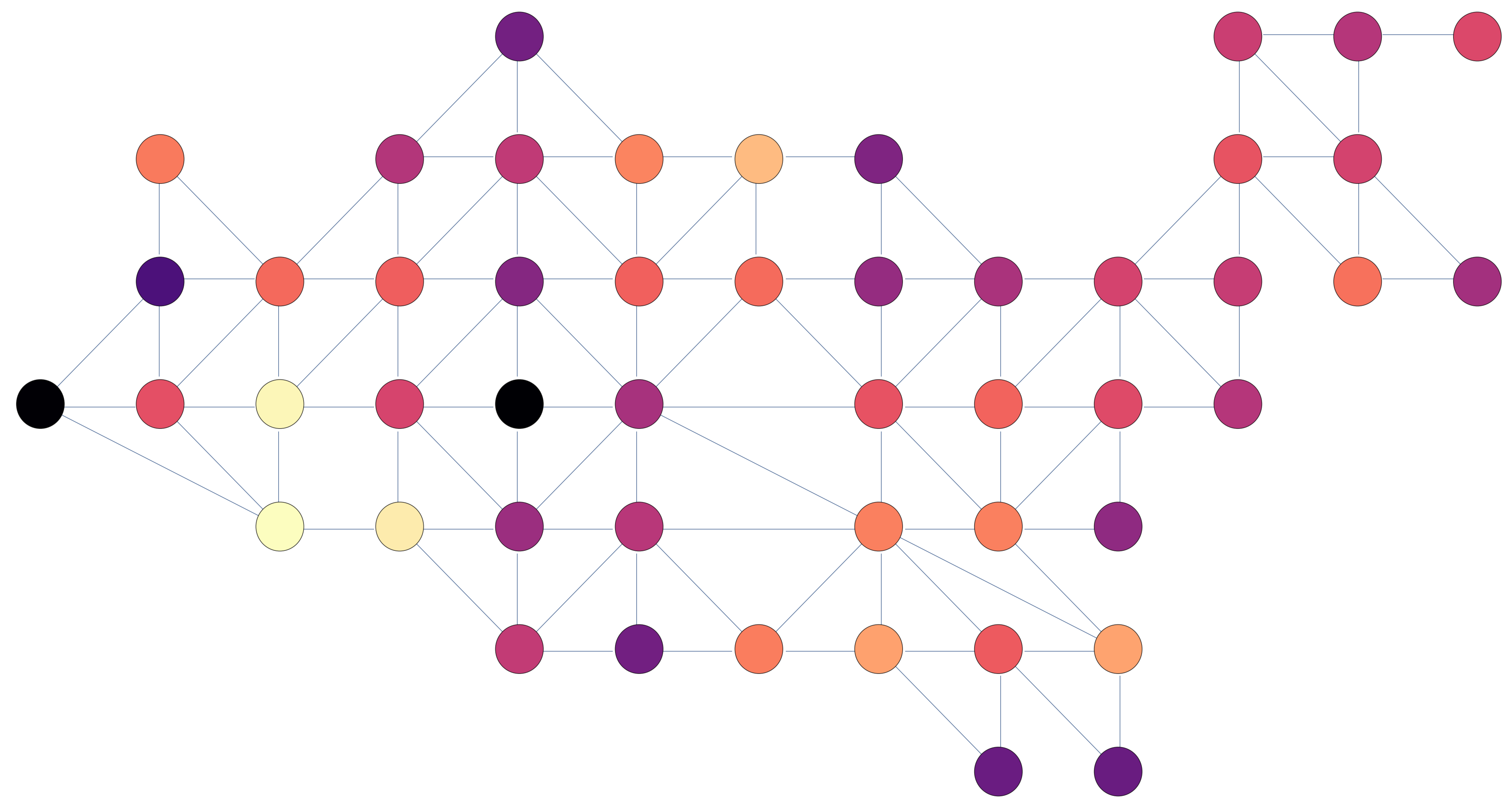
What Eigenspaces?



The first eigenvector by frequency

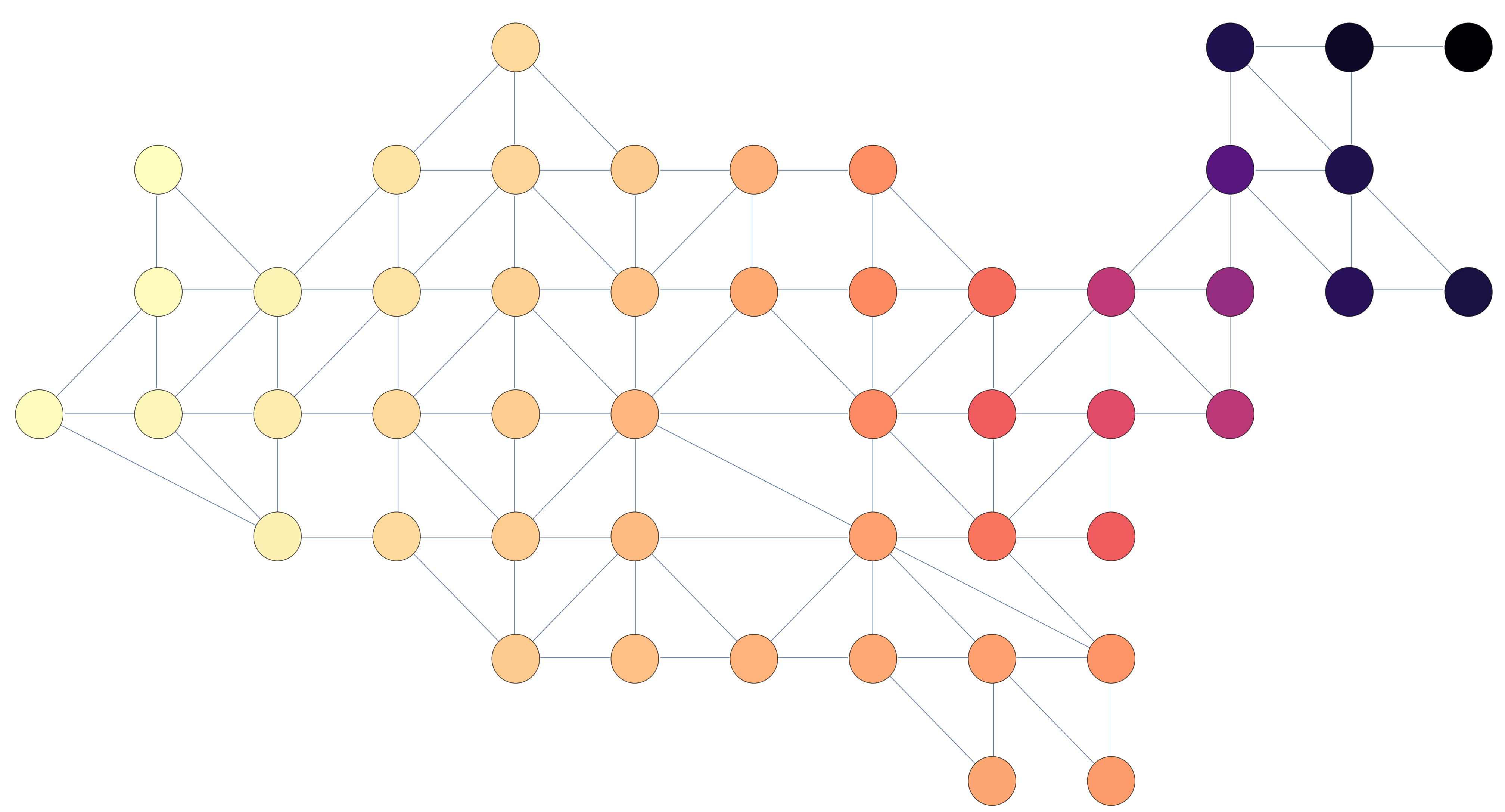


The second eigenvector by frequency

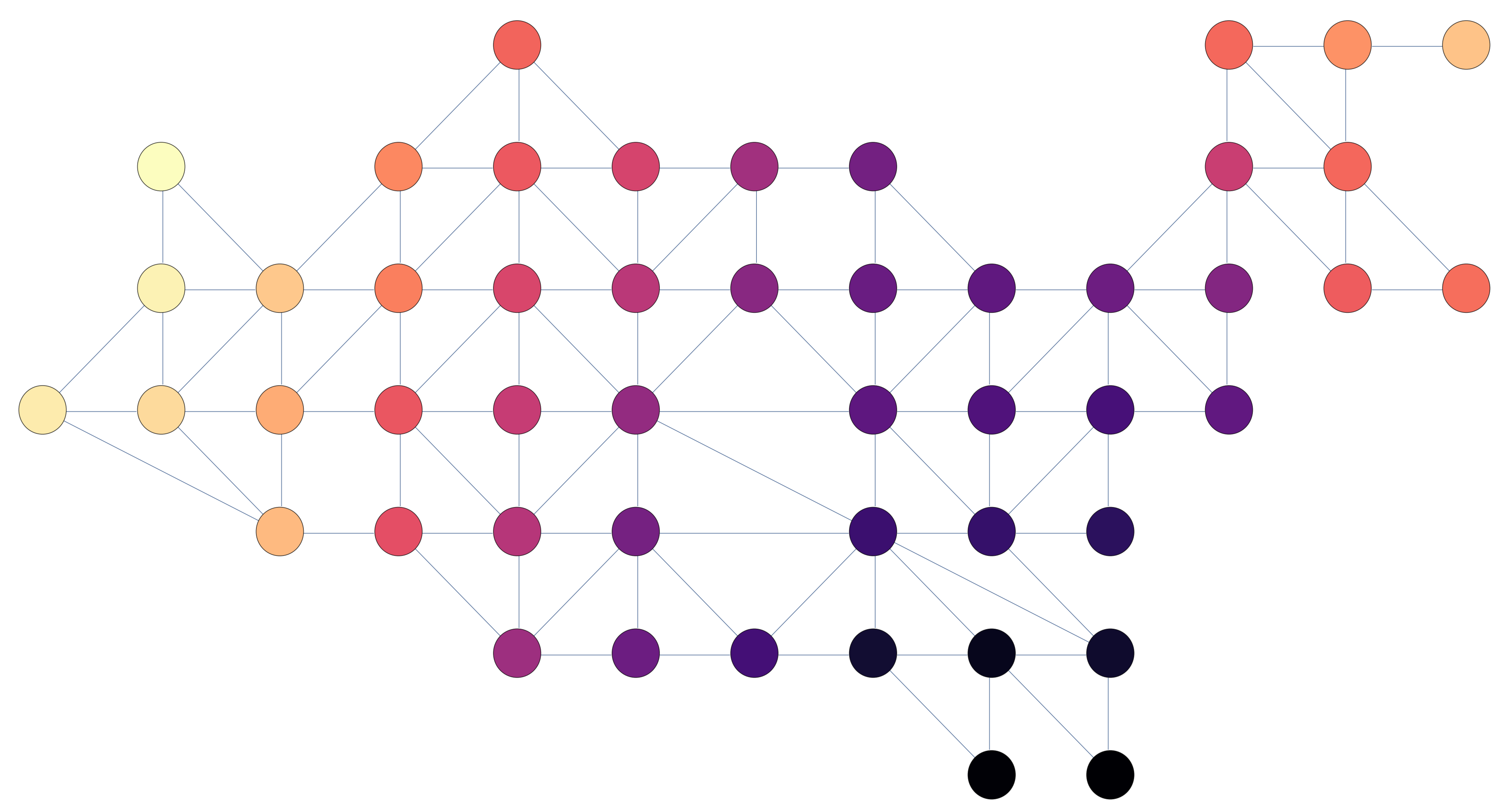


The 18th eigenvector by frequency

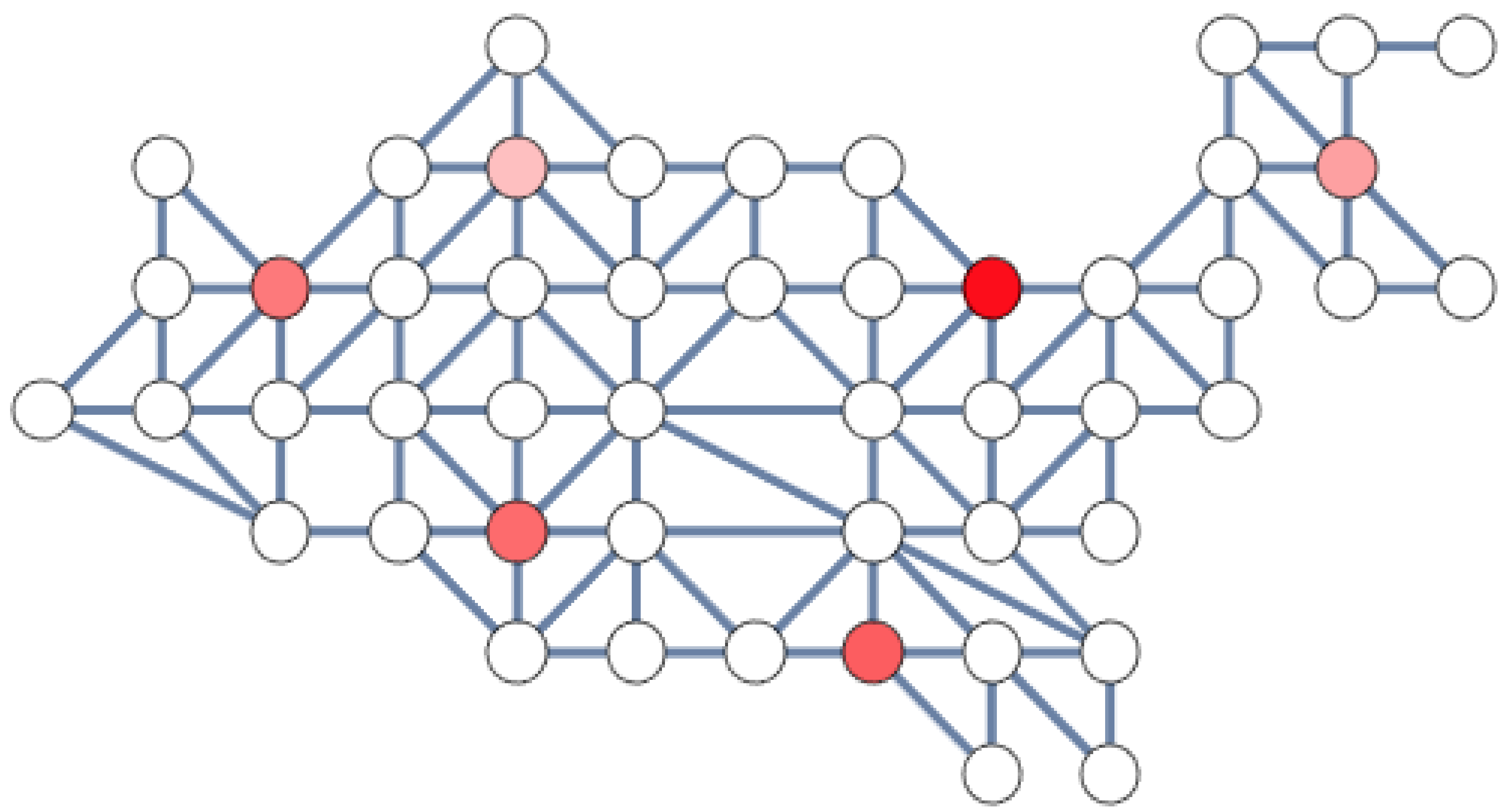
What Eigenspaces?



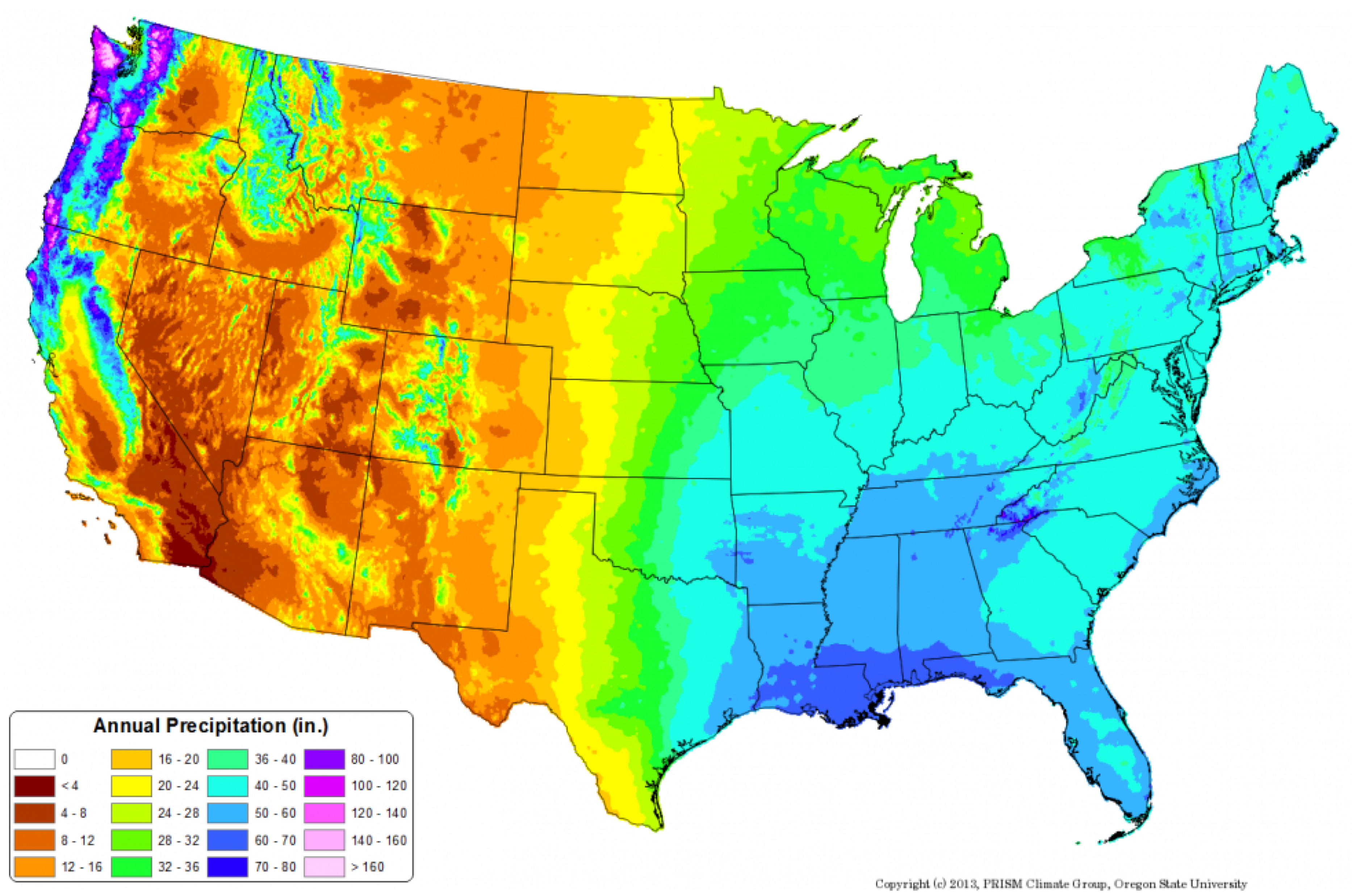
The first eigenvector by frequency



The second eigenvector by frequency



Averages the first 6 eigenvectors



Average annual precipitation, 1981-2010

Definitions

$G = ([n], E, w)$. Order the eigenspaces $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_m$

Definition: A k-graphical design is $S \subset [n]$ and weights $a_s \in \mathbb{R}$ such that S averages a basis for the first k eigenspaces with these weights.

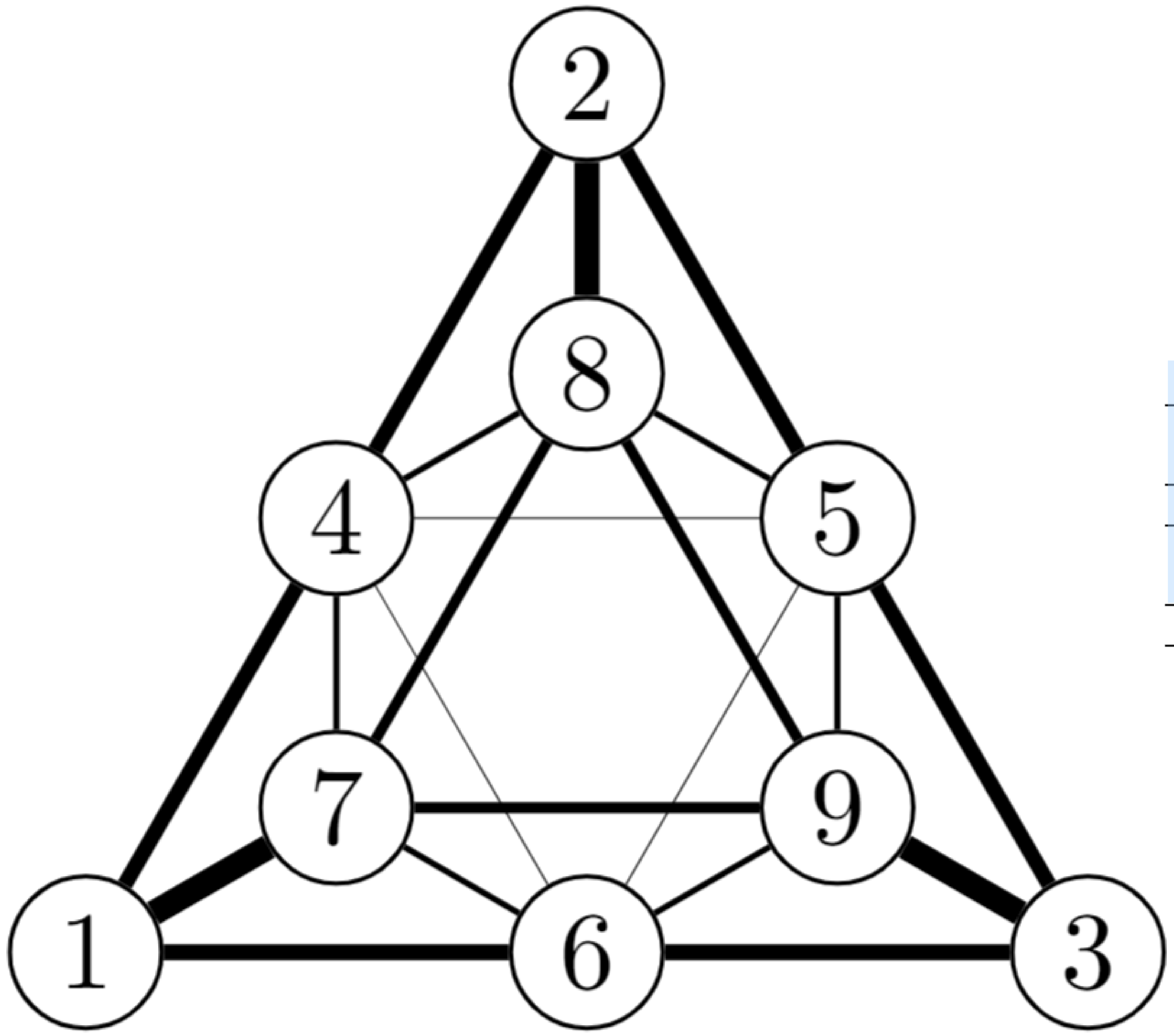
$w : E \to \mathbb{R}_{>0}$

Stefan Steinerberger

Definitions

$G = ([n], E, w)$. Order the eigenspaces $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_m$

Definition: A k-graphical design is $S \subset [n]$ and weights $a_s \in \mathbb{R}$ such that S averages a basis for the first k eigenspaces with these weights.

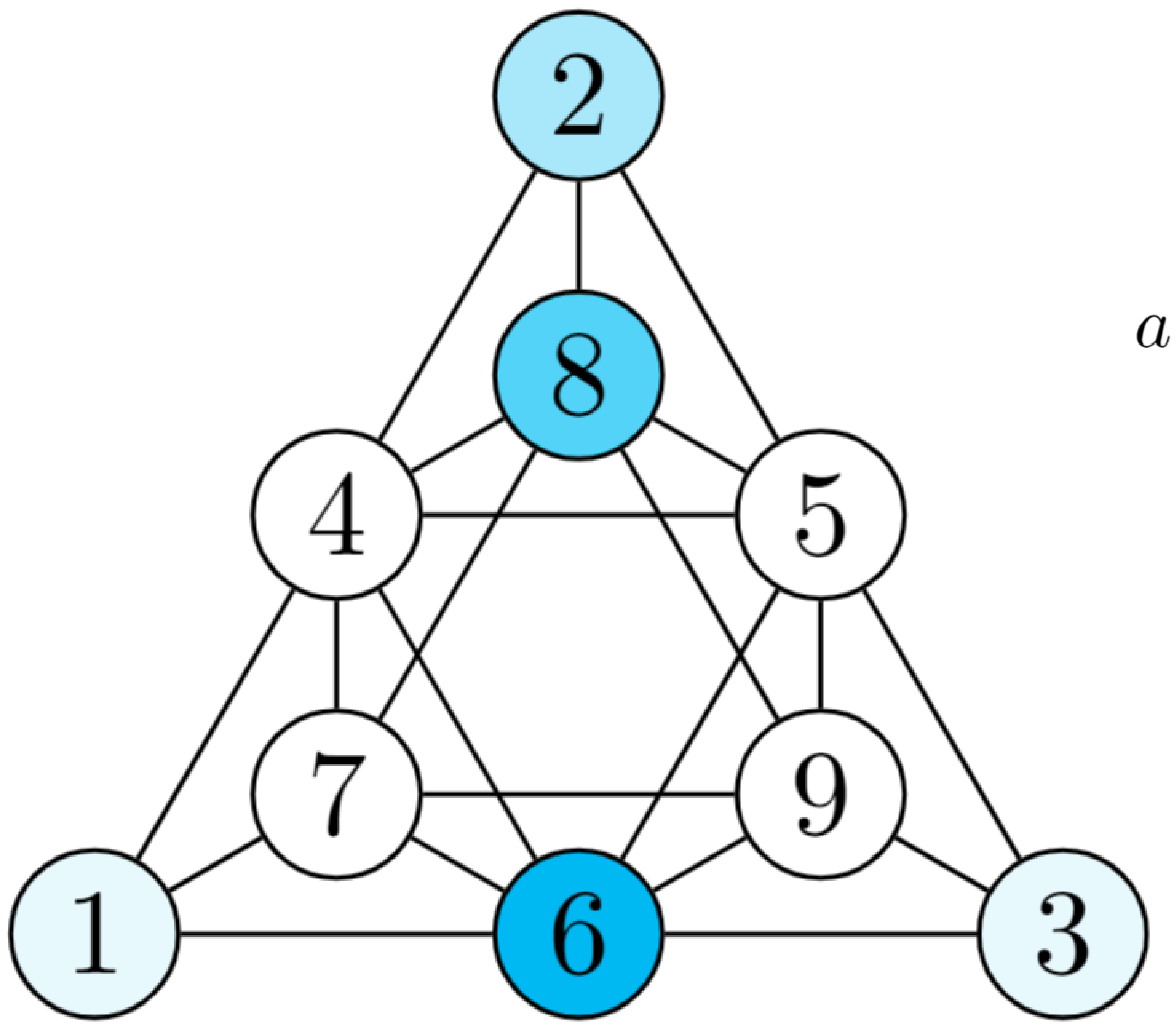


$\lambda_1 = 0$	1	1	1	1	1	1	1	1	1
$\lambda_2 = 1.069$	$-.6262$	$.5616$	$.0646$	$-.0264$	$.256$	$-.2291$	$-.3059$	$.2744$	$.0316$
	$-.2870$	$-.3988$	$.6858$	$-.2798$	$.1171$	$.1627$	$-.1402$	$-.1948$	$.3350$
$\lambda_3 = 1.861$	$.3193$	$.3193$	$.3193$	$.1406$	$.1406$	$.1406$	$-.4600$	$-.4600$	$-.4600$
$\lambda_4 = 2.661$	$.3248$	$-.4014$	$.0766$	$.0507$	$.2150$	$-.2658$	$-.4852$	$.5996$	$-.1144$
	$.2760$	$.1433$	$-.4193$	$-.2776$	$.1827$	$.0949$	$-.4123$	$-.2141$	$.6263$
$\lambda_5 = 2.672$	$.3468$	$.3458$	$.3468$	$-.4499$	$-.4499$	$-.4499$	$.1032$	$.1032$	$.1032$
$\lambda_6 = 3.003$	$.0724$	$-.0995$	$.0271$	$.1880$	$.5015$	$-.6894$	$.2707$	$-.3721$	$.1015$
	$.0731$	$.0261$	$-.0992$	$-.6876$	$.5066$	$.1810$	$.2734$	$.0977$	$-.3711$

Definitions

$G = ([n], E, w)$. Order the eigenspaces $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_m$

Definition: A k-graphical design is $S \subset [n]$ and weights $a_s \in \mathbb{R}$ such that S averages a basis for the first k eigenspaces with these weights.



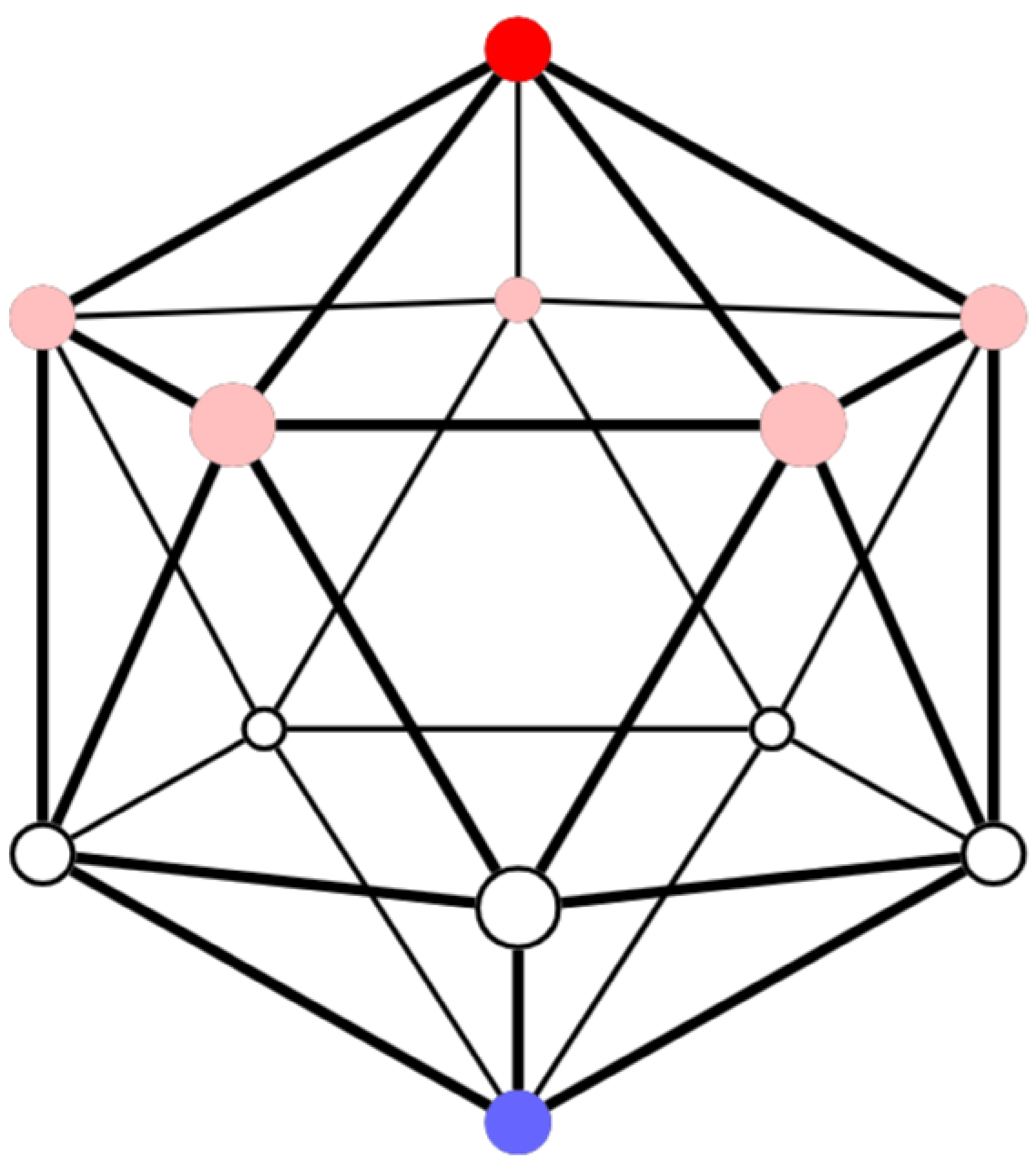
$$a_1 = a_3 = .0342,\ a_2 = .1111,\ a_6 = .5328,\ a_8 = .2876$$

$$S = \{1, 2, 3, 6, 8\}$$

What Kind Of Quadrature Weights?

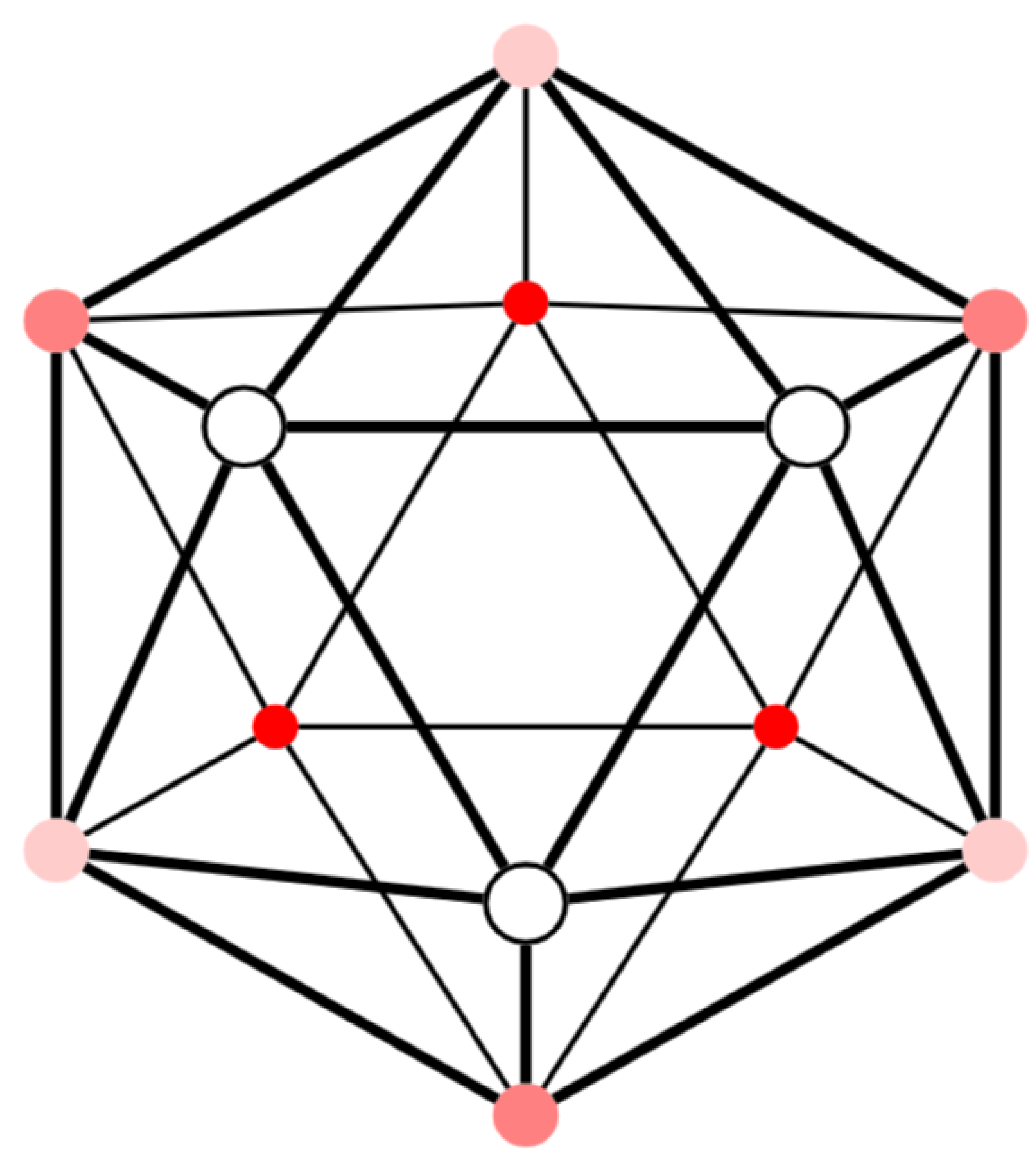
The unweighted icosahedral graph with

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$



An arbitrarily weighted 3-design

$$a_w \in \mathbb{R}$$



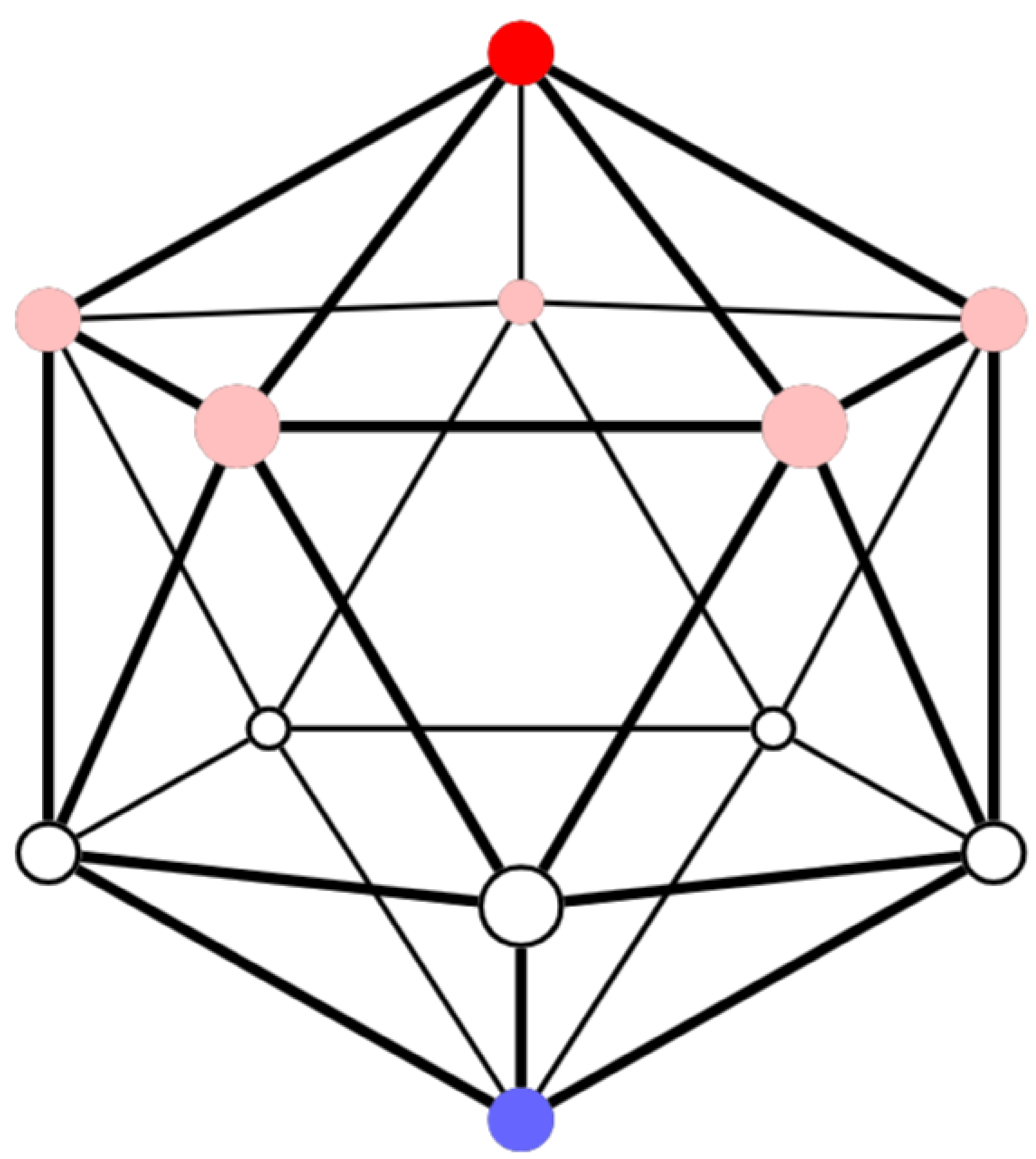
A positively weighted 3-design

$$a_w \geq 0$$

What Kind Of Quadrature Weights?

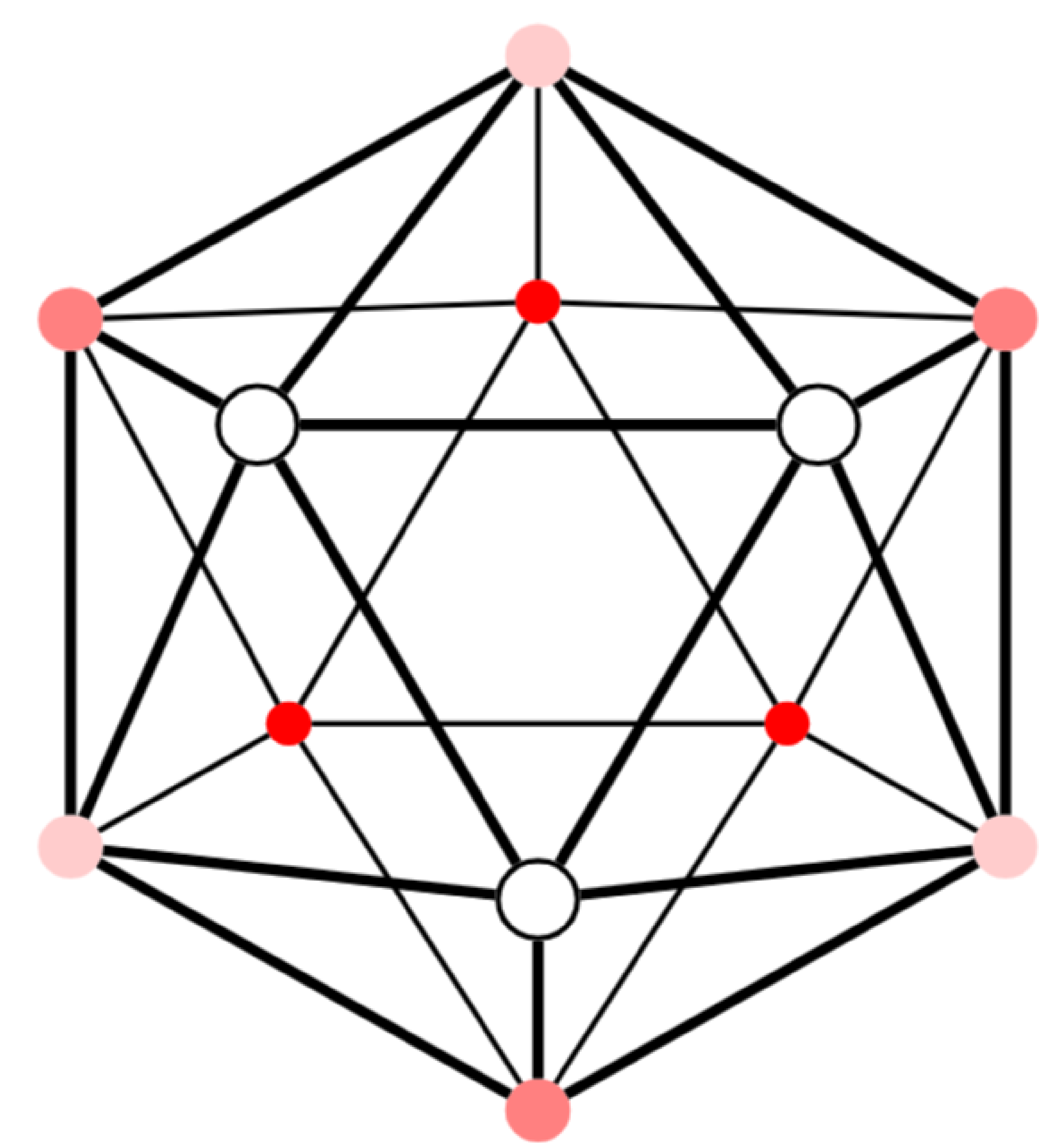
The unweighted icosahedral graph with

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$



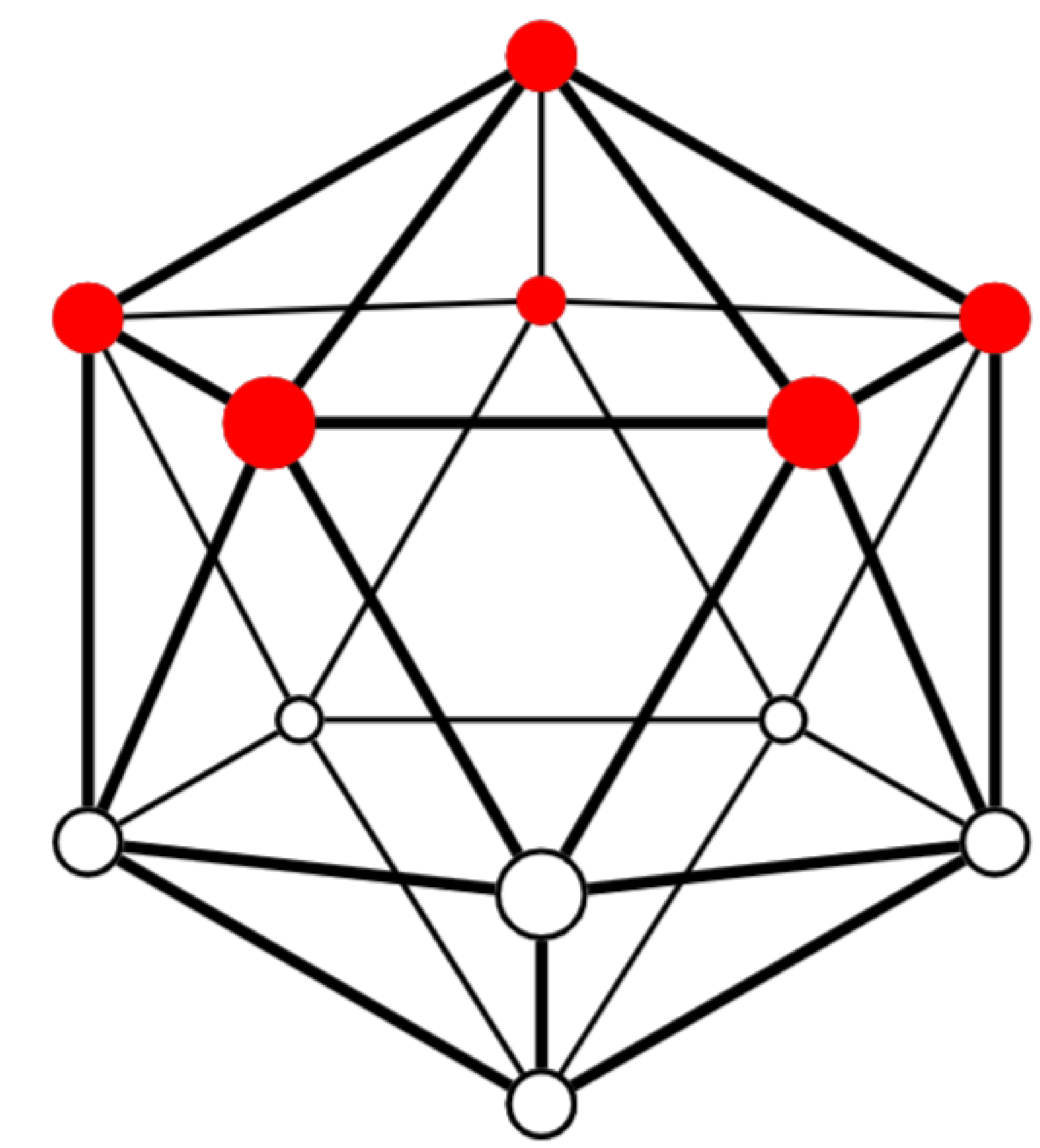
An arbitrarily weighted 3-design

$$a_w \in \mathbb{R}$$



A positively weighted 3-design

$$a_w \geq 0$$



A combinatorial 2-design

$$a_w \in \{0, 1\}$$

Main Structure Theorem

Notation:

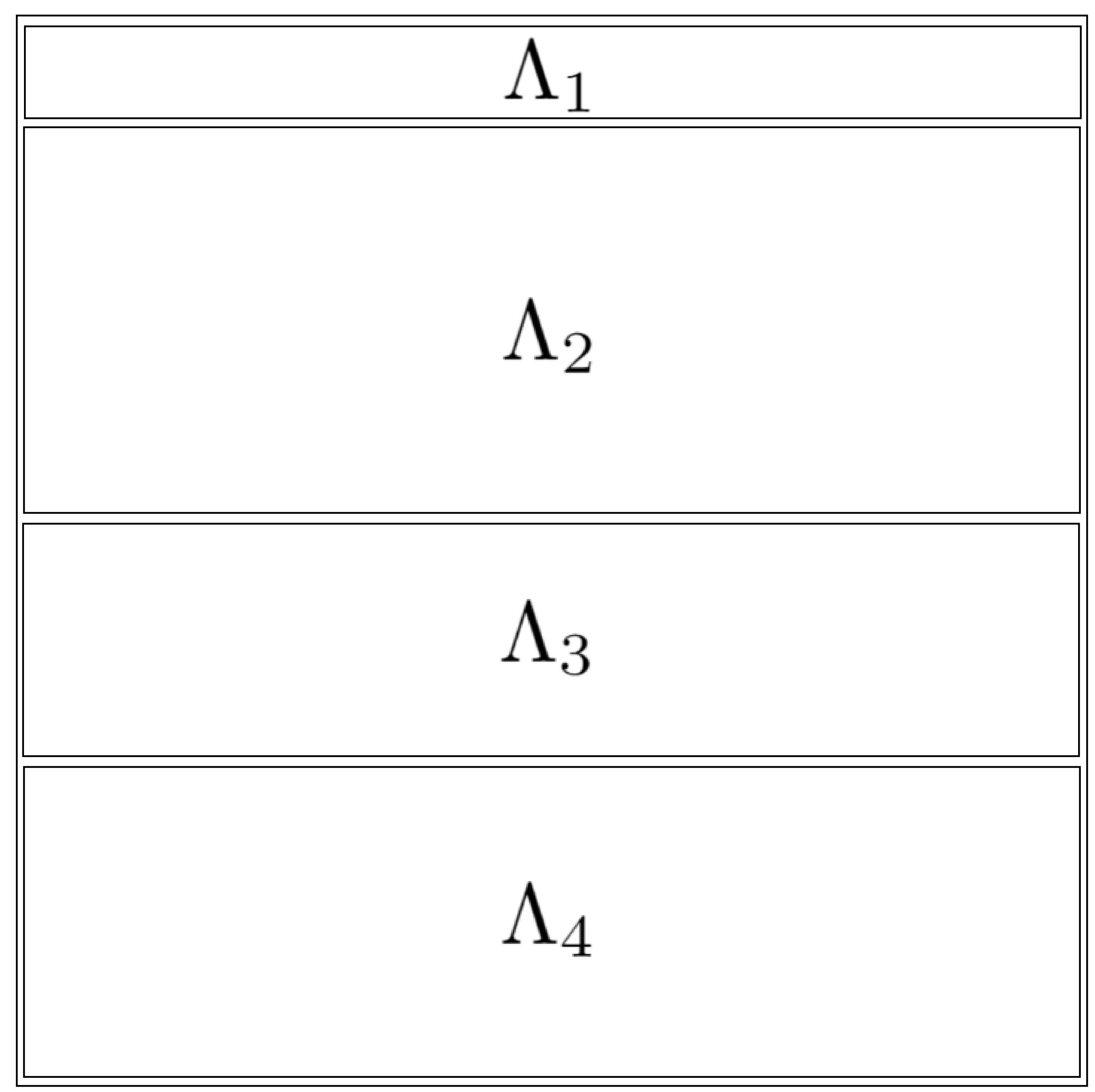
- matrix U, rows are the eigenvectors
- $\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$
- $\overline{\mathbf{k}} = \{\lambda_1, \lambda_{k+1}, \ldots, \lambda_m\}$.
- $U_{\boldsymbol{k}}$, $U_{\overline{\mathbf{k}}}$ = submatrices of corresponding eigenvectors
- $\mathcal{U}_{\boldsymbol{k}}$, $\mathcal{U}_{\overline{\mathbf{k}}}$ = columns of corresponding submatrices
- $P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})$ is an *eigenpolytope* (Chris Godsil)

Main Structure Theorem

Notation:

- matrix U, rows are the eigenvectors
- $\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$
- $\overline{\mathbf{k}} = \{\lambda_1, \lambda_{k+1}, \ldots, \lambda_m\}$.
- $U_{\boldsymbol{k}}$, $U_{\overline{\mathbf{k}}}$ = submatrices of corresponding eigenvectors
- $\mathcal{U}_{\boldsymbol{k}}$, $\mathcal{U}_{\overline{\mathbf{k}}}$ = columns of corresponding submatrices
- $P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})$ is an *eigenpolytope* (Chris Godsil)

U =

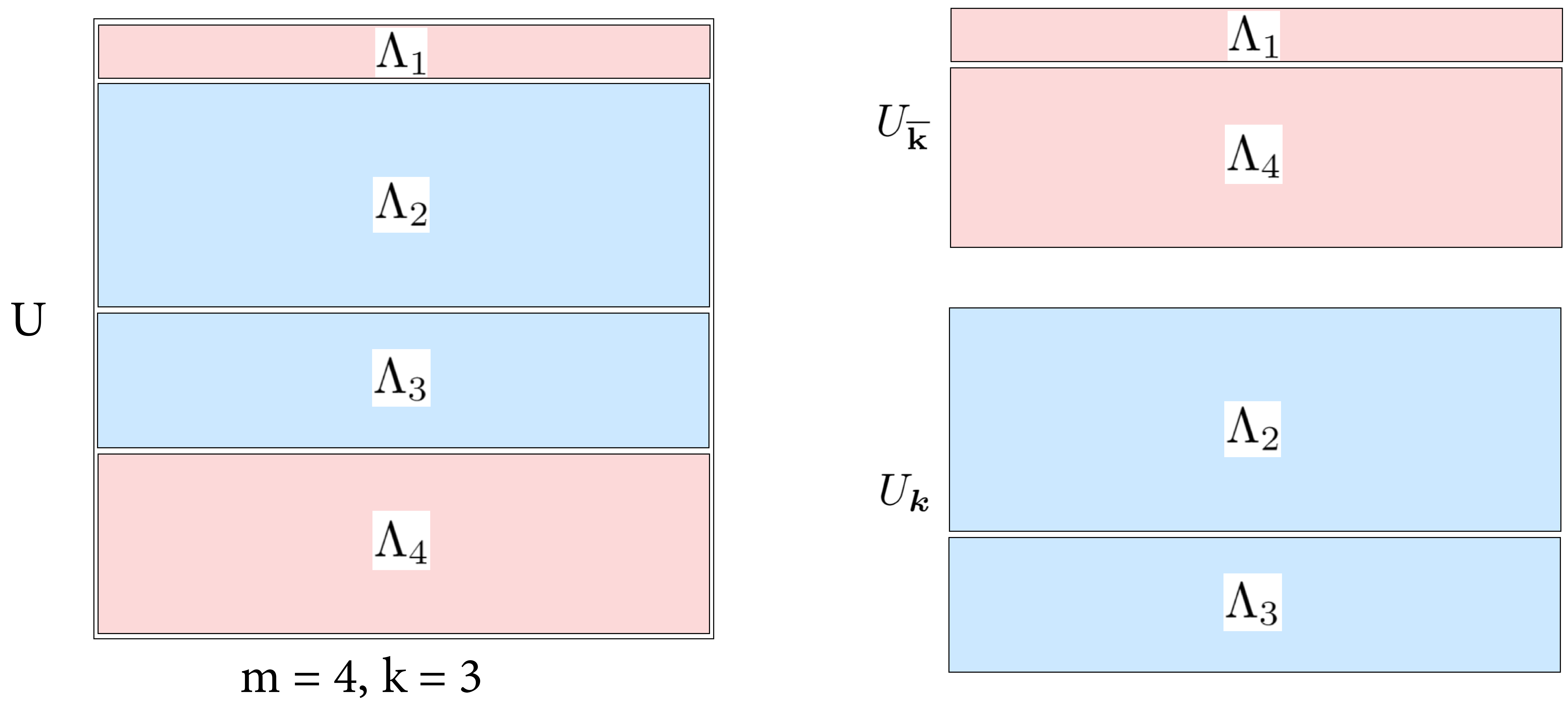


m = 4

Main Structure Theorem

Notation:

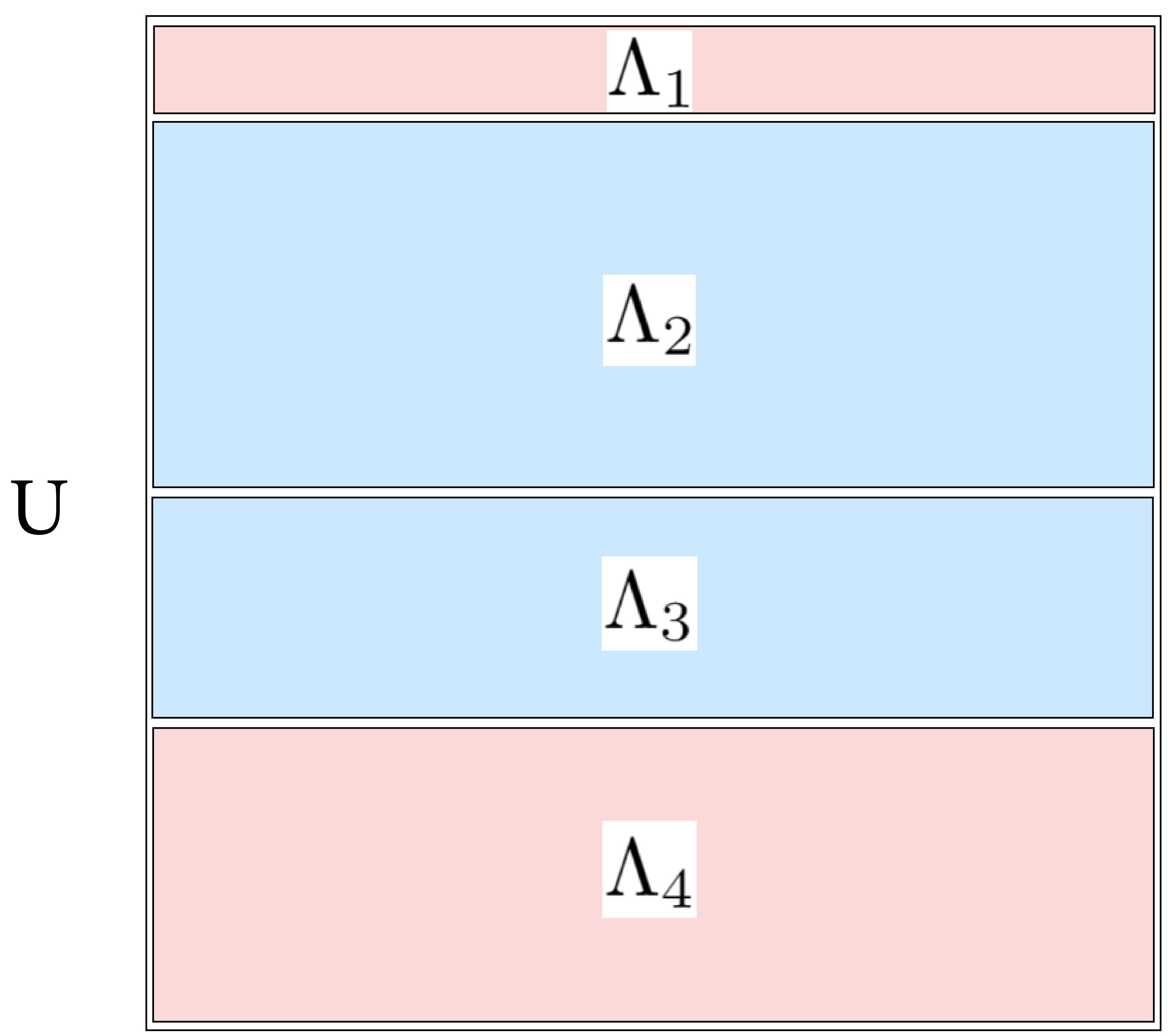
- matrix U, rows are the eigenvectors
- $\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$
- $\overline{\mathbf{k}} = \{\lambda_1, \lambda_{k+1}, \ldots, \lambda_m\}$.
- $U_{\boldsymbol{k}}$, $U_{\overline{\mathbf{k}}}$ = submatrices of corresponding eigenvectors
- $\mathcal{U}_{\boldsymbol{k}}$, $\mathcal{U}_{\overline{\mathbf{k}}}$ = columns of corresponding submatrices
- $P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})$ is an *eigenpolytope* (Chris Godsil)



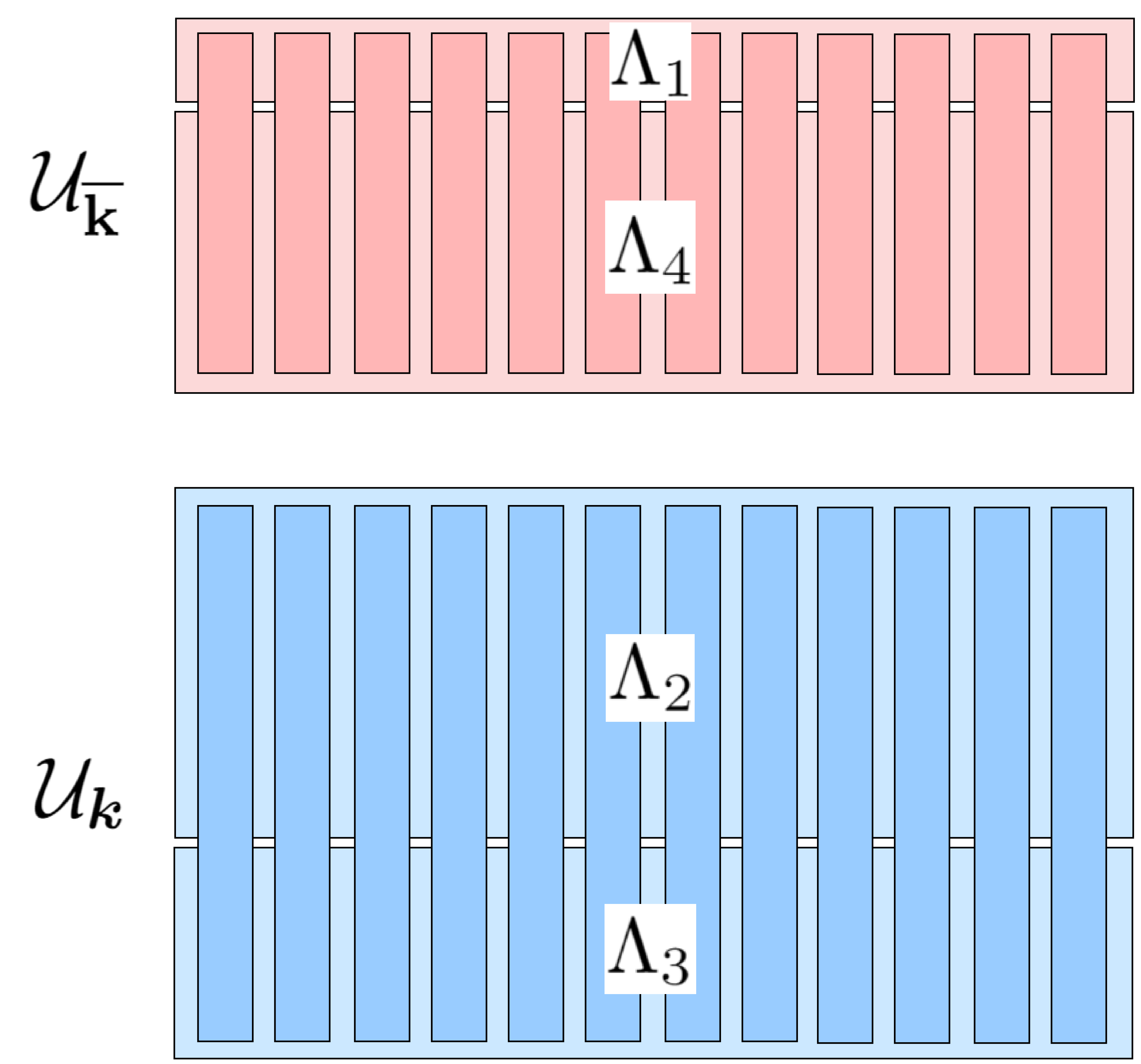
Main Structure Theorem

Notation:

- matrix U, rows are the eigenvectors
- $\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$
- $\overline{\mathbf{k}} = \{\lambda_1, \lambda_{k+1}, \ldots, \lambda_m\}$.
- $U_{\boldsymbol{k}}$, $U_{\overline{\mathbf{k}}}$ = submatrices of corresponding eigenvectors
- $\mathcal{U}_{\boldsymbol{k}}$, $\mathcal{U}_{\overline{\mathbf{k}}}$ = columns of corresponding submatrices
- $P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})$ is an *eigenpolytope* (Chris Godsil)



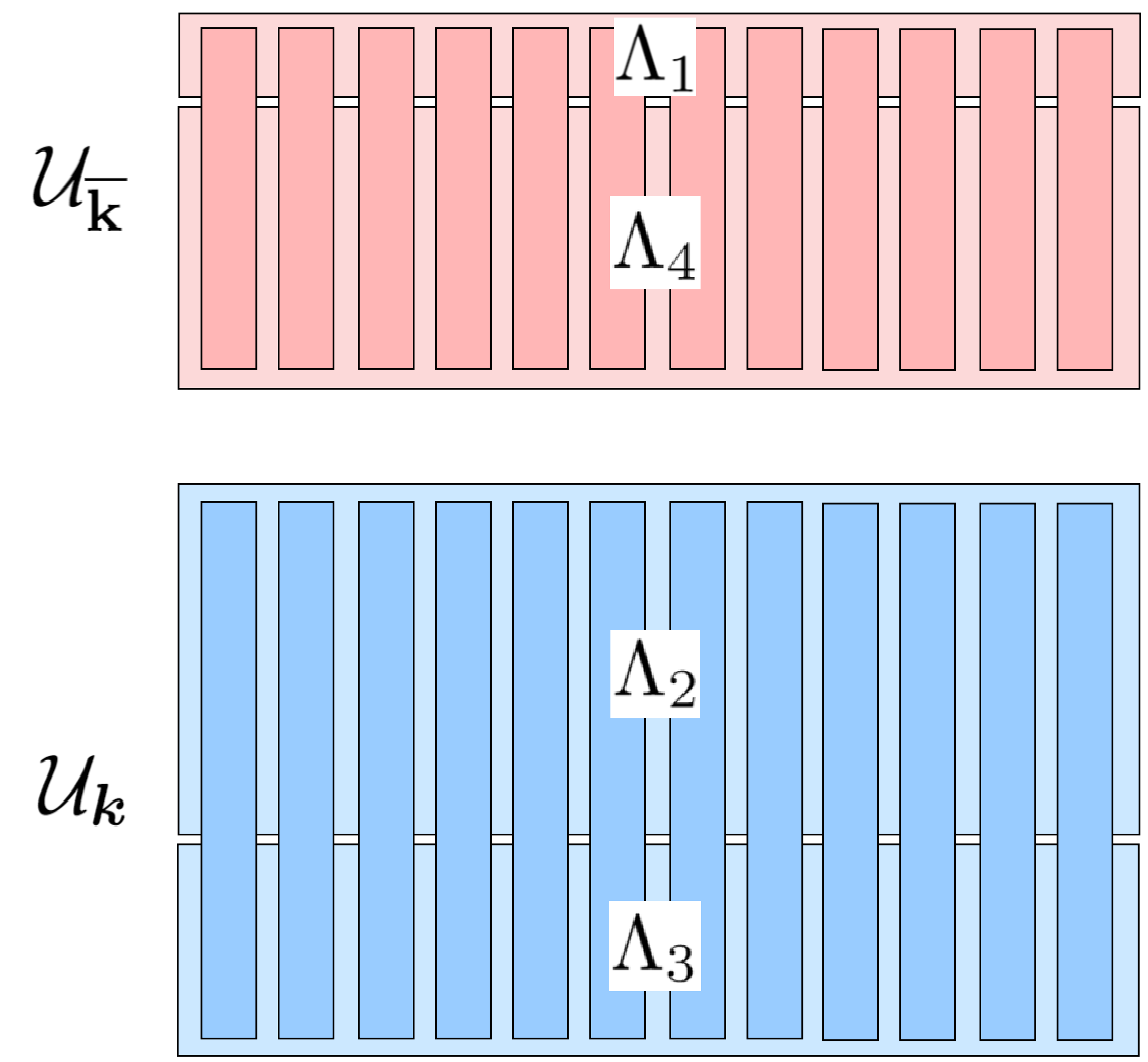
m = 4, k = 3



Main Structure Theorem

Notation:

- matrix U, rows are the eigenvectors
- $\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$
- $\overline{\mathbf{k}} = \{\lambda_1, \lambda_{k+1}, \ldots, \lambda_m\}.$
- $U_{\boldsymbol{k}}$, $U_{\overline{\mathbf{k}}}$ = submatrices of corresponding eigenvectors
- $\mathcal{U}_{\boldsymbol{k}}$, $\mathcal{U}_{\overline{\mathbf{k}}}$ = columns of corresponding submatrices
- $P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})$ is an *eigenpolytope* (Chris Godsil)



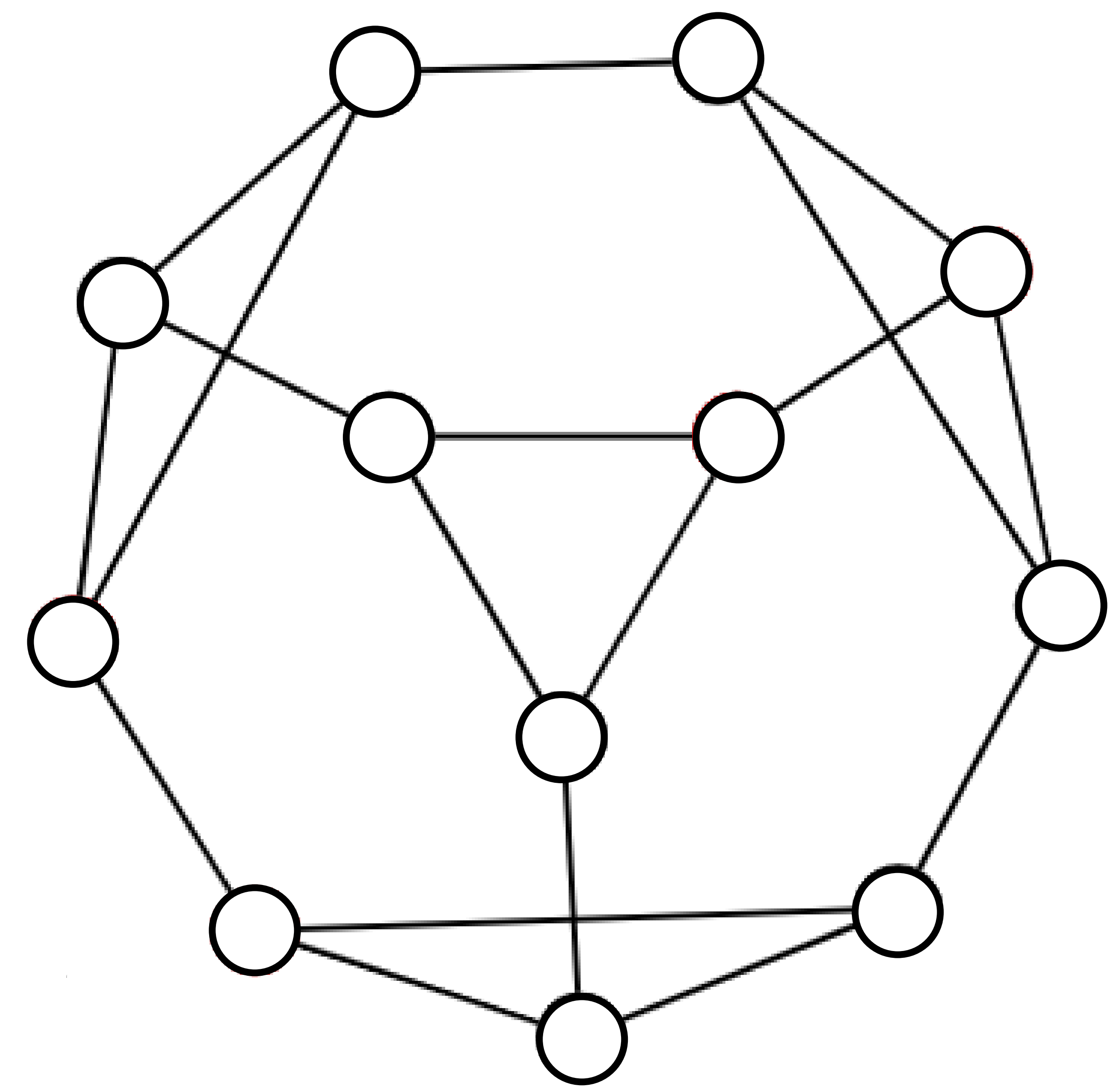
Theorem (B., Thomas, Shiroma):

$$\left\{\begin{array}{c}\text{Minimal positively}\\ \text{weighted k-designs}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Facets of}\\ P_{\overline{\boldsymbol{k}}} = \mathrm{conv}(\mathcal{U}_{\overline{\boldsymbol{k}}})\end{array}\right\}$$

$S \subset [n]$ is a k-design $\longleftrightarrow$ $V \setminus S$ is a facet

The Truncated Tetrahedral Graph

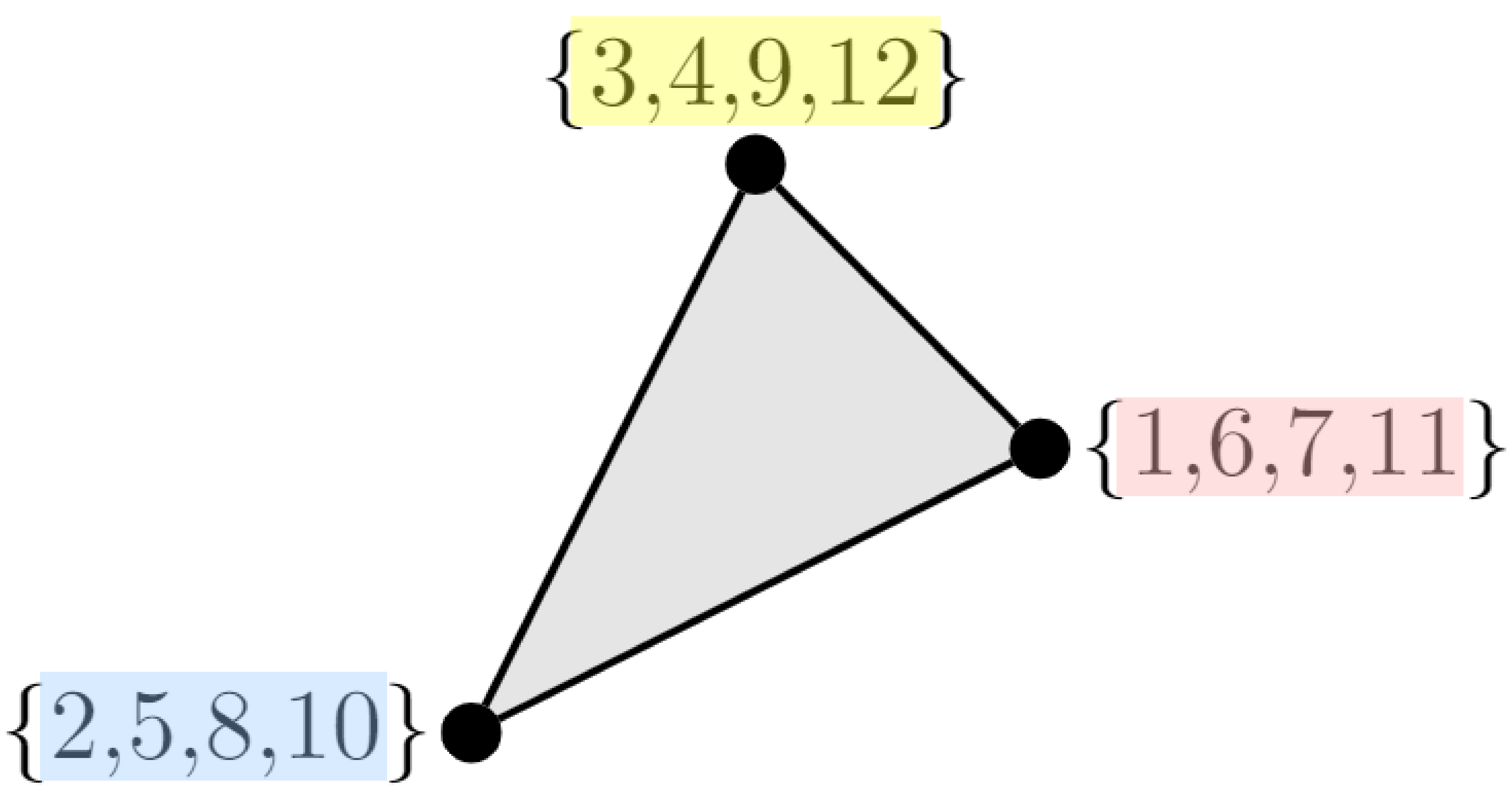
find a minimal 4-graphical design?



$$U = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
-1 & -1.5 & -0.5 & 1.5 & 2 & 1.5 & -0.5 & -1.5 & -1 & 1 & 0 & 0 \\
2 & 1.5 & 1.5 & -0.5 & -1 & -1.5 & -1.5 & -0.5 & -1 & 0 & 1 & 0 \\
-1 & -0.5 & -1.5 & -1.5 & -1 & -0.5 & 1.5 & 1.5 & 2 & 0 & 0 & 1 \\
\hline
0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 \\
\hline
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\
\hline
1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1
\end{array}\right]
\quad
\begin{array}{c}
\text{eigenvalue} \\
0 \\
\\
1 \\
\\
\\
5 \\
\\
\\
4 \\
\\
3 \\
\\
\end{array}$$

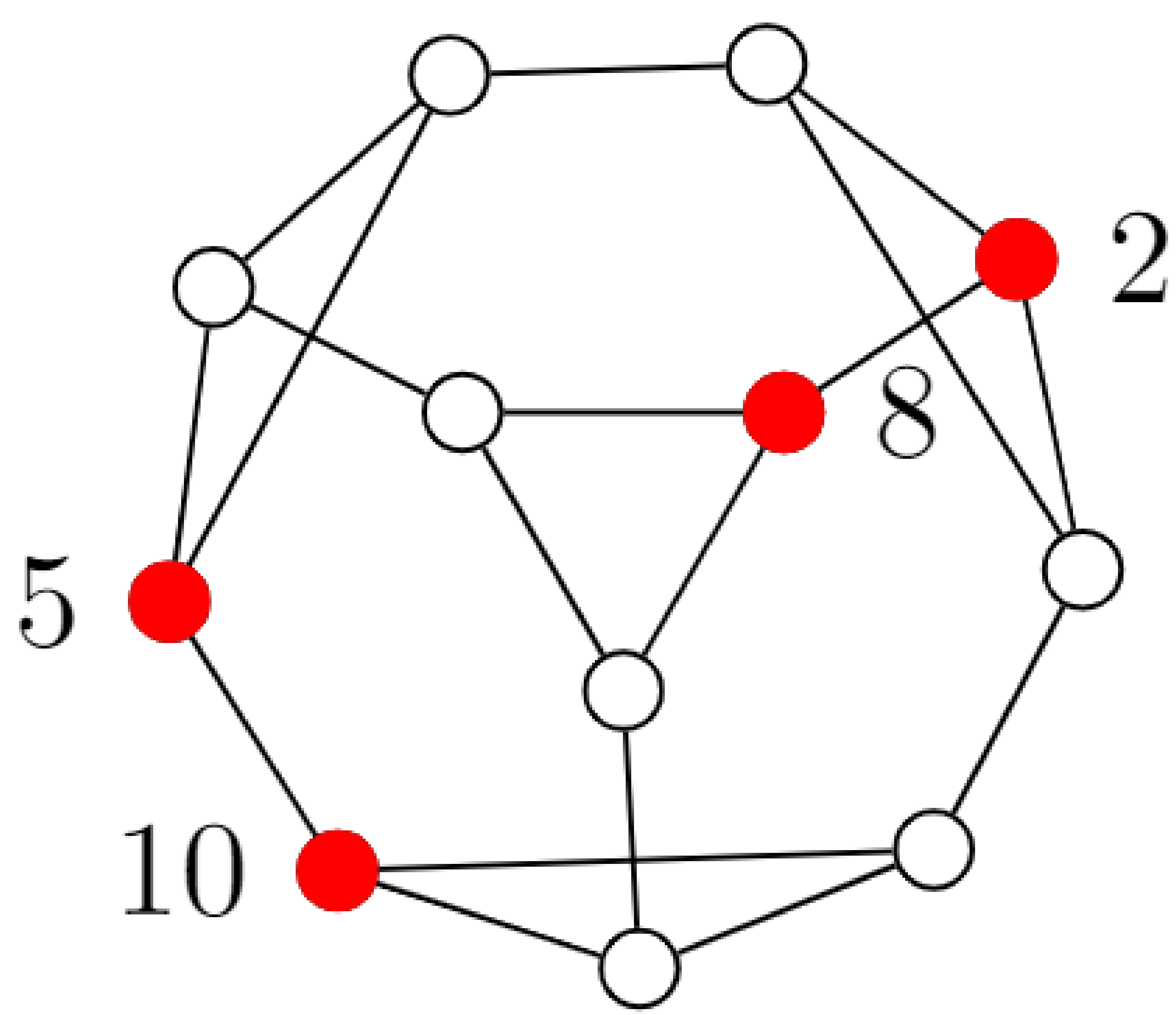
The Truncated Tetrahedral Graph

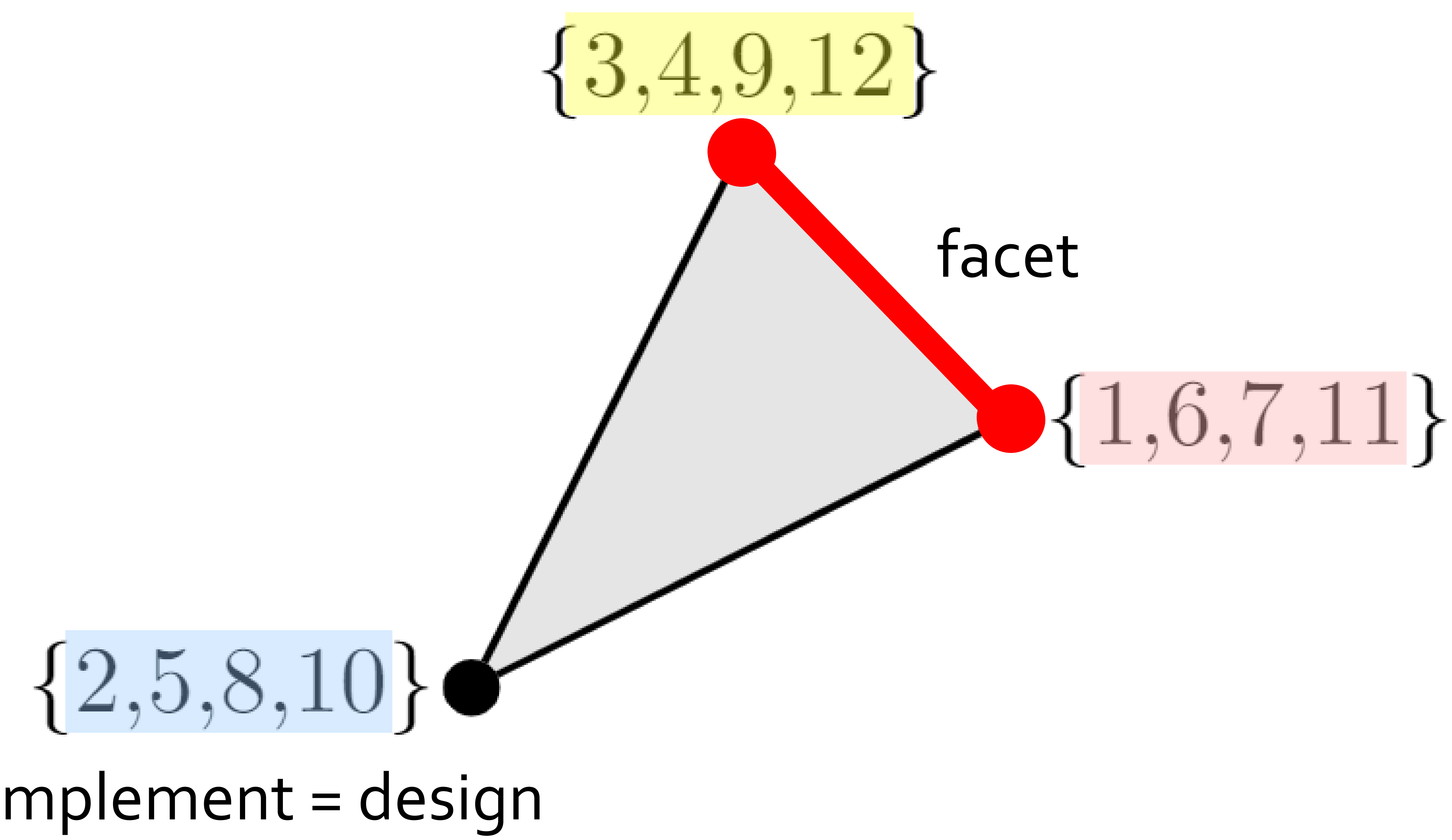
$$U_{\bar{\mathbf{4}}} = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right].$$



The Truncated Tetrahedral Graph

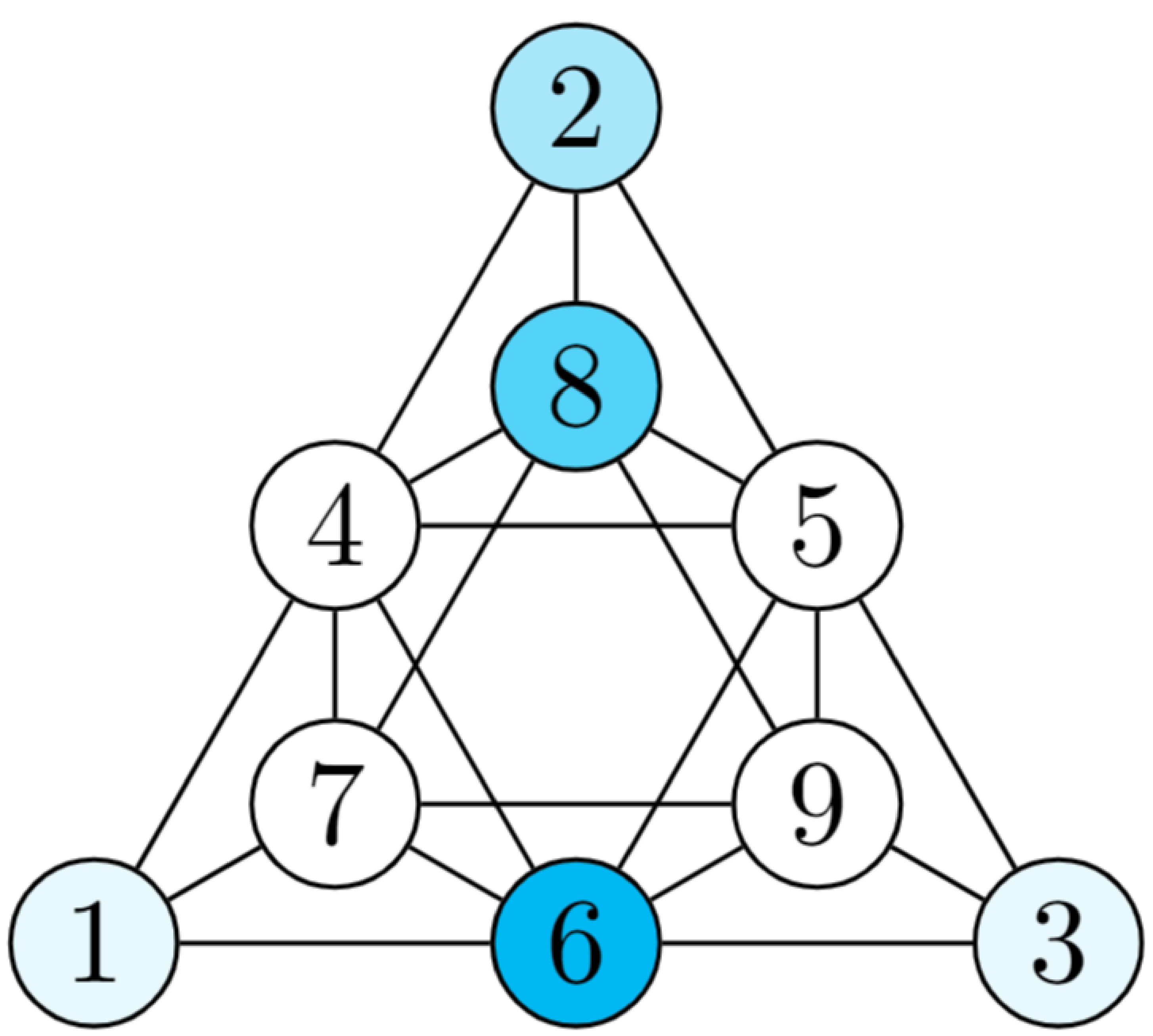
$$U_{\bar{\mathbf{4}}} = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right].$$

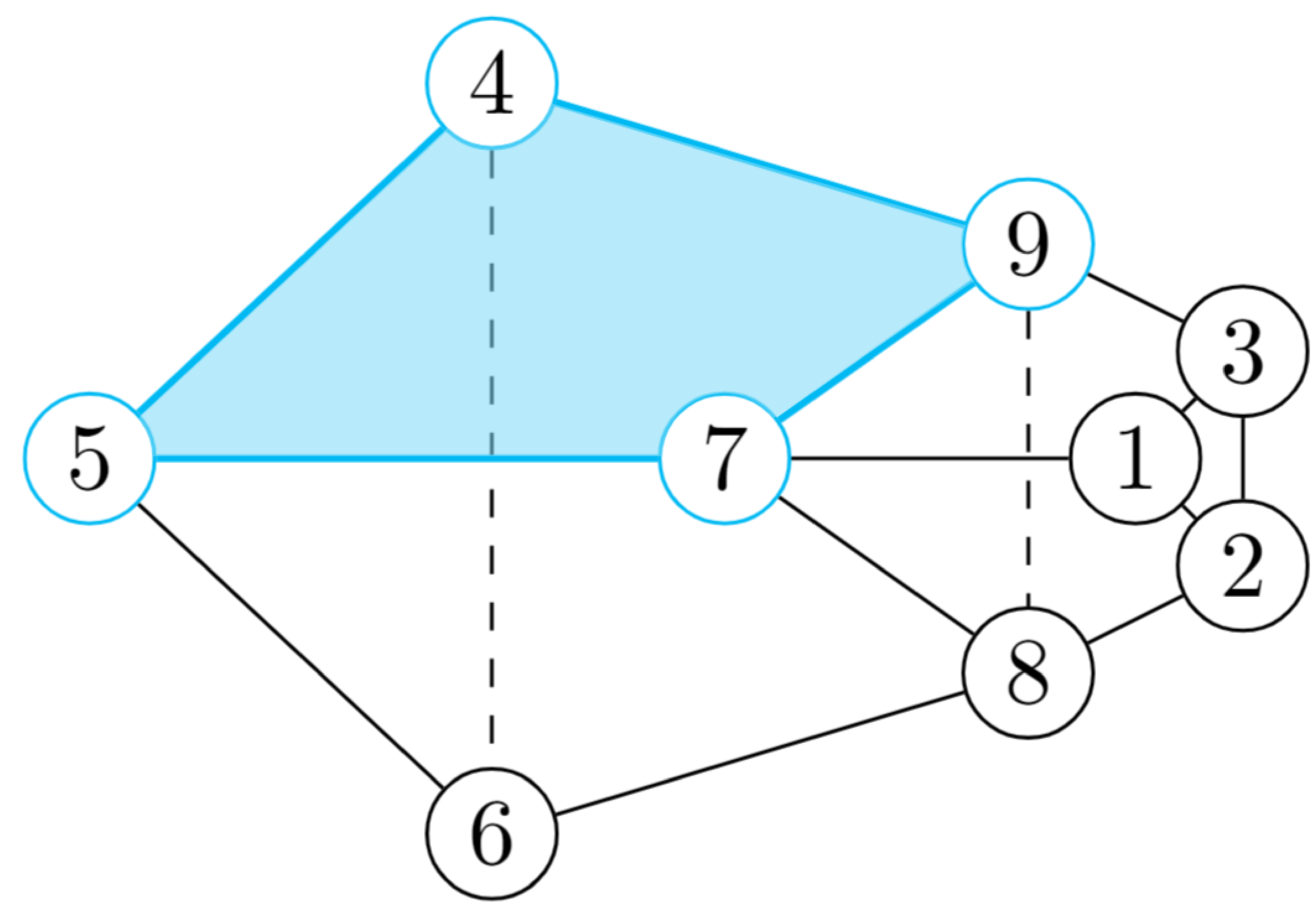


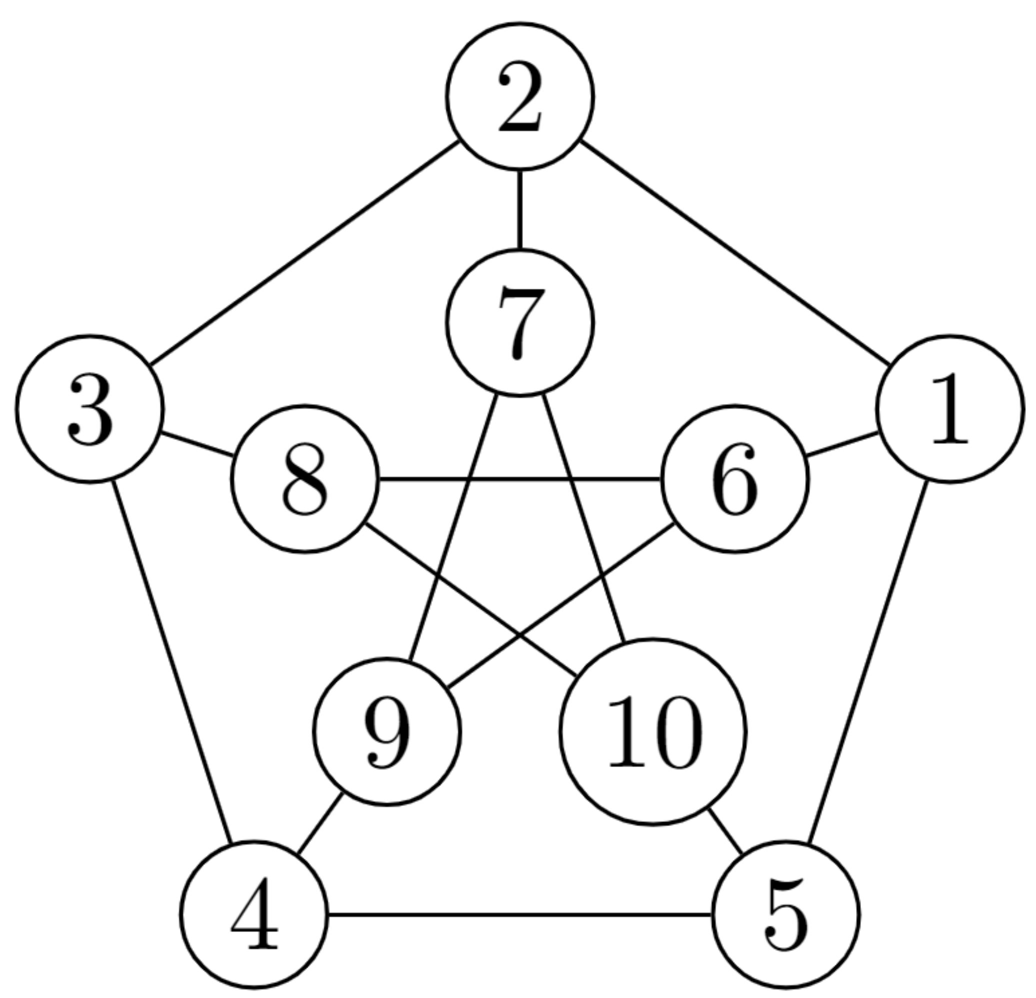


That Weighted Graph From Before

$\lambda_1 = 0$	1	1	1	1	1	1	1	1	1
$\lambda_2 = 1.069$	$-.6262$	$.5616$	$.0646$	$-.0264$	$.256$	$-.2291$	$-.3059$	$.2744$	$.0316$
	$-.2870$	$-.3988$	$.6858$	$-.2798$	$.1171$	$.1627$	$-.1402$	$-.1948$	$.3350$
$\lambda_3 = 1.861$	$.3193$	$.3193$	$.3193$	$.1406$	$.1406$	$.1406$	$-.4600$	$-.4600$	$-.4600$
$\lambda_4 = 2.661$	$.3248$	$-.4014$	$.0766$	$.0507$	$.2150$	$-.2658$	$-.4852$	$.5996$	$-.1144$
	$.2760$	$.1433$	$-.4193$	$-.2776$	$.1827$	$.0949$	$-.4123$	$-.2141$	$.6263$
$\lambda_5 = 2.672$	$.3468$	$.3458$	$.3468$	$-.4499$	$-.4499$	$-.4499$	$.1032$	$.1032$	$.1032$
$\lambda_6 = 3.003$	$.0724$	$-.0995$	$.0271$	$.1880$	$.5015$	$-.6894$	$.2707$	$-.3721$	$.1015$
	$.0731$	$.0261$	$-.0992$	$-.6876$	$.5066$	$.1810$	$.2734$	$.0977$	$-.3711$



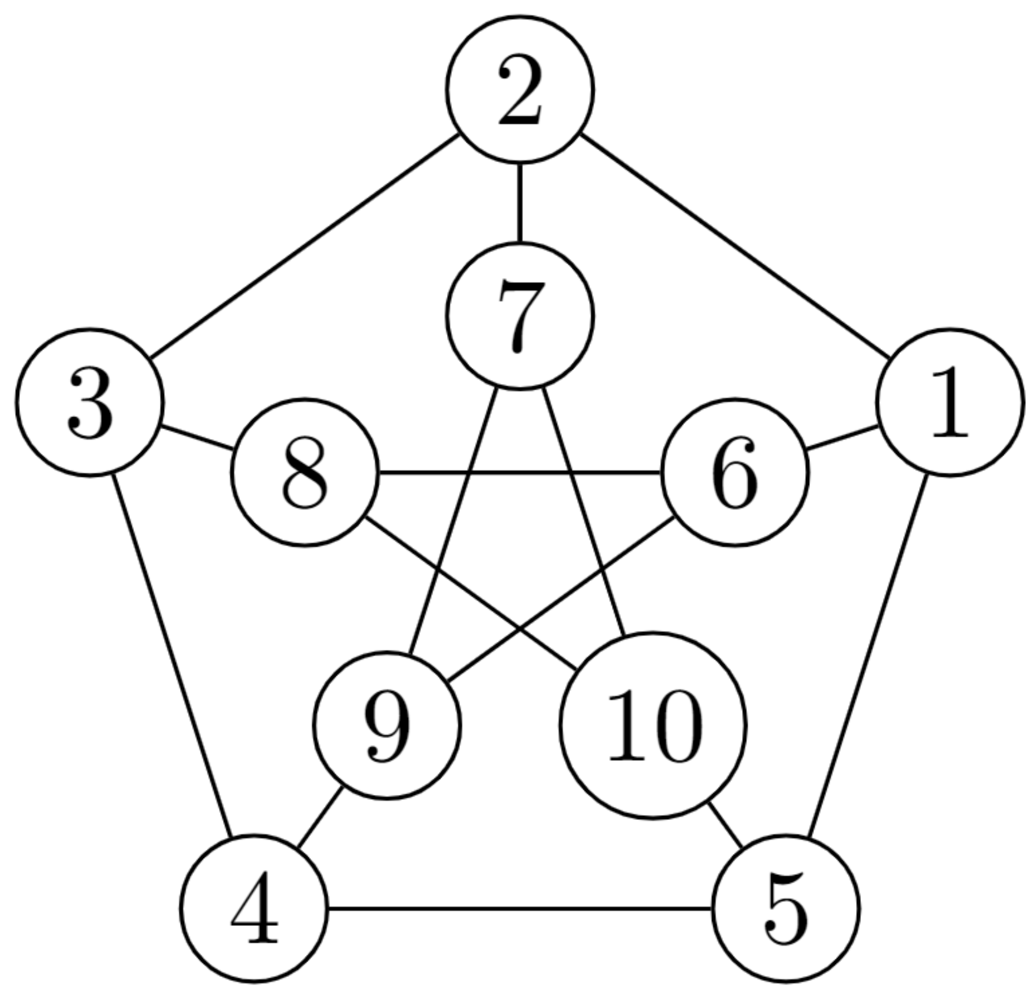




$$\lambda_1 = 0^{(1)}, \lambda_2 = 5^{(4)}, \text{ and } \lambda_3 = 2^{(5)}$$

$$U_{\overline{\mathbf{2}}} = \left[\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The Petersen Graph



$$\lambda_1 = 0^{(1)}, \lambda_2 = 5^{(4)}, \text{ and } \lambda_3 = 2^{(5)}$$

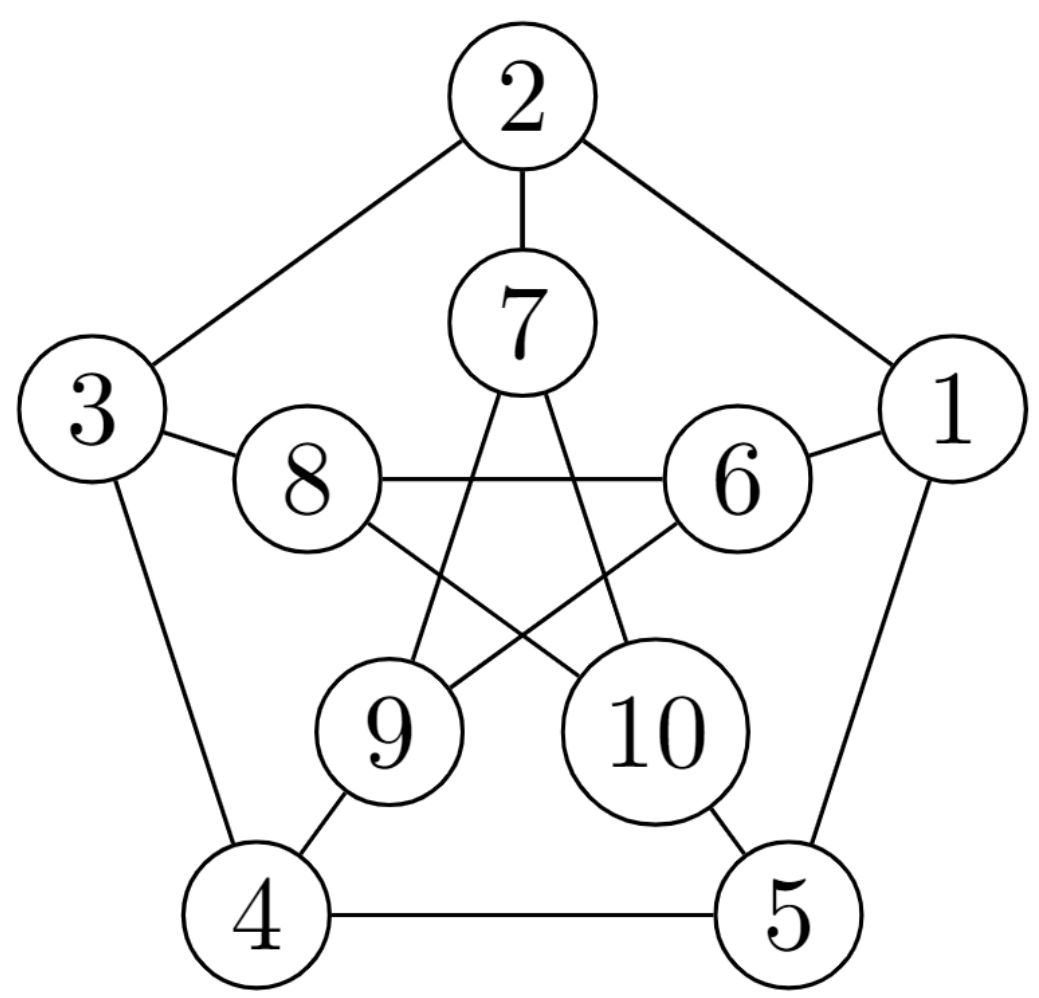
$$U_{\overline{\mathbf{2}}} = \left[\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The eigenpolytope is 5-dimensional and has face vector

$$(10, 45, 90, 75, 22)$$

22 facets come in two types: 12 are simplices and 10 have 6 vertices

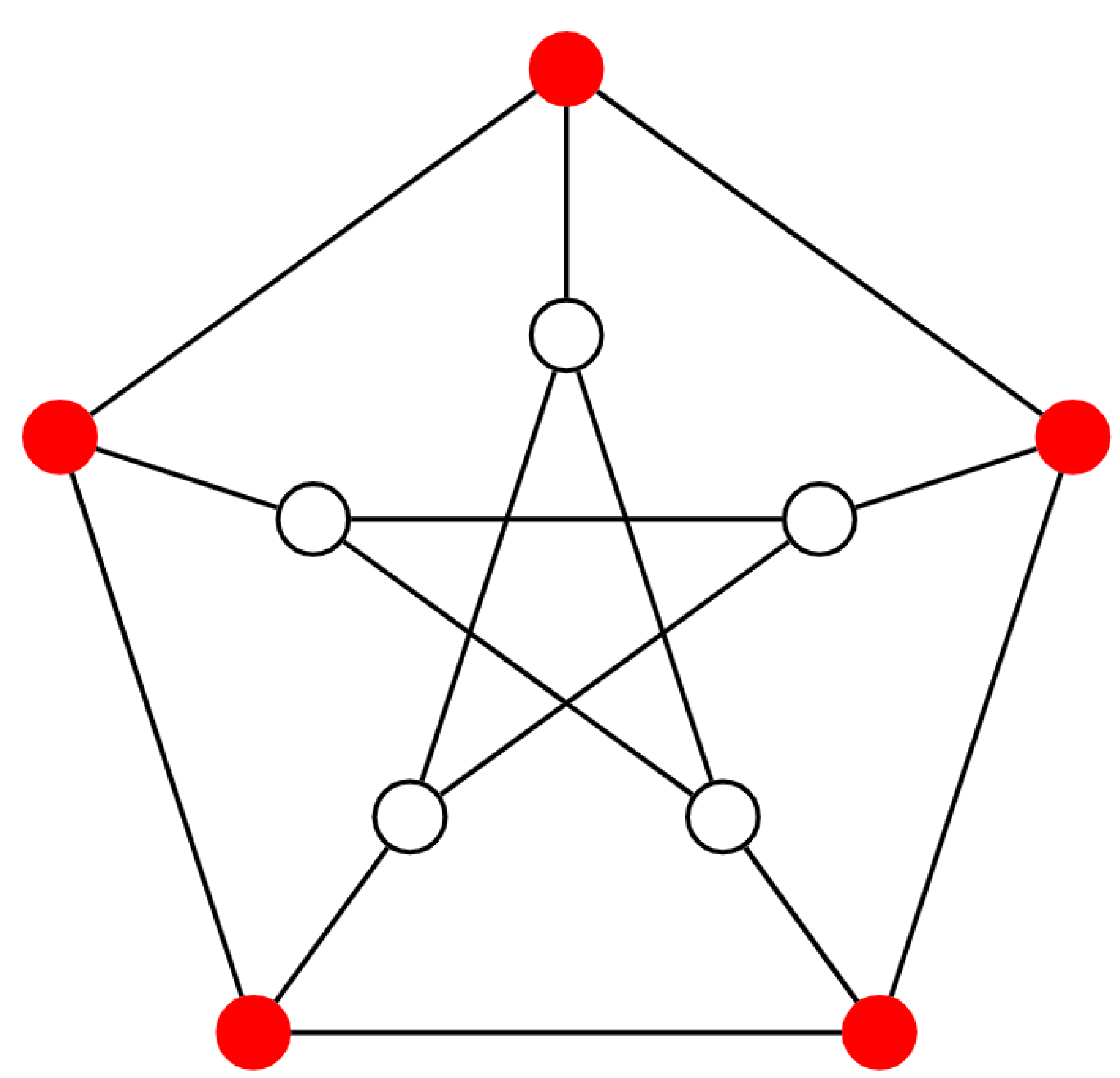
The Petersen Graph

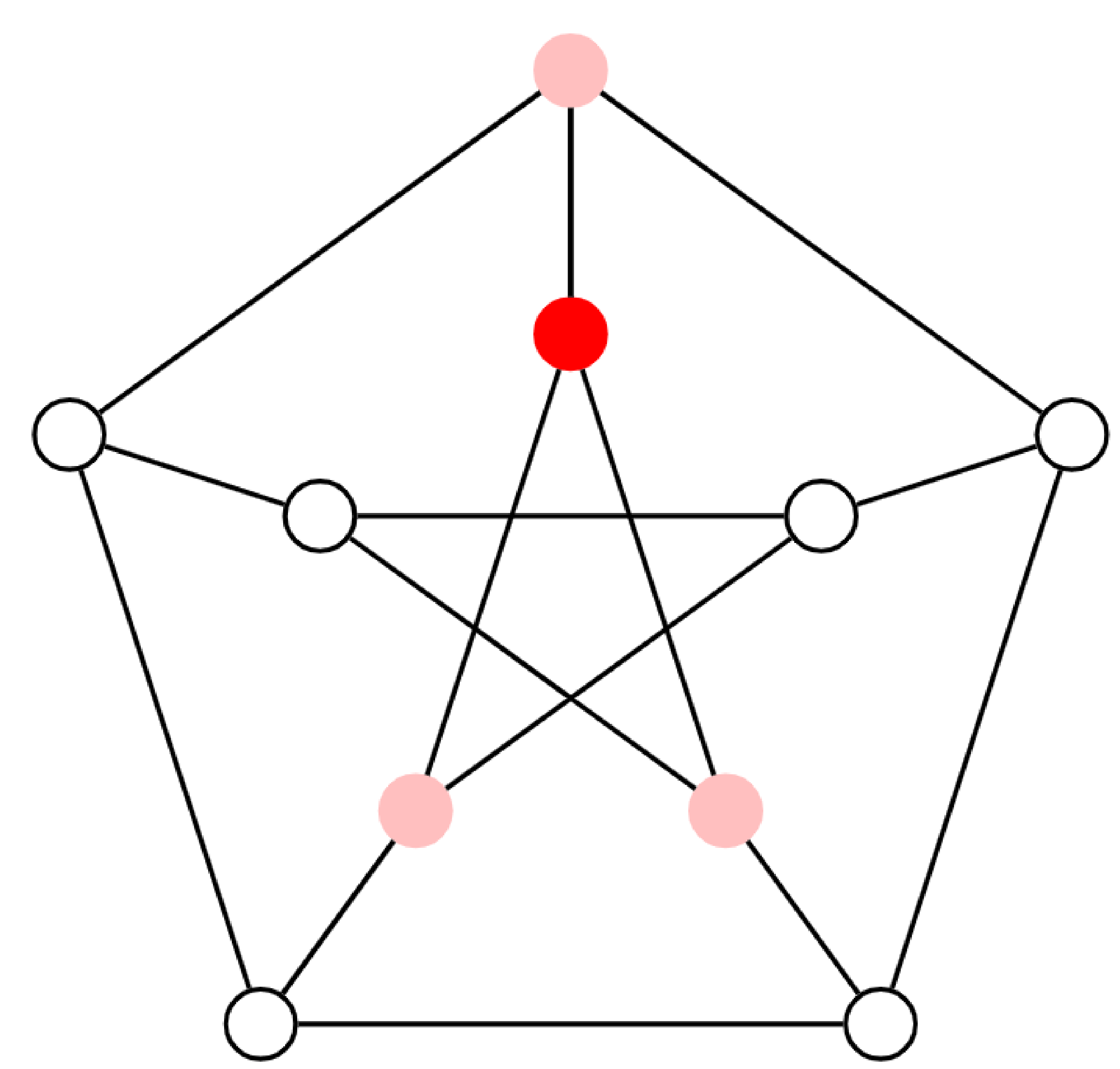


$$\lambda_1 = 0^{(1)}, \lambda_2 = 5^{(4)}, \text{ and } \lambda_3 = 2^{(5)}$$

$$U_{\overline{\mathbf{2}}} = \left[\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

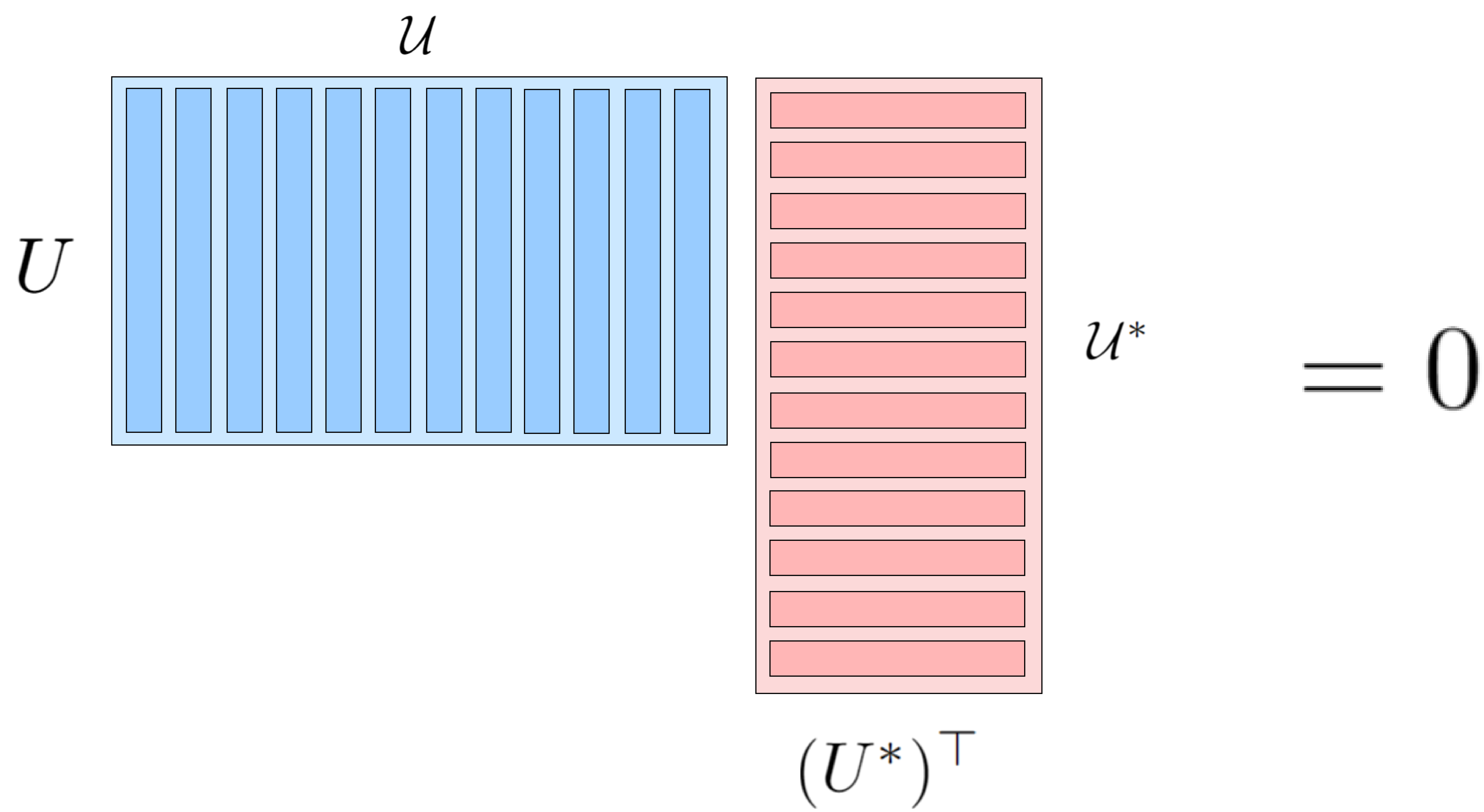
22 facets come in two types: 12 are simplices and 10 have 6 vertices





Why?

Oriented matroid duality for vector configurations:

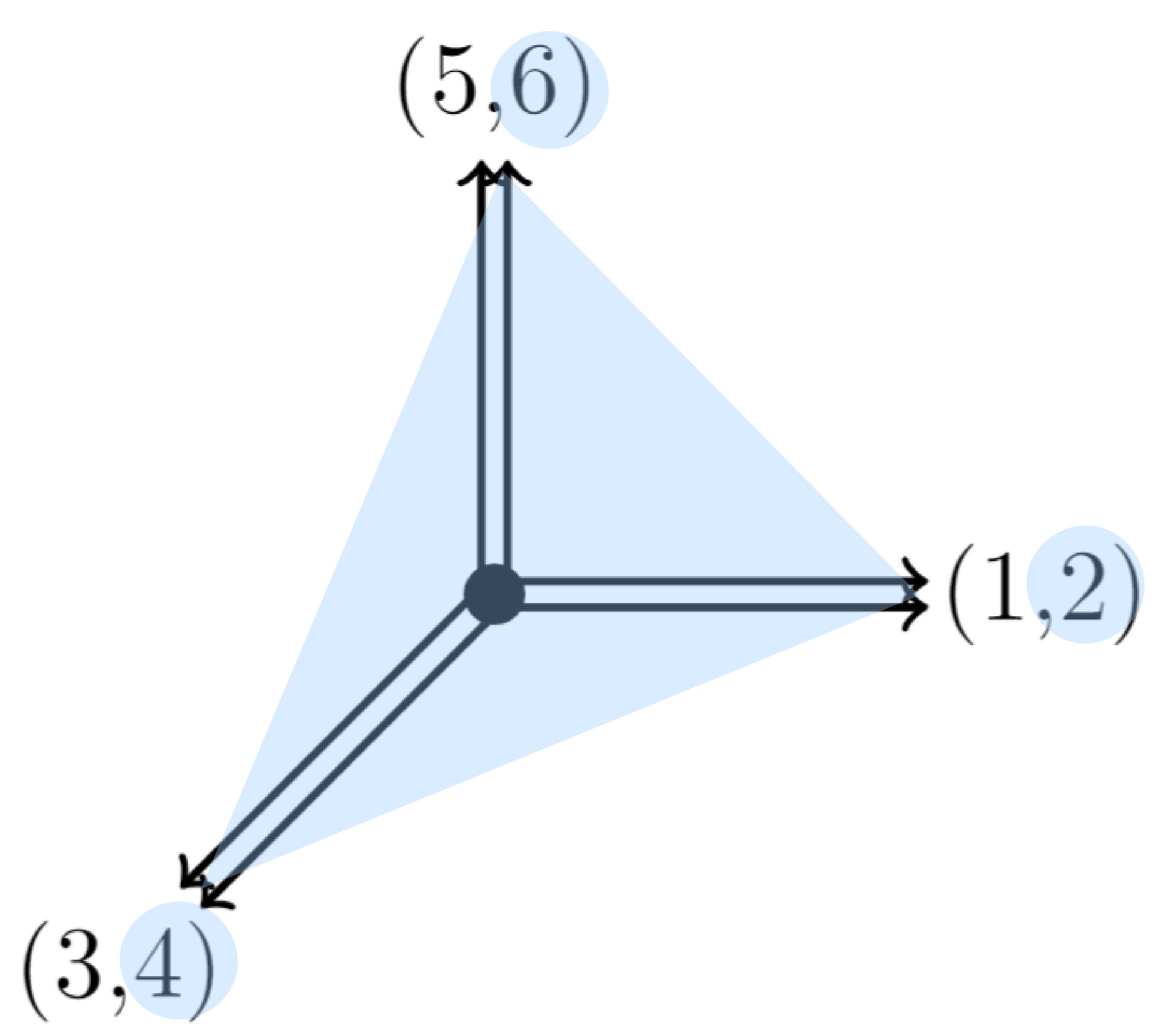


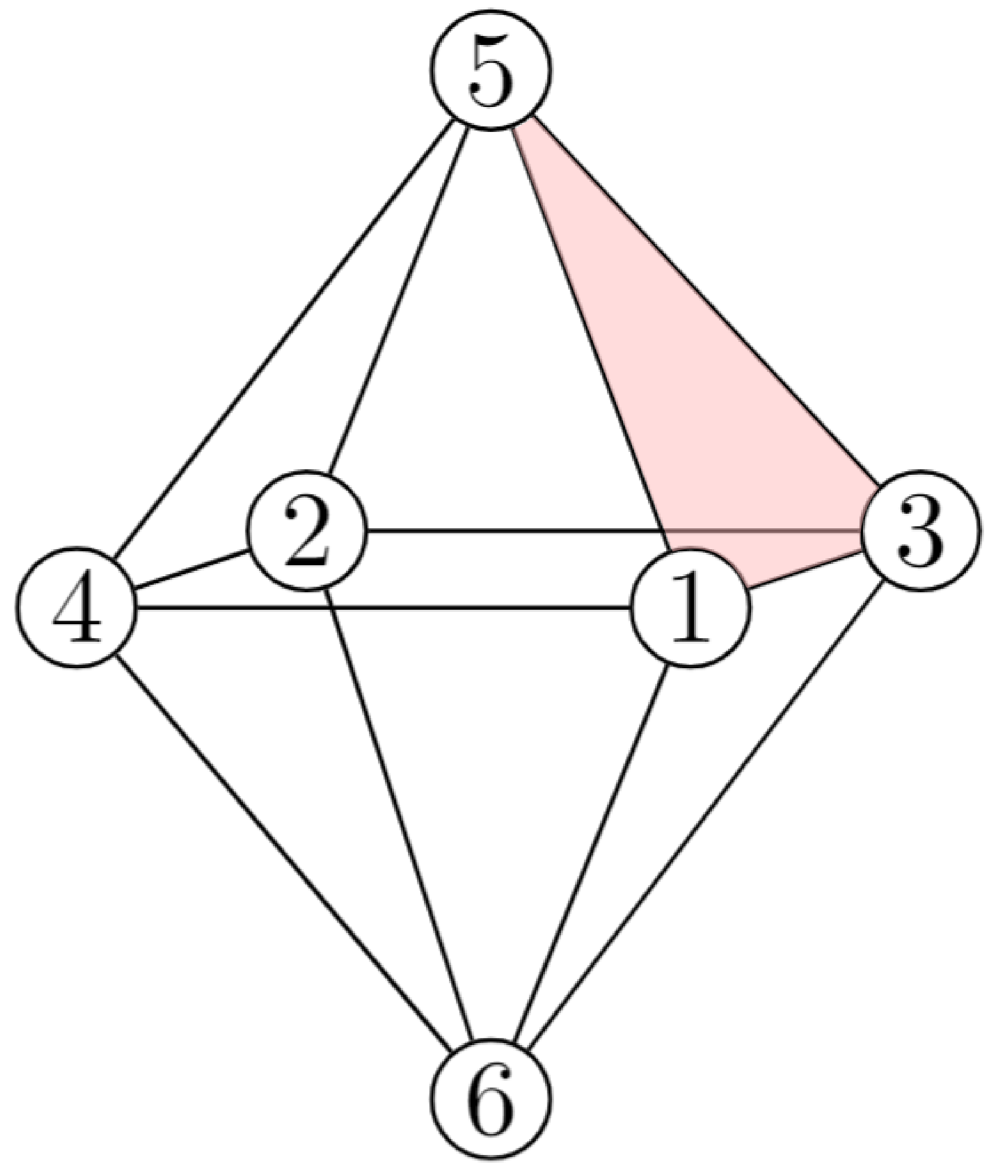
Why?

Gale duality for polytopes:

$$U = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \end{array}$$

$$U^* = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \end{array}$$



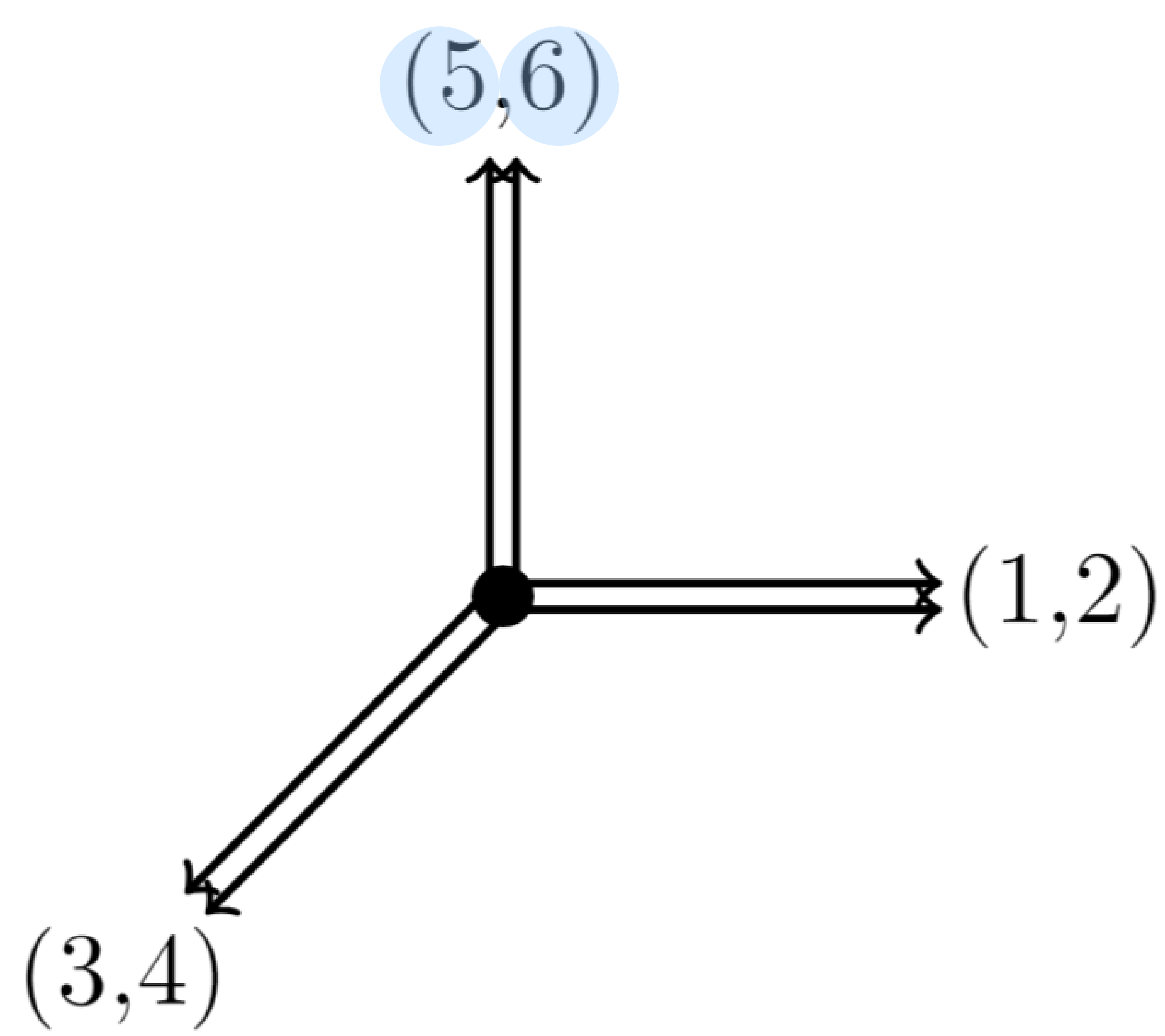


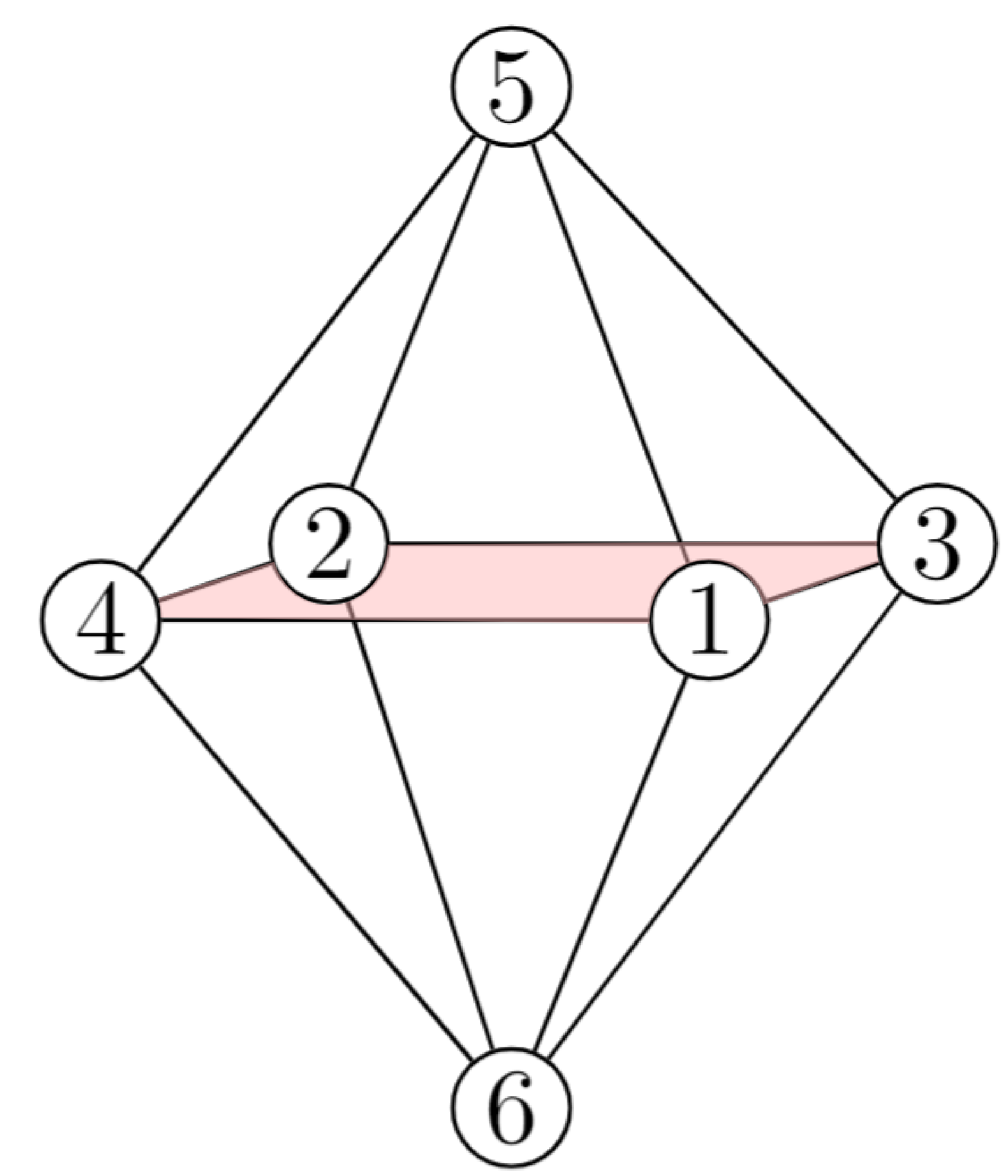
Why?

Gale duality for polytopes:

$$U = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \\ \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \end{array}$$

$$U^* = \begin{array}{c} \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \\ \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \end{array}$$

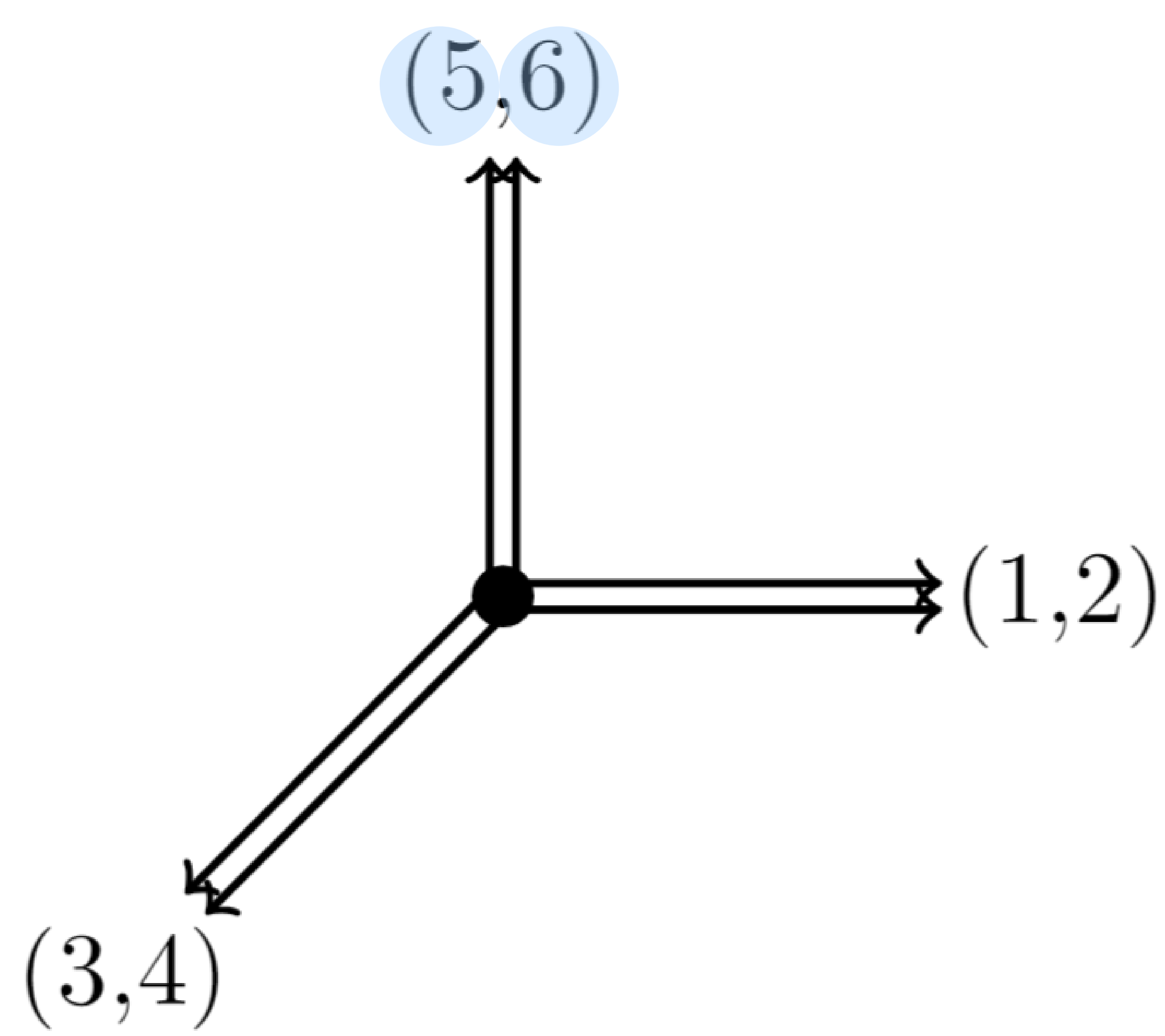


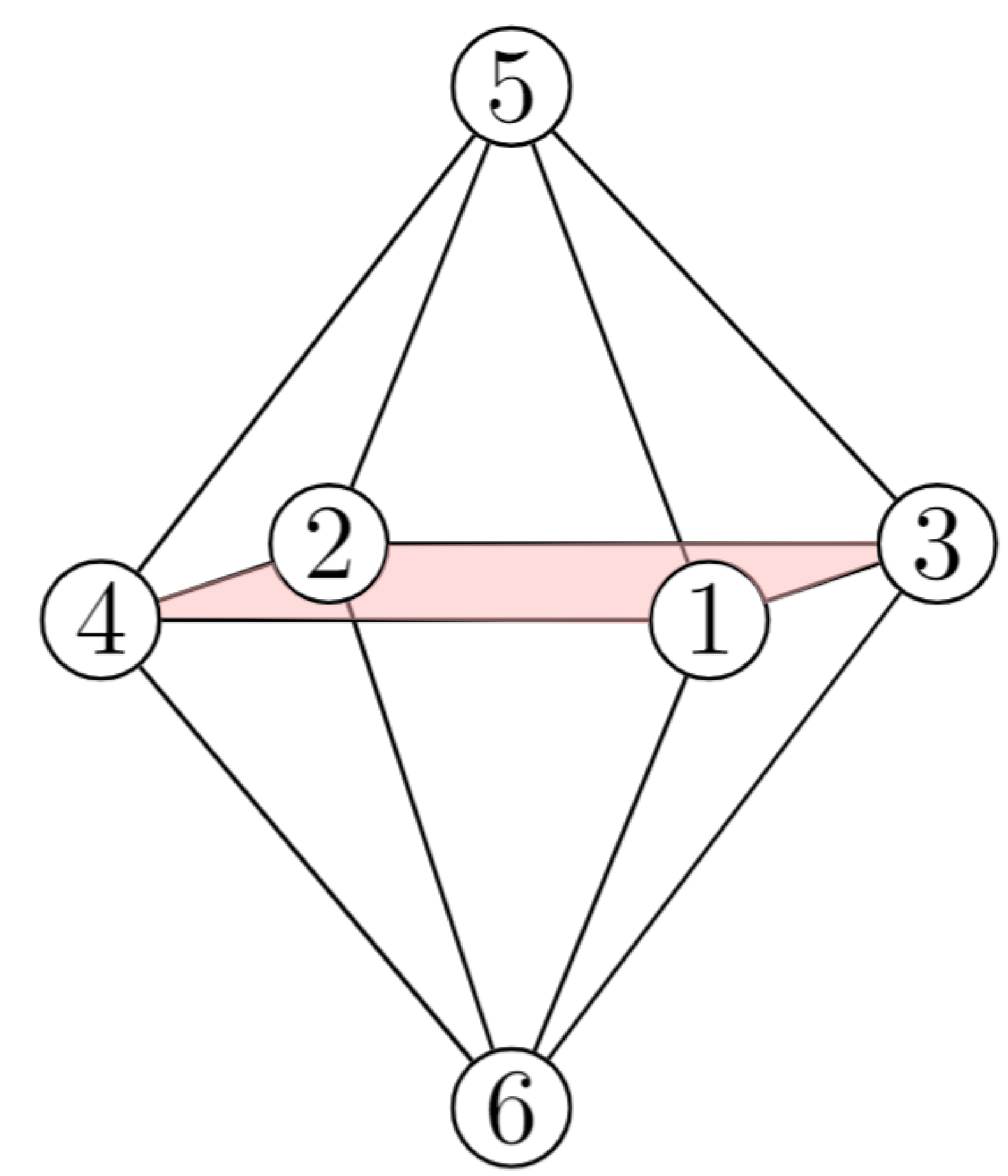


Why?

Gale duality for polytopes:

$$U = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array} \qquad U^* = \begin{array}{c} \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array}$$





Theorem: For $I \subseteq [n]$, $\operatorname{conv}\{u_i^* : i \in [n] \setminus I\}$ is a face of $\operatorname{conv}(\mathcal{U}^*)$ if and only if 0 is in the relative interior of $\operatorname{conv}\{u_i : i \in I\}$.

Two Consequences

Existence and Bounds

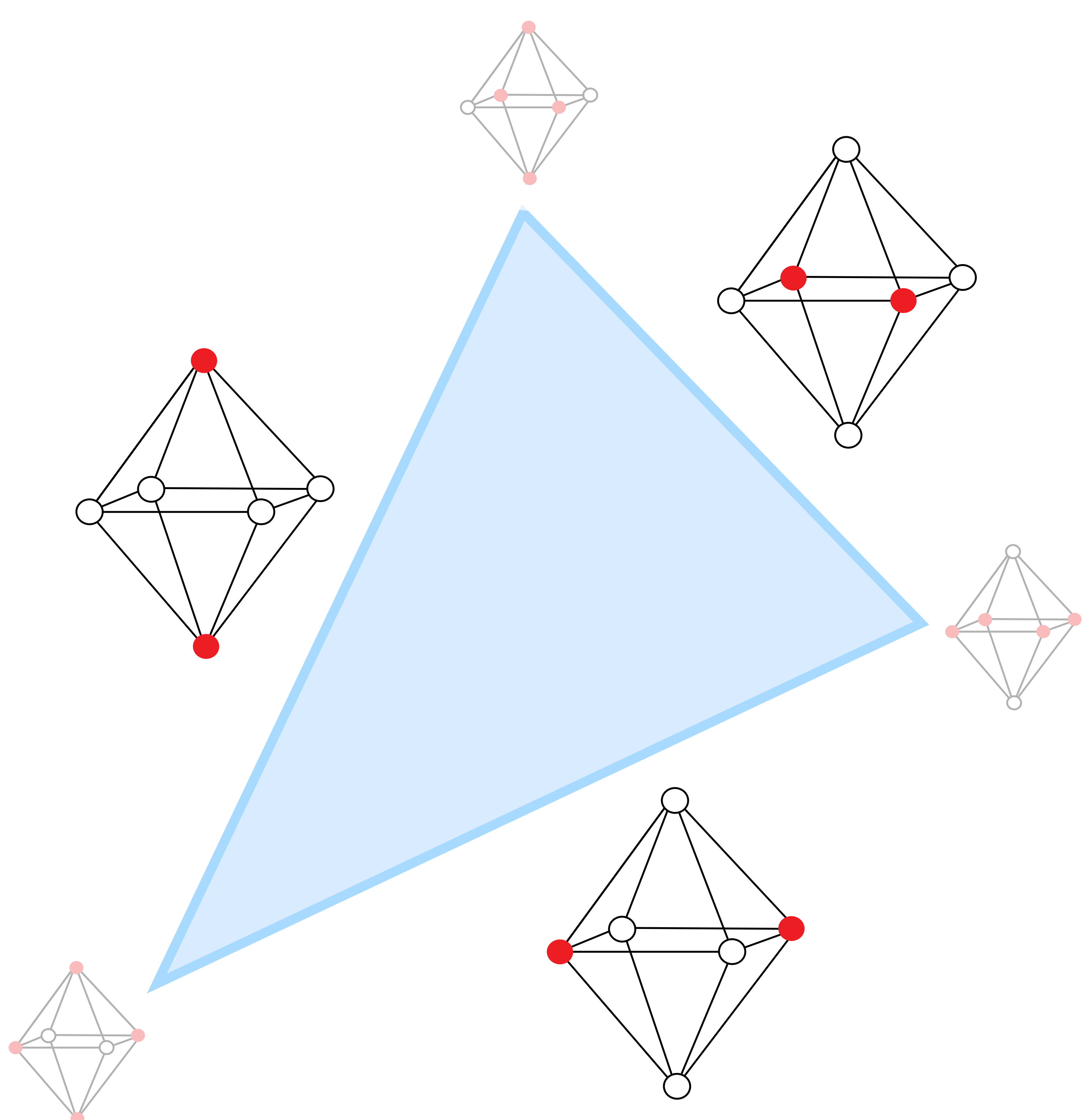
Theorem (B., Thomas, Shiroma): For $k = 1, \ldots, m-1$ there is a positively weighted k-designof size at most $\sum_{i=1}^{k} \dim \Lambda_i$

Two Consequences

Existence and Bounds

Theorem (B., Thomas, Shiroma): For $k = 1, \ldots, m-1$ there is a positively weighted k-design of size at most $\sum_{i=1}^{k} \dim \Lambda_i$

Organization

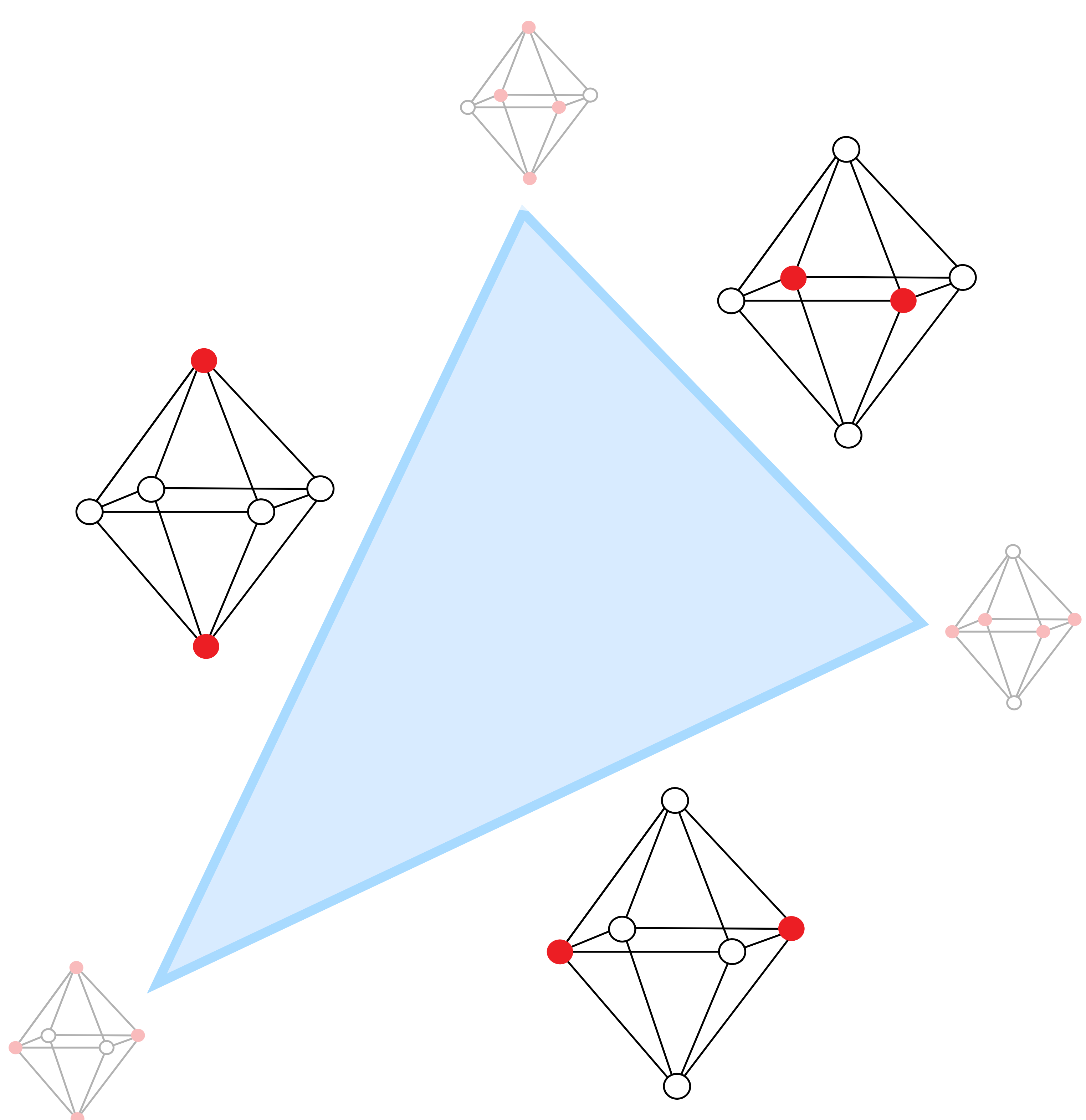


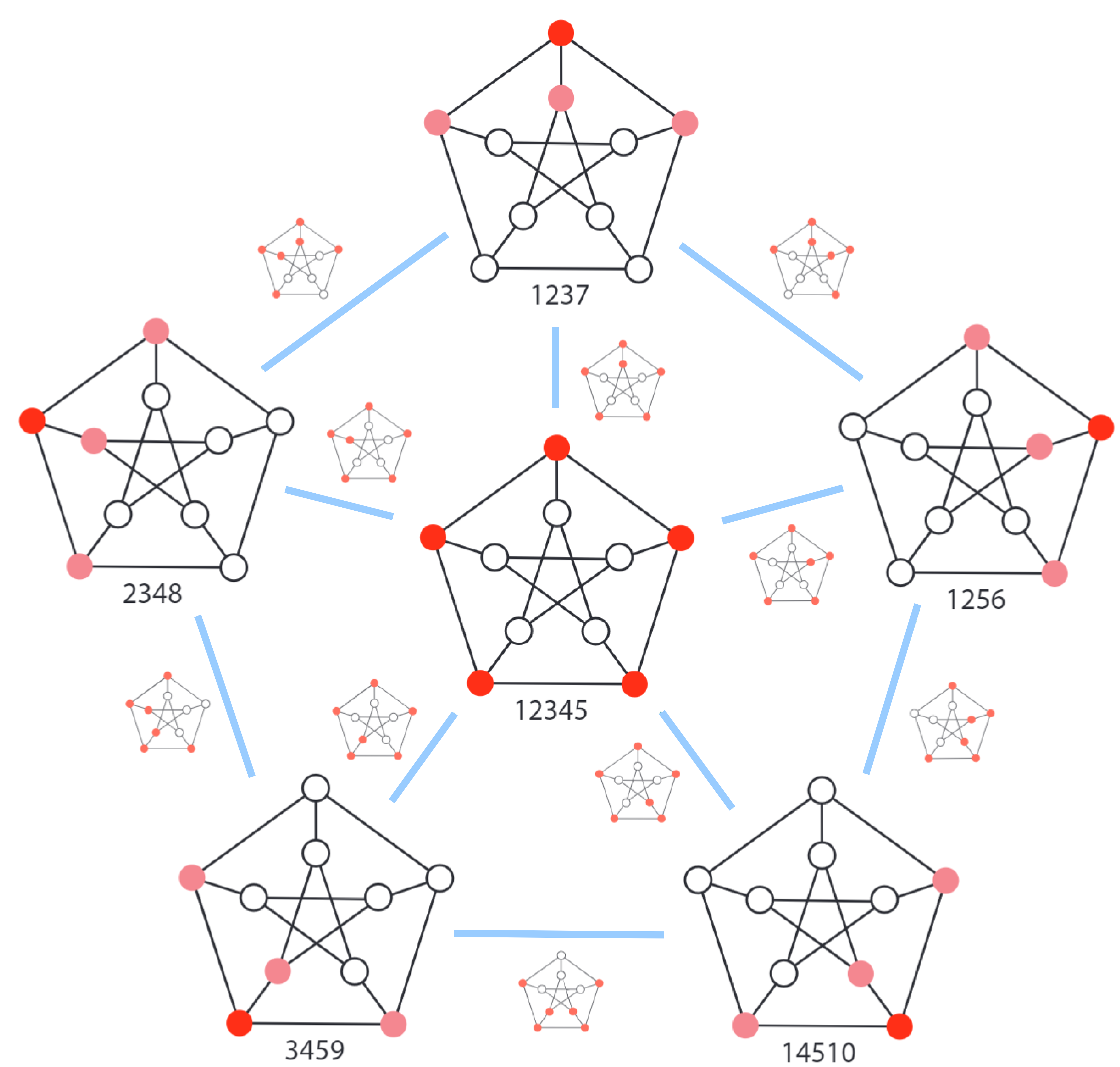
Two Consequences

Existence and Bounds

Theorem (B., Thomas, Shiroma): For $k = 1, \ldots, m-1$ there is a positively weighted k-design of size at most $\sum_{i=1}^{k} \dim \Lambda_i$

Organization

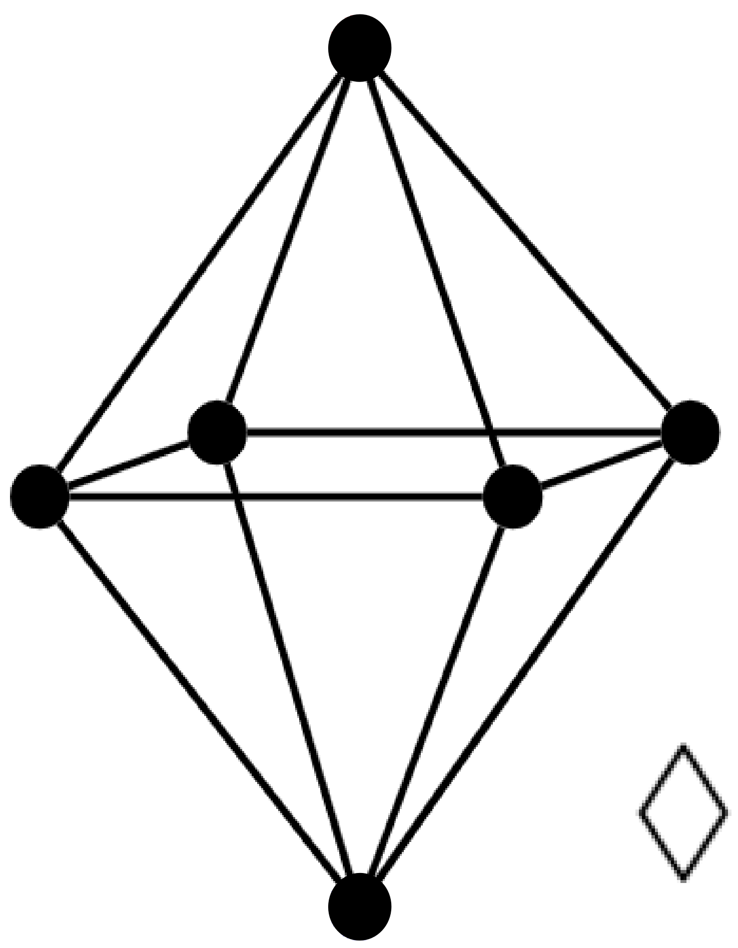




Cross-Polytopes

The d-dimensonal cross polytope is $\lozenge_d = \mathrm{conv}\{\pm e_i : i \in [d]\}$.

	vertex						
	e_1	$-e_1$	$\cdots$	$\cdots$	$\cdots$	e_d	$-e_d$
$\lambda_1 = 0$				$\mathbb{1}^\top$			
$\lambda_2 = 2d$	e_1	e_1	$\cdots$	e_{d-1}	e_{d-1}	$-\mathbb{1}$	$-\mathbb{1}$
$\lambda_3 = 2d-2$	e_1	$-e_1$	$\cdots$	$\cdots$	$\cdots$	e_d	$-e_d$

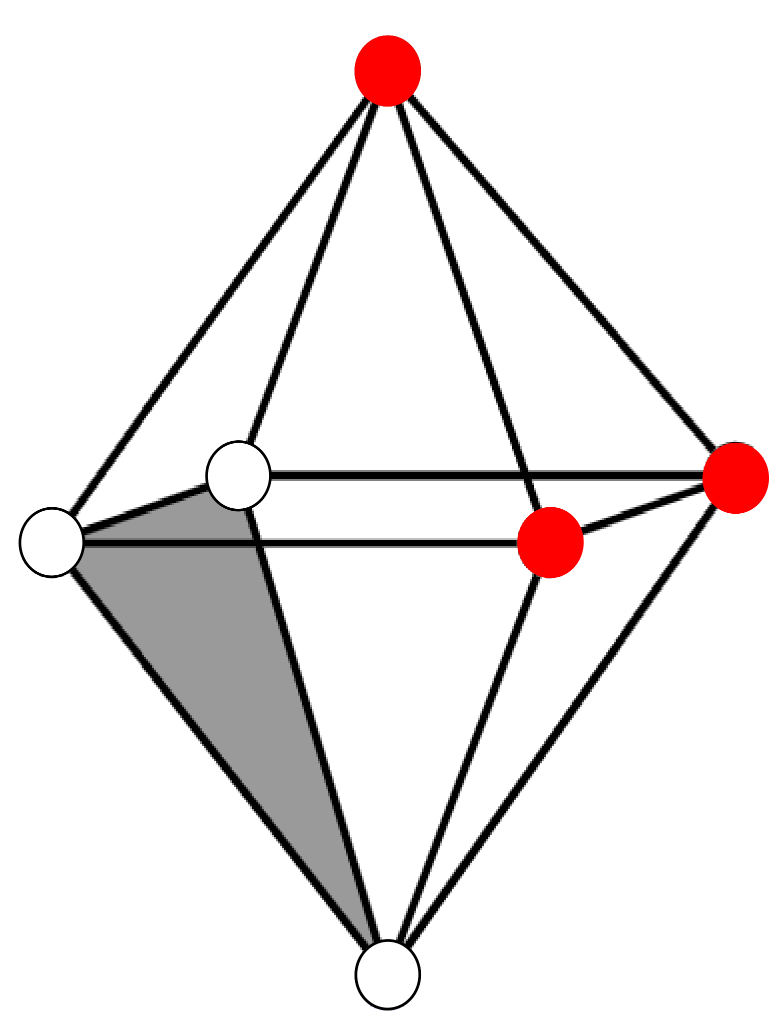


$\lozenge_3$

Cross-Polytopes

The d-dimensonal cross polytope is $\Diamond_d = \mathrm{conv}\{\pm e_i : i \in [d]\}$.

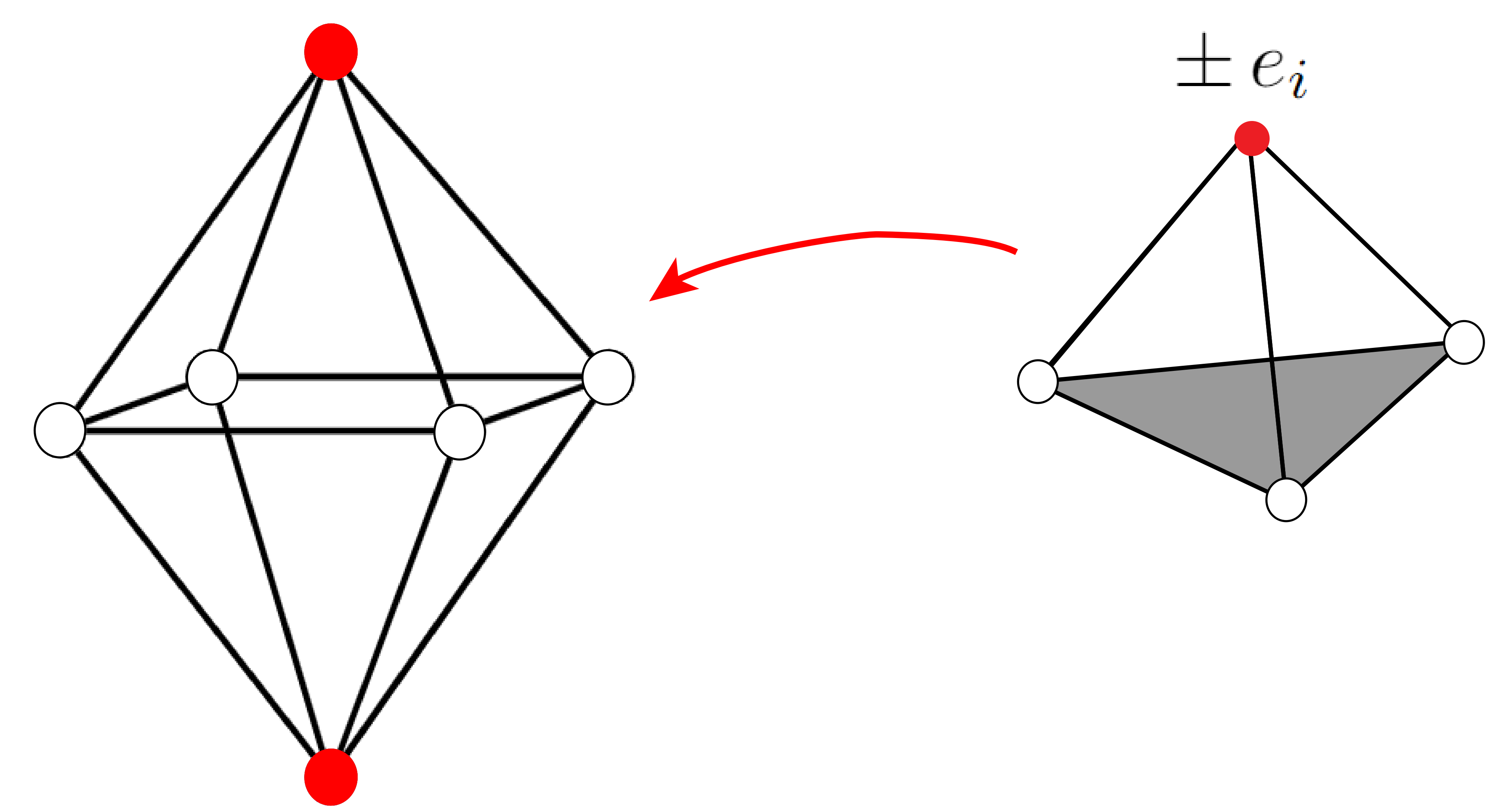
	vertex						
	e_1	$-e_1$	$\cdots$	$\cdots$	$\cdots$	e_d	$-e_d$
$\lambda_1 = 0$	$\mathbb{1}^\top$						
$\lambda_2 = 2d$	e_1	e_1	$\cdots$	e_{d-1}	e_{d-1}	$-\mathbb{1}$	$-\mathbb{1}$
$\lambda_3 = 2d-2$	e_1	$-e_1$	$\cdots$	$\cdots$	$\cdots$	e_d	$-e_d$



$\Lambda_1 < \Lambda_2 < \Lambda_3$

$P_{\bar{\mathbf{2}}} \simeq \Diamond_d$

2-designs are facets

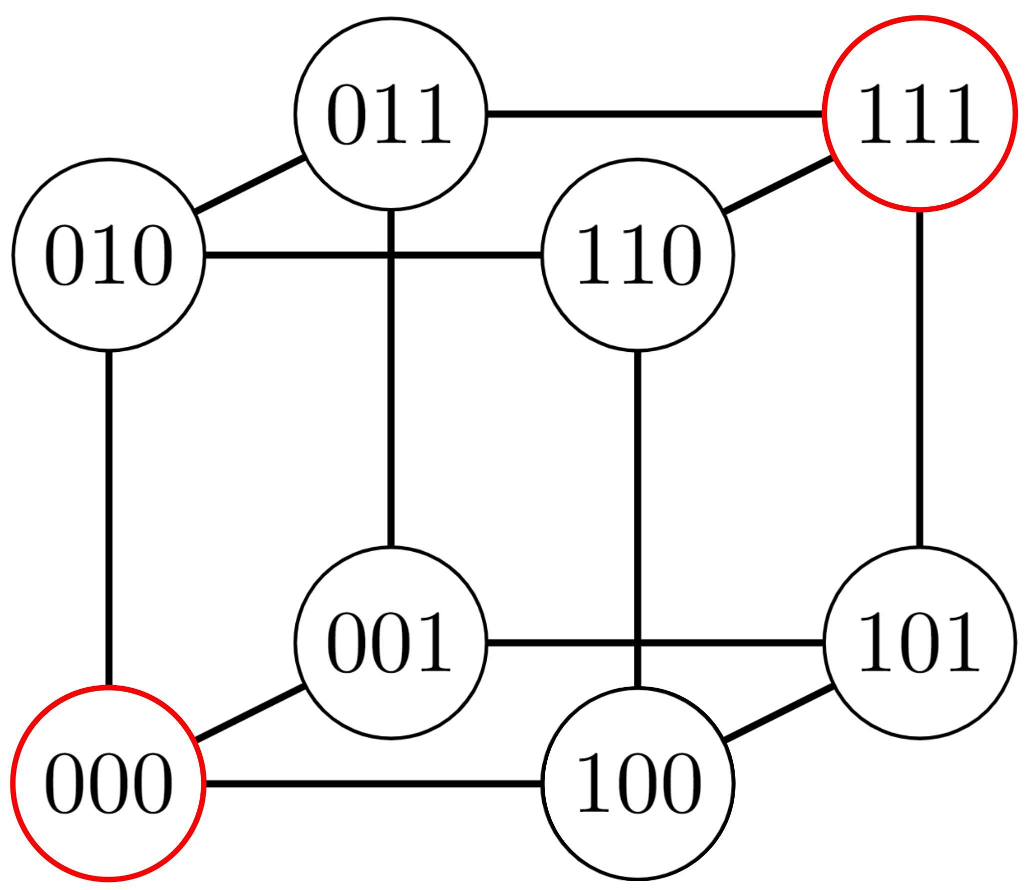


$\Lambda_1 < \Lambda_3 < \Lambda_2$

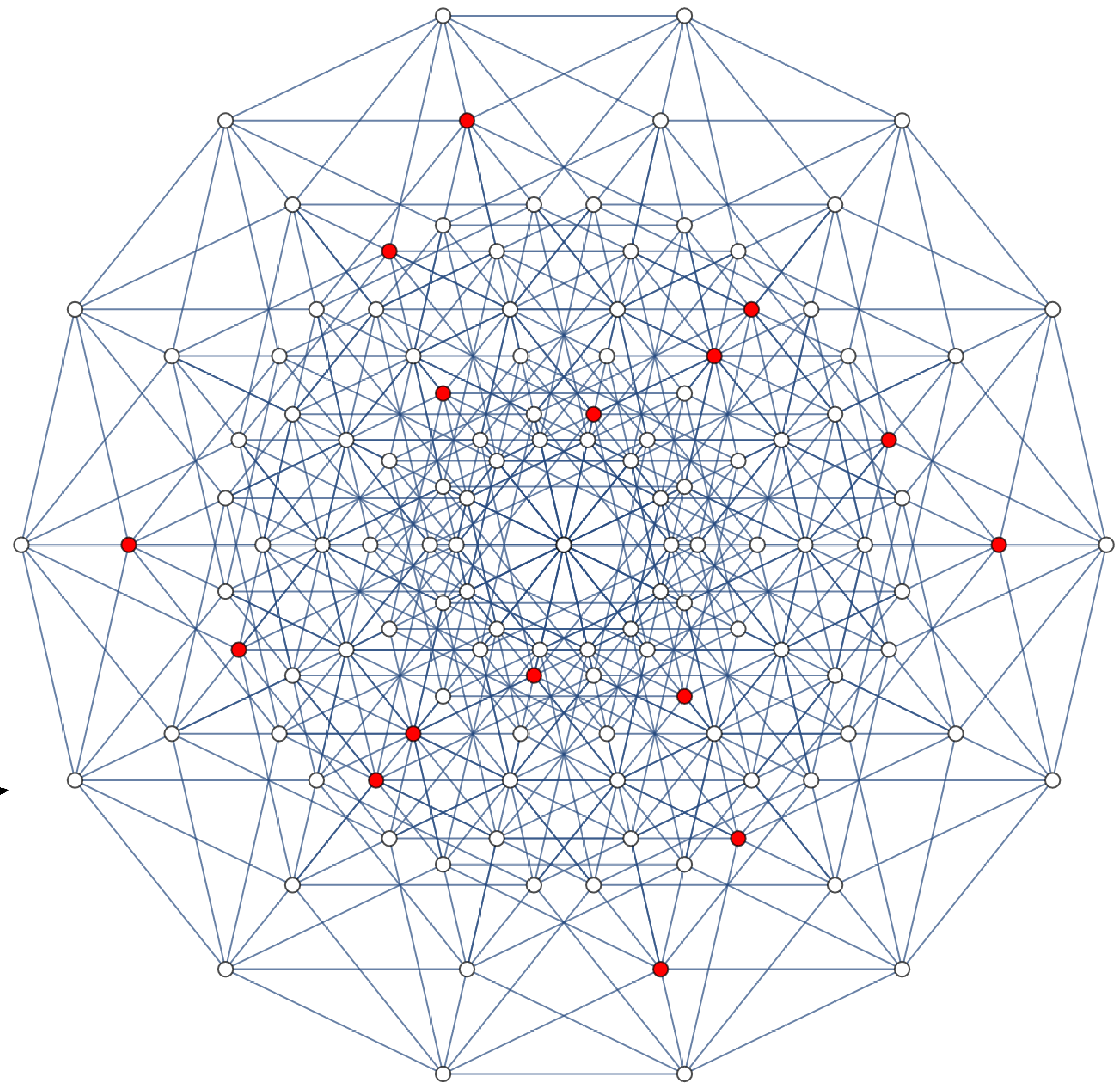
$P_{\bar{\mathbf{2}}}$ is the d-simplex

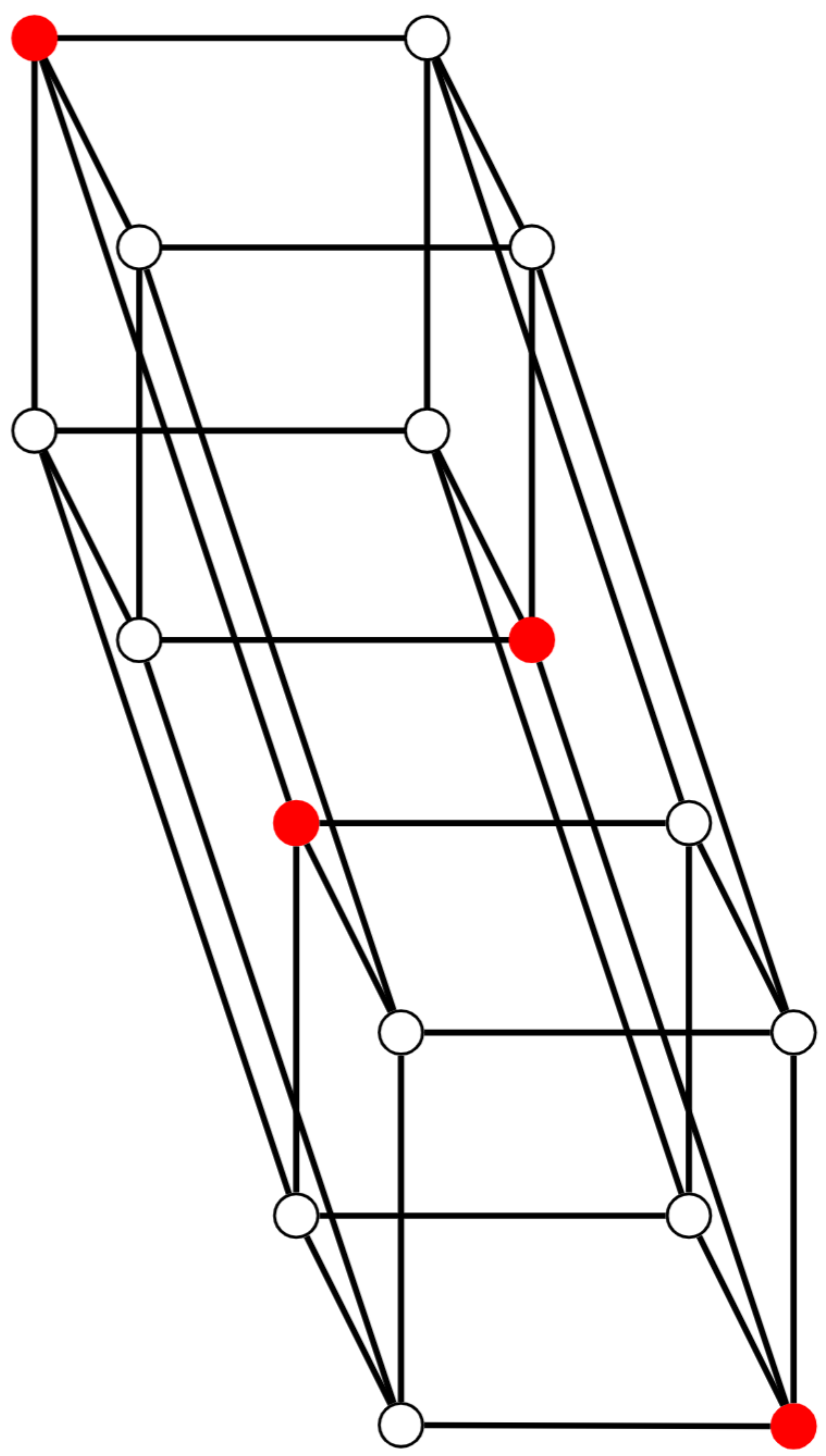
2-designs are antipodal points

Cubes



The Hamming code!

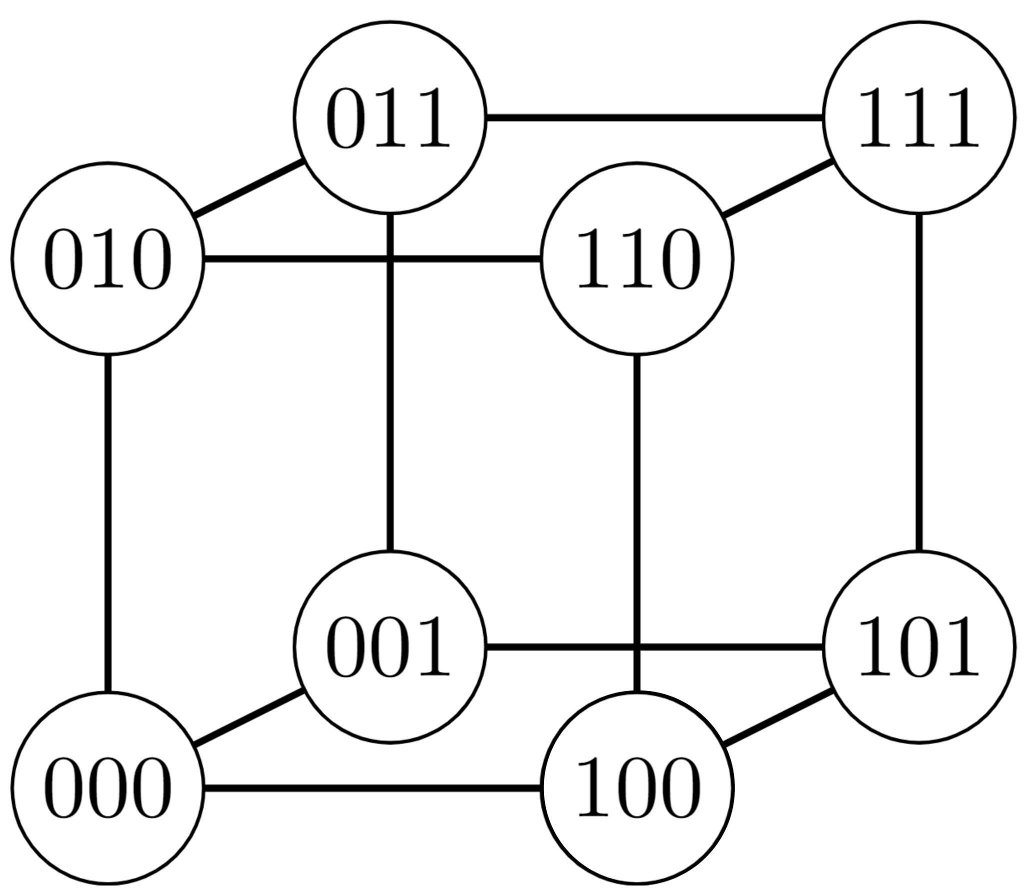




Linear error correcting codes make 'good' graphical designs on the graphs of cubes

Cubes

The eigenpolytopes of the cube can be
thought of as generalized cut polytopes

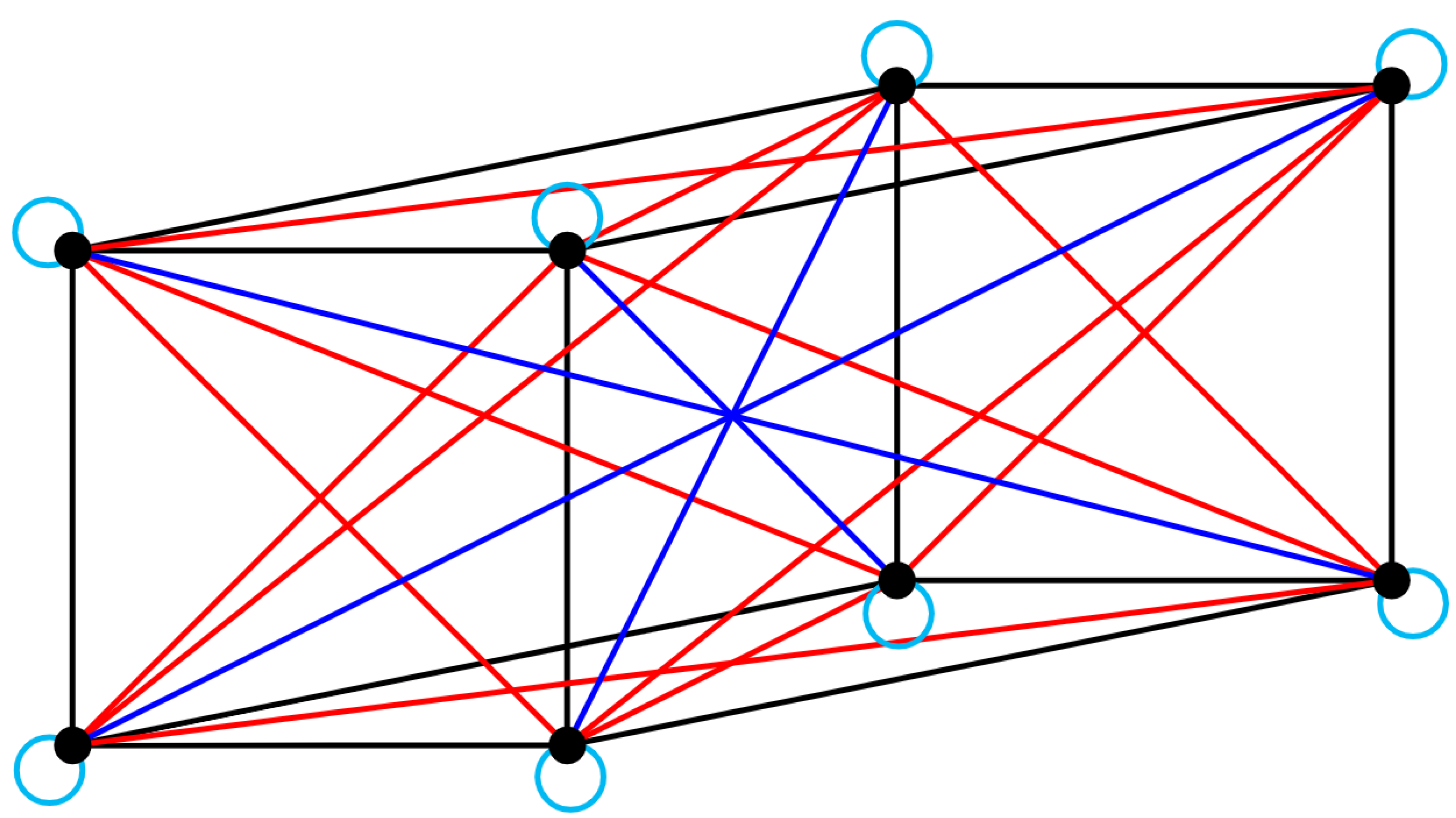


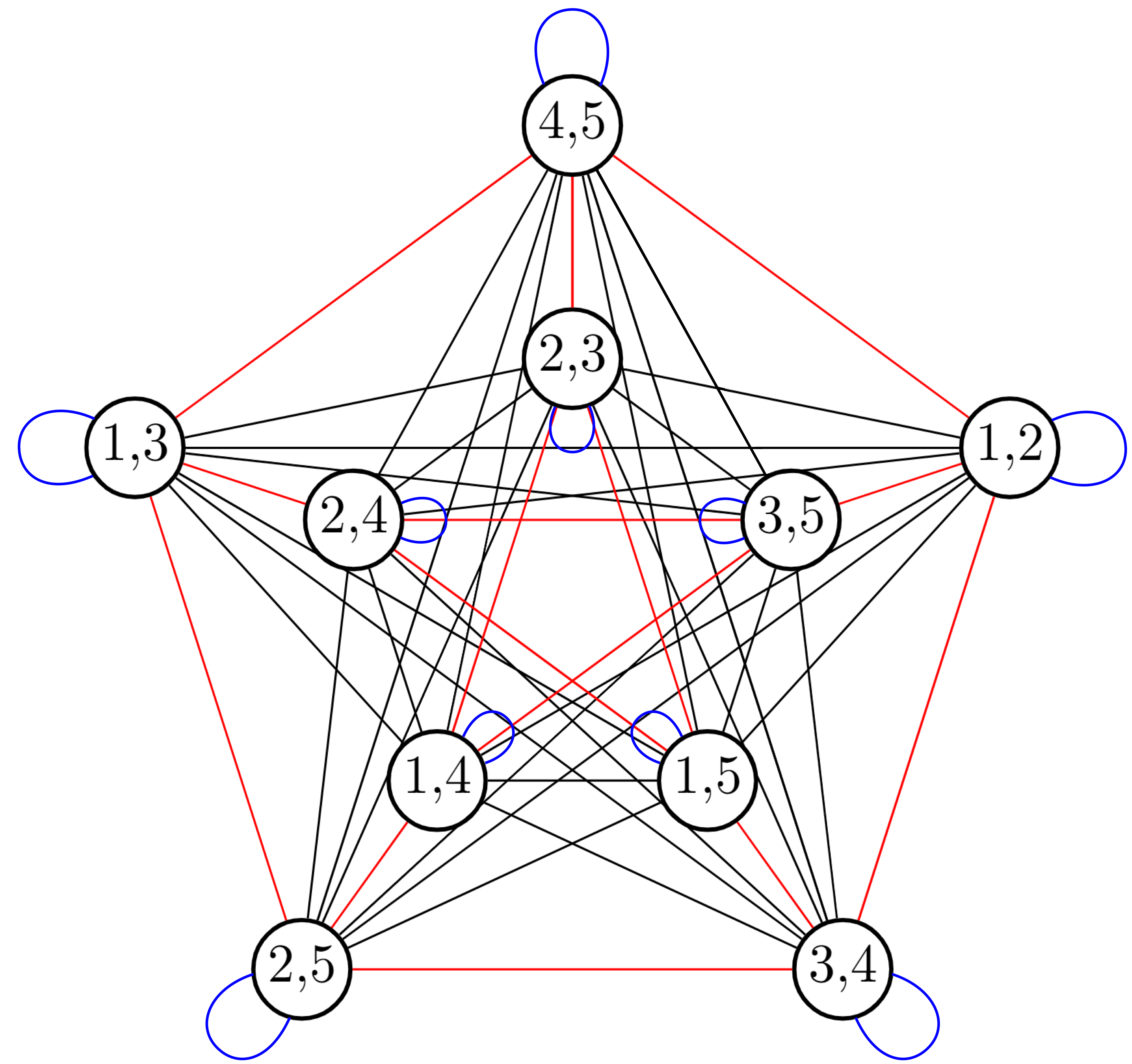
$\mathrm{CUT}_3^{\square}$

$$U = \left[\begin{array}{rrrrrrrr} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ \hline 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{array}\right]$$

Bonisoli's Theorem for linear equidistant codes provides surprisingly small graphical designs on the cube, equivalently, very large faces of these polytopes

Cubes

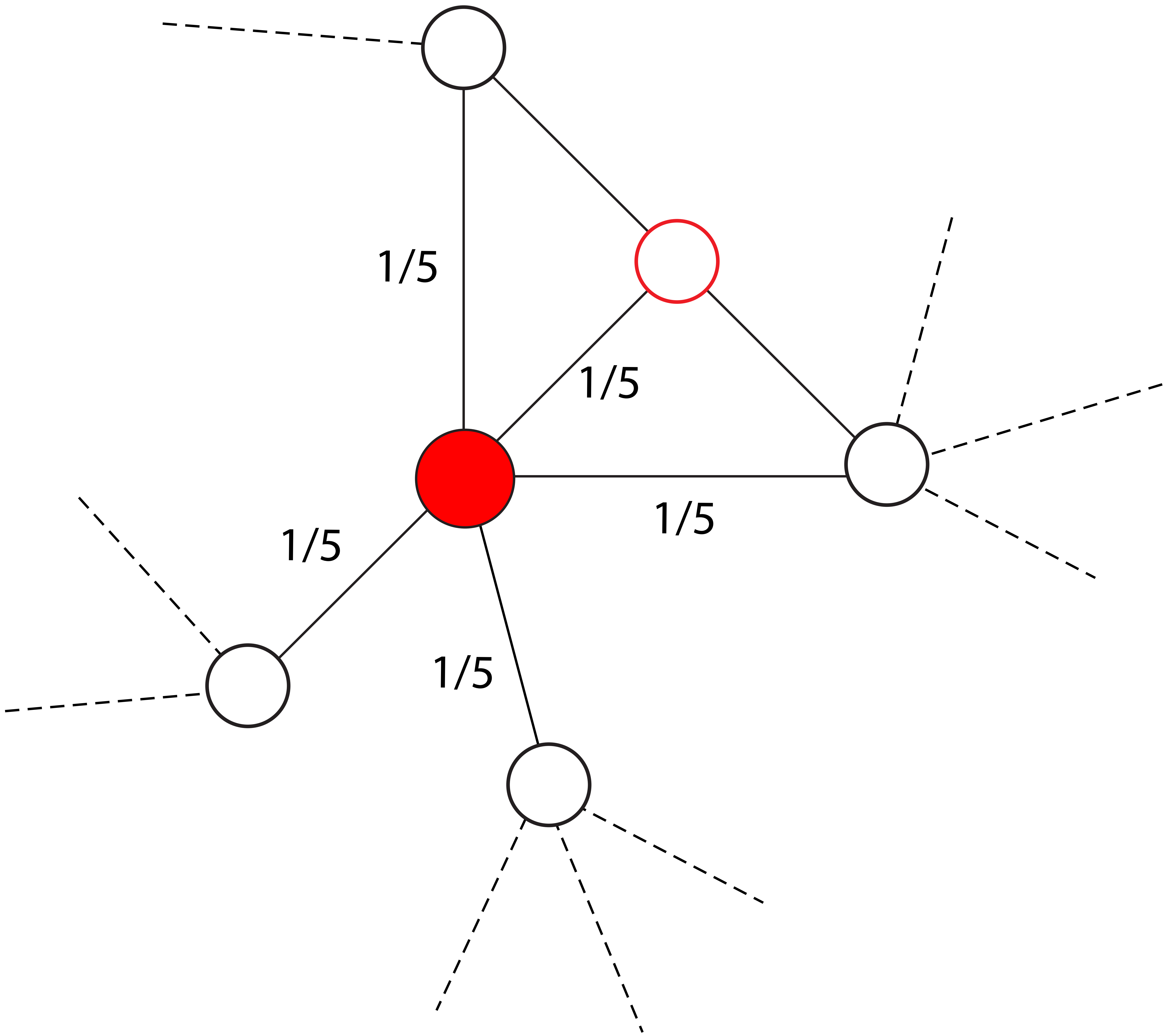




More generally, if a graph comes from an association scheme, graphical designs are a generalization of classical t-designs

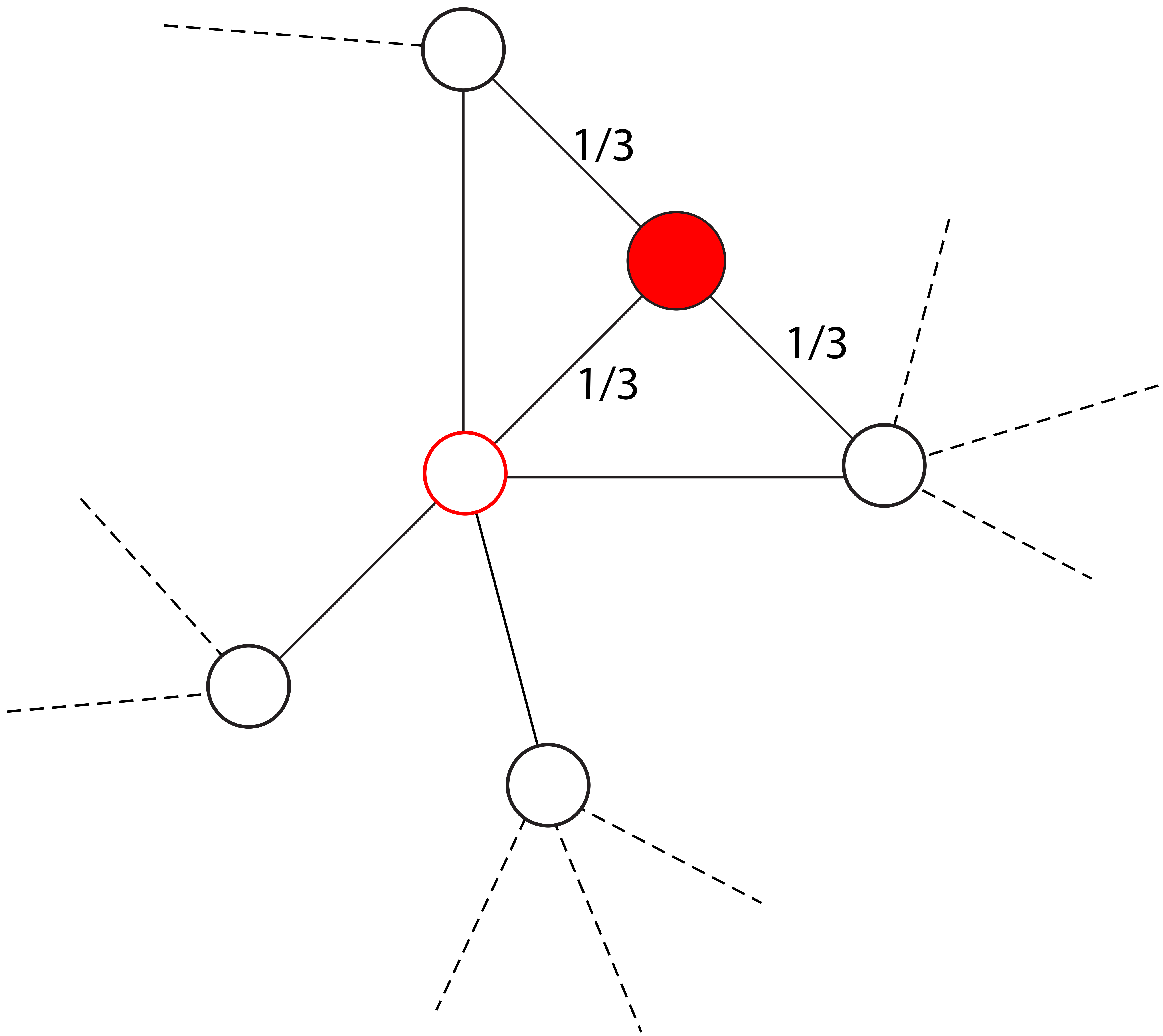
Random Walks on Graphs

A random neighbor is chosen with equal probability



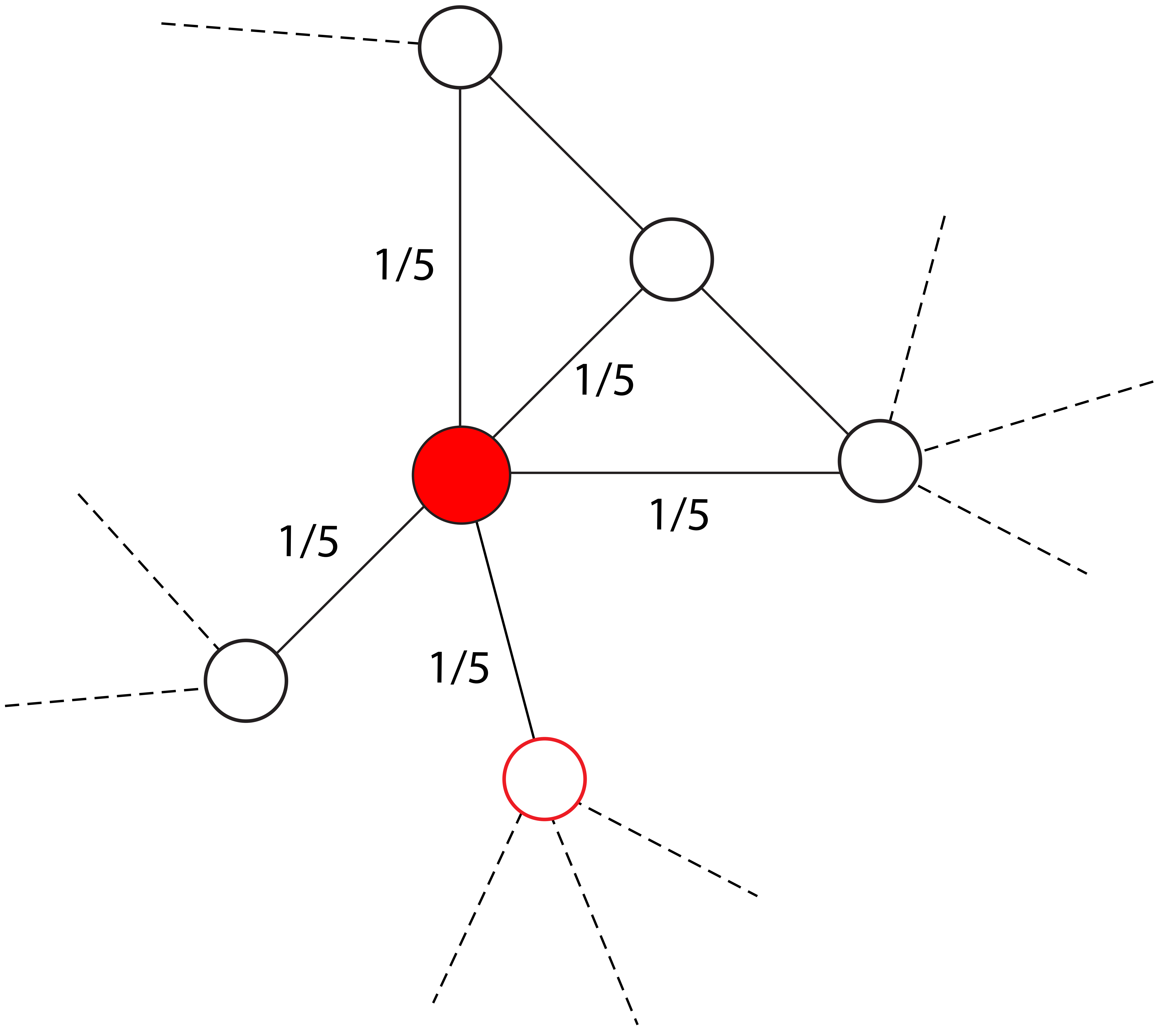
Random Walks on Graphs

A random neighbor is chosen with equal probability



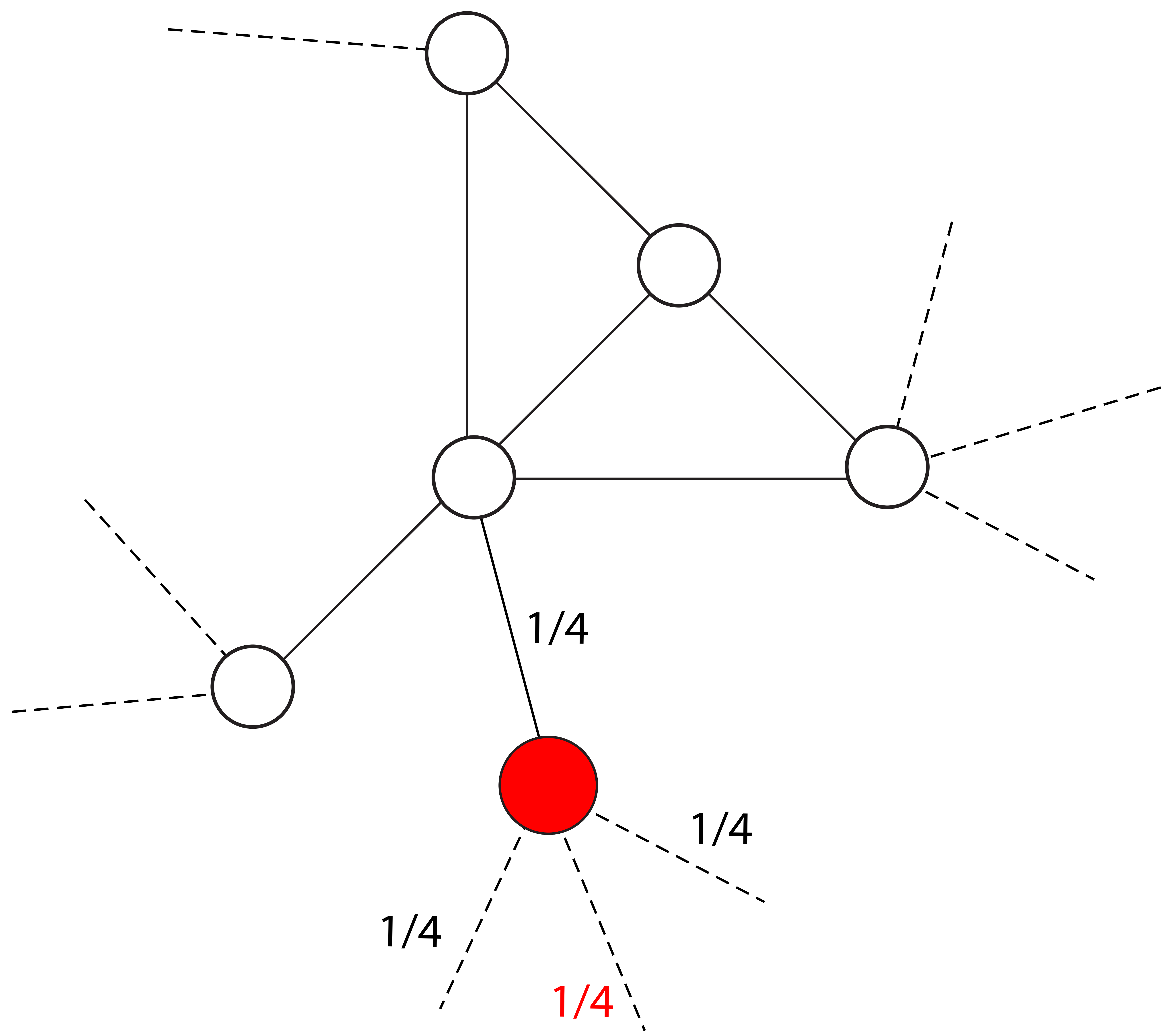
Random Walks on Graphs

A random neighbor is chosen with equal probability



Random Walks on Graphs

A random neighbor is chosen with equal probability



Random Walks on Graphs

Initial measure: μ_0

$\mu_{k+1} = AD^{-1}\mu_k$

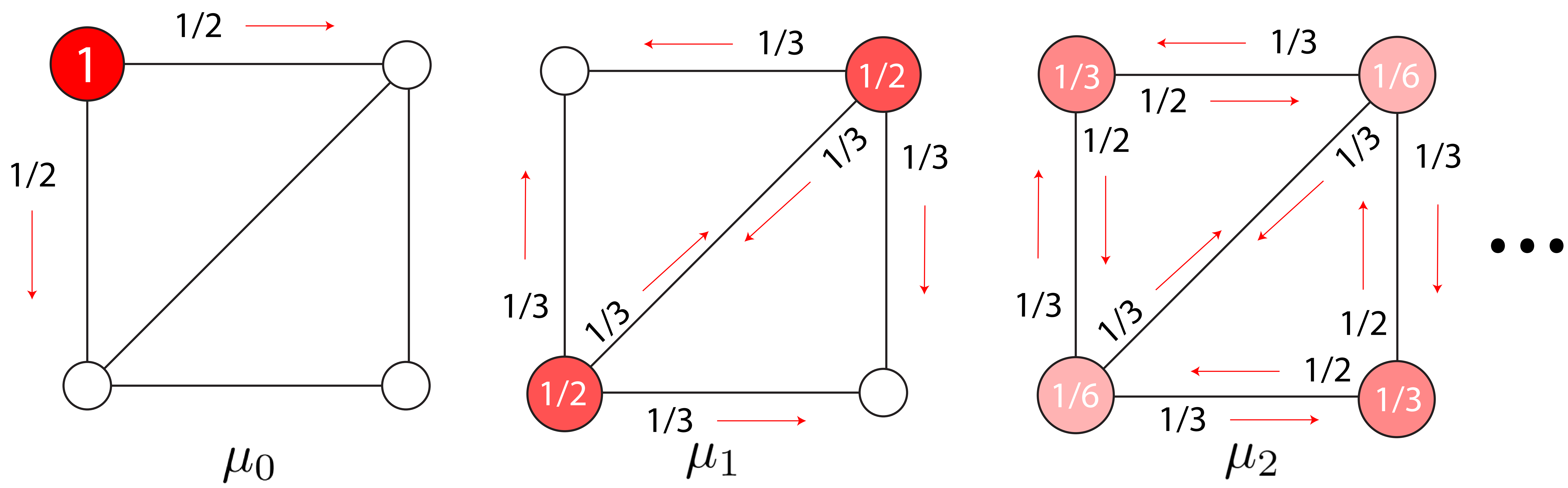
Stationary distribution: $D^{-1}\mathbb{1}$ normalized

Random Walks on Graphs

Initial measure: μ_0

$\mu_{k+1} = AD^{-1}\mu_k$

Stationary distribution: $D^{-1}\mathbb{1}$ normalized

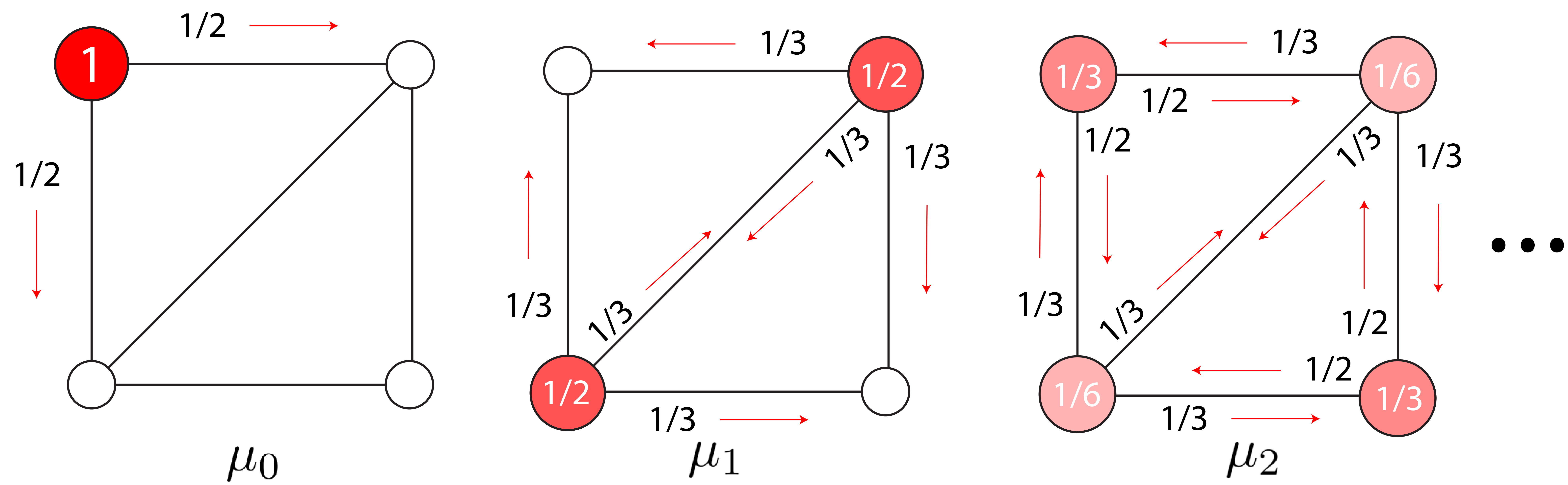


Random Walks on Graphs

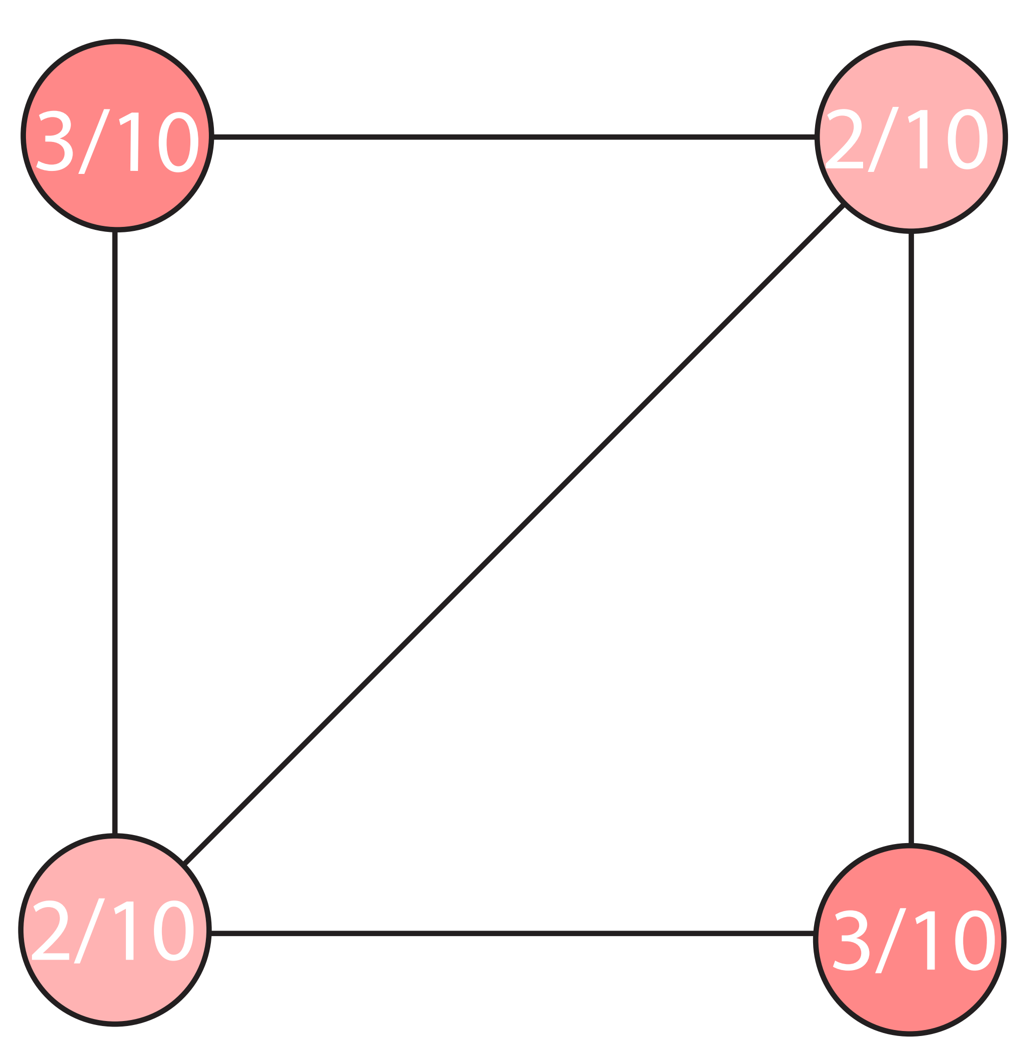
Initial measure: μ_0

$$\mu_{k+1} = AD^{-1}\mu_k$$

Stationary distribution: $D^{-1}\mathbb{1}$ normalized



$\frac{3}{5}D^{-1}\mathbb{1}$



Convergence to the stationary distribution is controlled by $|\lambda_2|$

Random Walks + Graphical Designs

Theorem (Steinerberger & Thomas): If μ_0 defines an ℓ-graphical design, then

$$\sum_{v \in V} \left| \mu_k(v) - \frac{1}{n} \right|^2 \leq \lambda_{\ell+1}^{2k}.$$

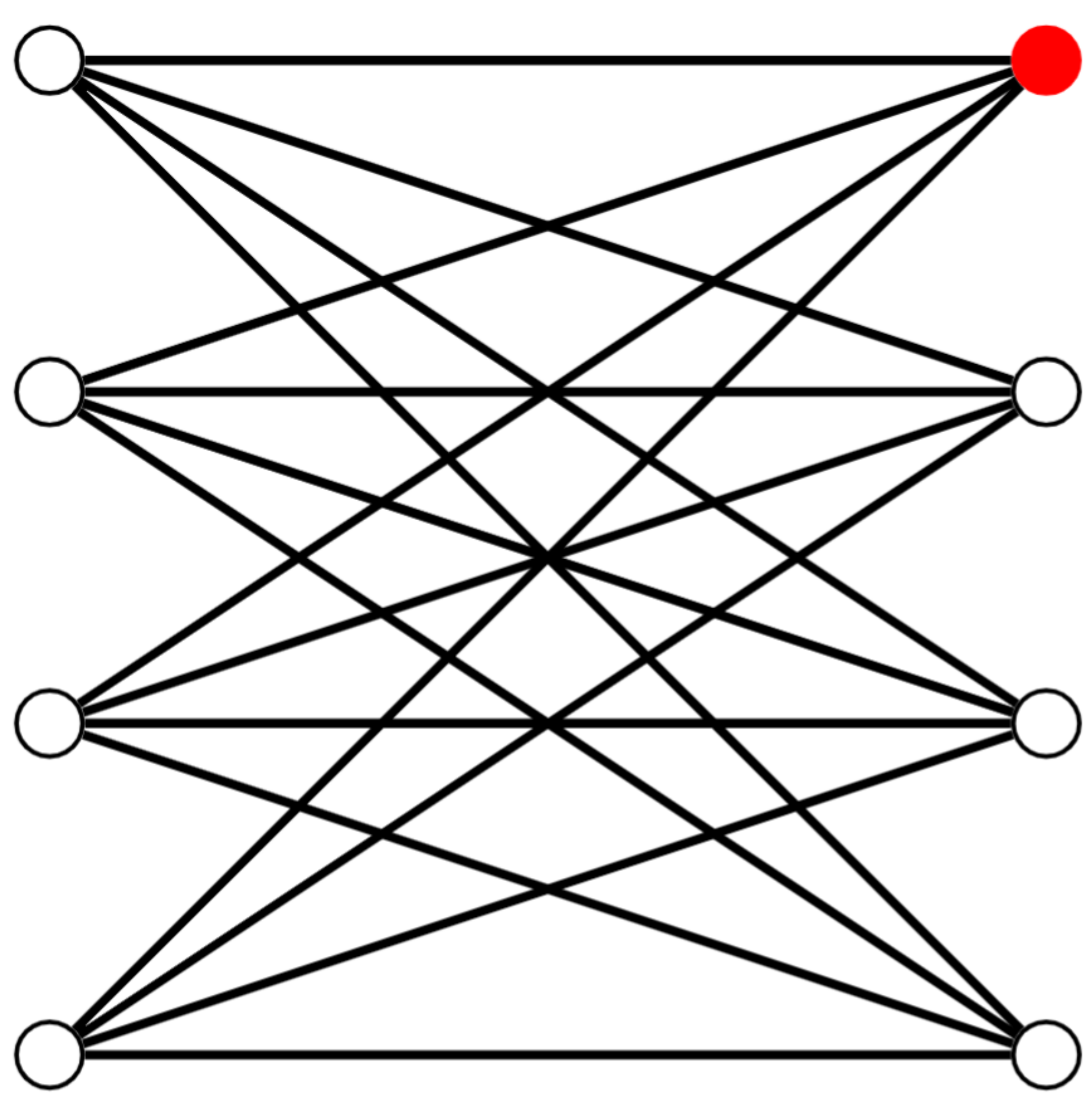
Convergence to the stationary distribution controlled by $|\lambda_{l+1}|$!

Random Walks + Graphical Designs

Theorem (Steinerberger & Thomas): If μ_0 defines an ℓ-graphical design, then

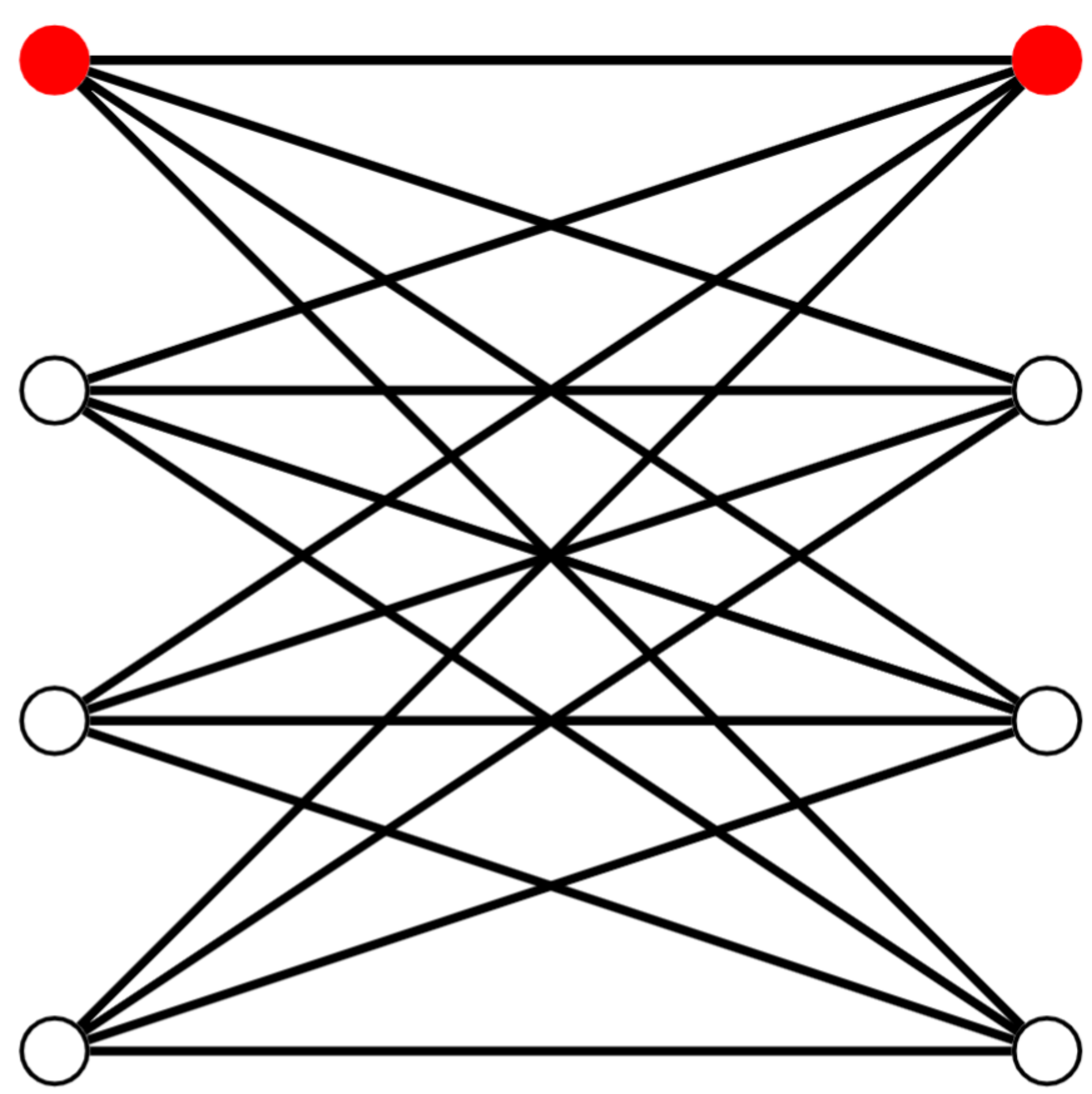
$$\sum_{v \in V} \left| \mu_k(v) - \frac{1}{n} \right|^2 \leq \lambda_{\ell+1}^{2k}.$$

Convergence to the stationary distribution controlled by $|\lambda_{l+1}|$!



$\lambda_2 = -1$

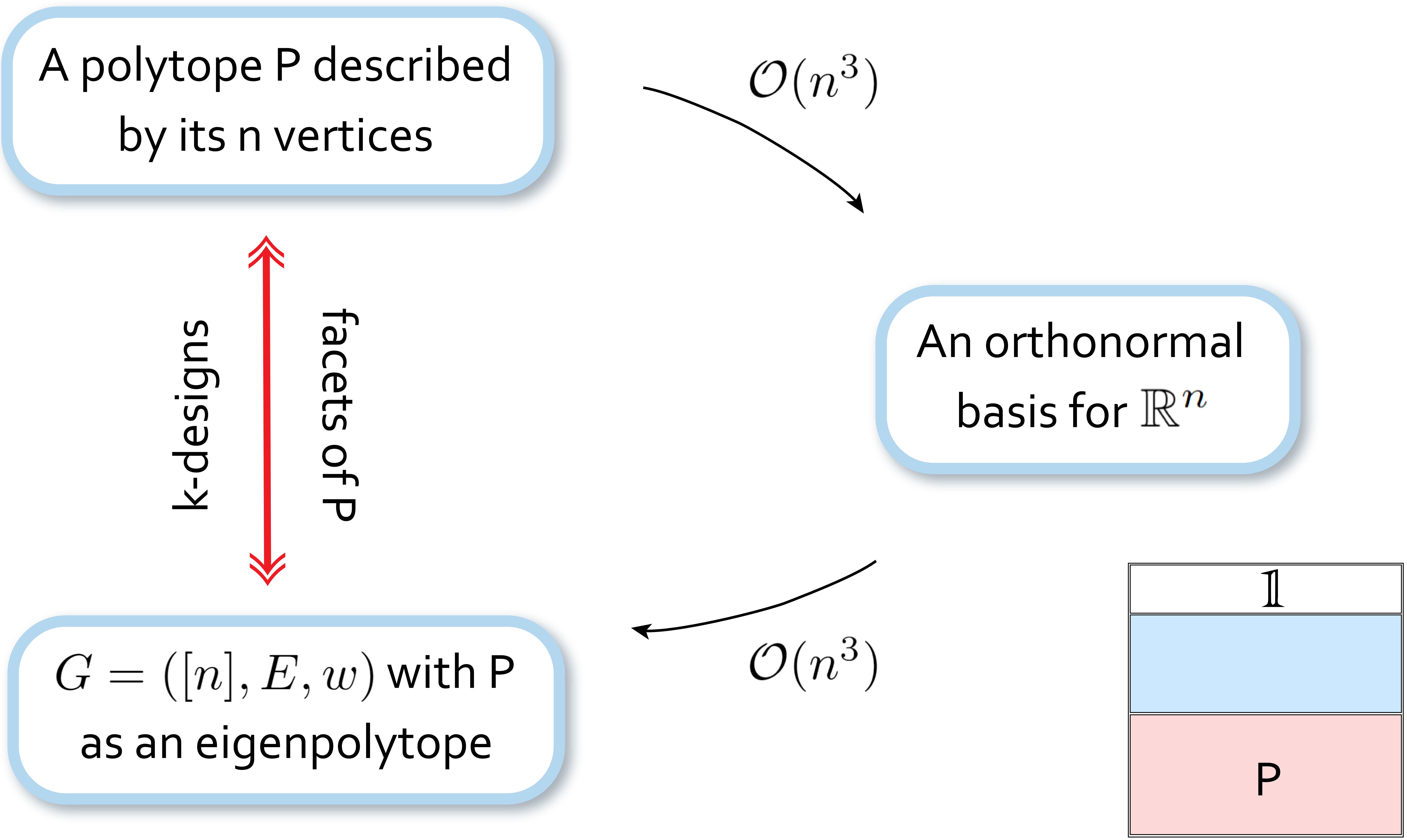
Never converges :(



$\lambda_3 = 0$

Converges in one step :)

Complexity Results: A Road Map



Creating Graphs from Orthonormal Bases

Lemma (B. and Shiroma): The algorithm

Input: an orthonormal basis $\mathcal{B}$ of $\mathbb{1}_n^\perp$ and a set partition of $[n-1]$

Output: $G = ([n], E, w)$, with nontrivial eigenvectors $\mathcal{B}$, eigenspaces specified by the partition

can be done in $\mathcal{O}(n^3)$ time.

Creating Graphs from Orthonormal Bases

Lemma (B. and Shiroma): The algorithm

Input: an orthonormal basis $\mathcal{B}$ of $\mathbb{1}_n^\perp$ and a set partition of $[n-1]$

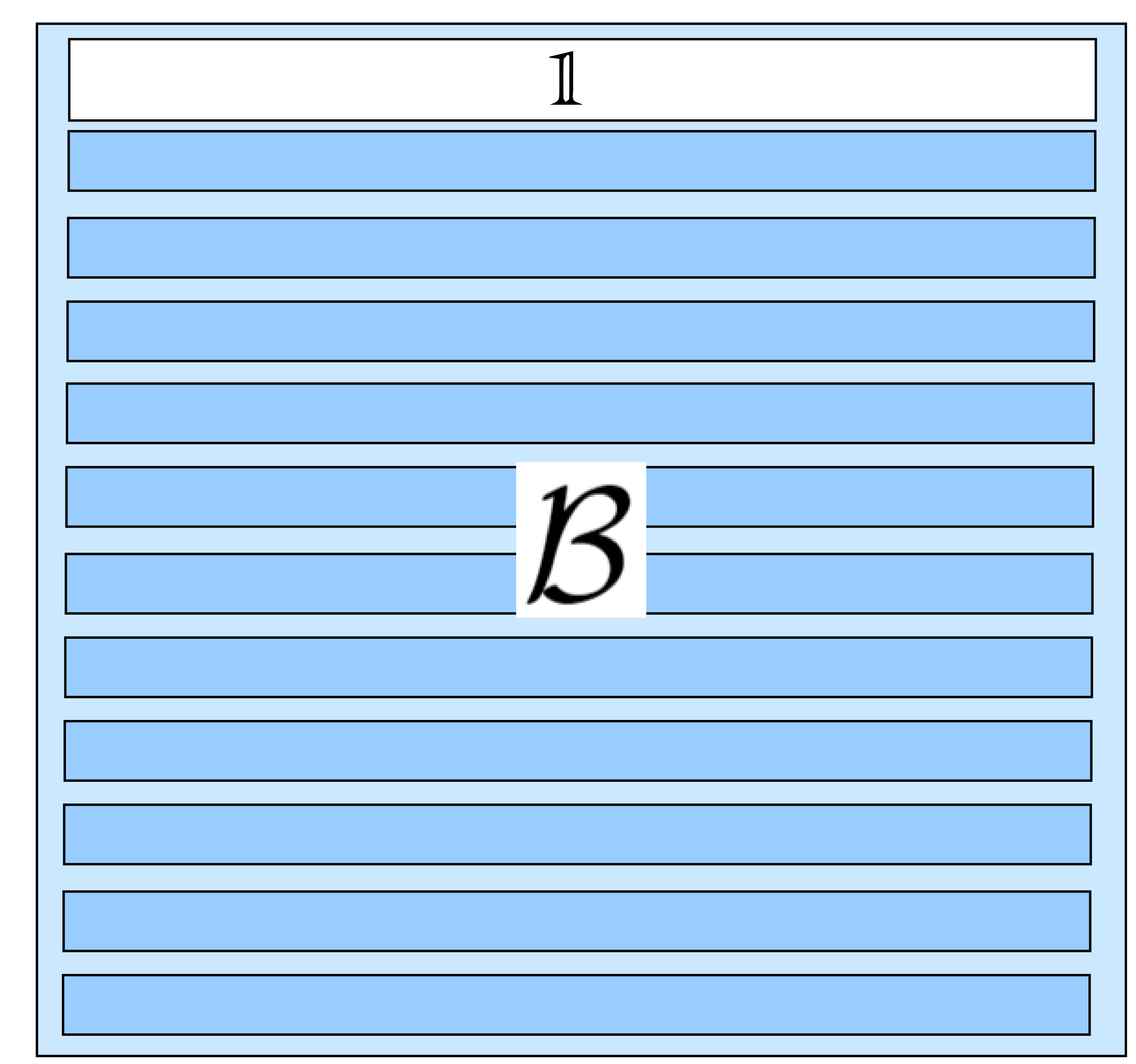
Output: $G = ([n], E, w)$, with nontrivial eigenvectors $\mathcal{B}$, eigenspaces specified by the partition

can be done in $\mathcal{O}(n^3)$ time.

Step 1: derive constraints on the eigenvalues from expressing

$$D - A = B^\top M B$$

$$M = \mathrm{Diag}(0, \lambda_2, \ldots, \lambda_n)$$

$B =$ 

Creating Graphs from Orthonormal Bases

Lemma (B. and Shiroma): The algorithm

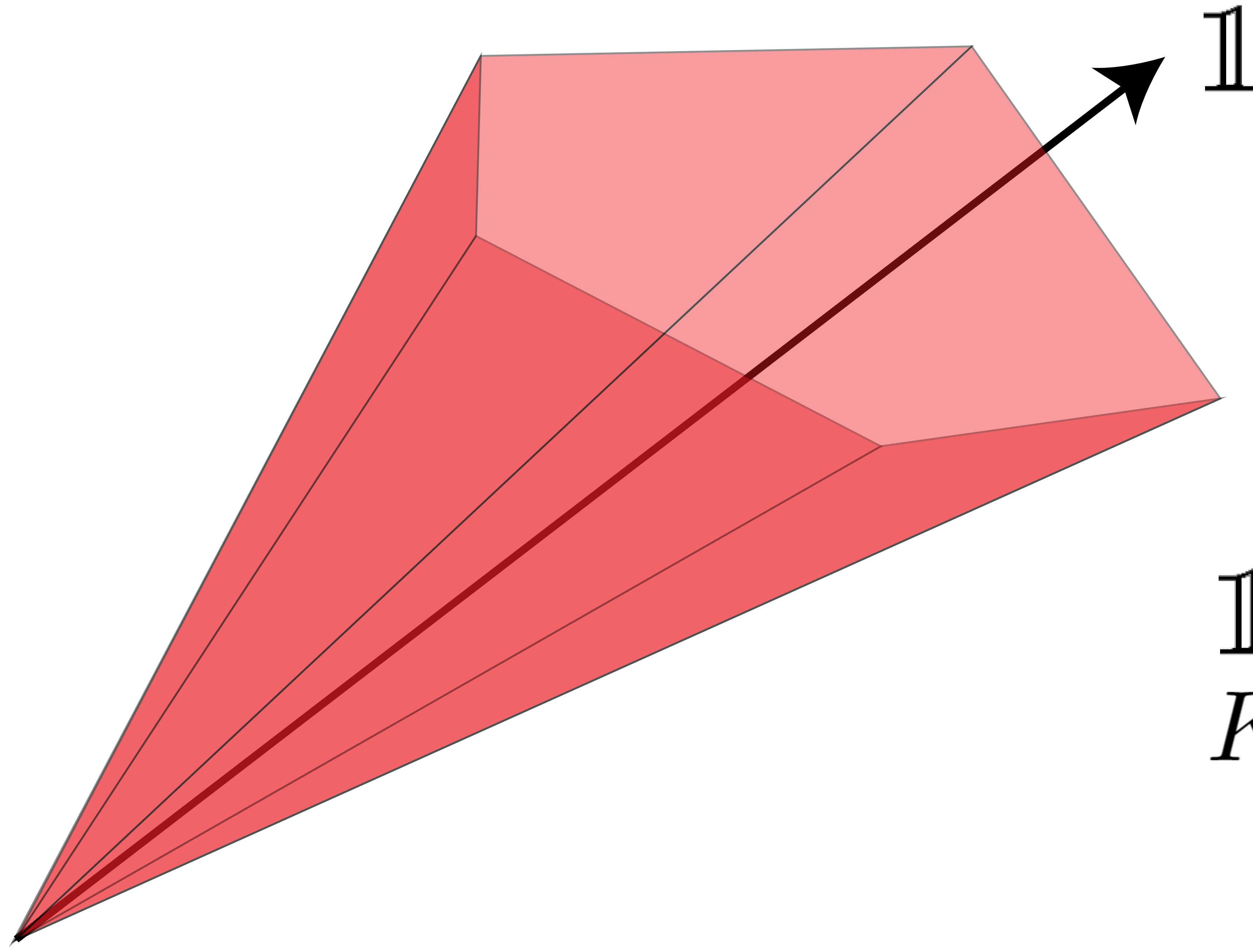
Input: an orthonormal basis $\mathcal{B}$ of $\mathbb{1}_n^\perp$ and a set partition of $[n-1]$

Output: $G = ([n], E, w)$, with nontrivial eigenvectors $\mathcal{B}$, eigenspaces specified by the partition

can be done in $\mathcal{O}(n^3)$ time.

Step 2: wiggle!

full-dimensional polyhedral cone of eigenvalues which provide valid graphs



$\mathbb{1}$ corresponding to K_n lies in the interior

Embedding Polytopes in Graphs

Theorem (B. and Shiroma): The algorithm

Input: A polytope P described by its n vertices

Output: $G = ([n], E, w)$ which has an eigenpolytope combinatorially equivalent to P.

can be done in $\mathcal{O}(n^3)$ time.

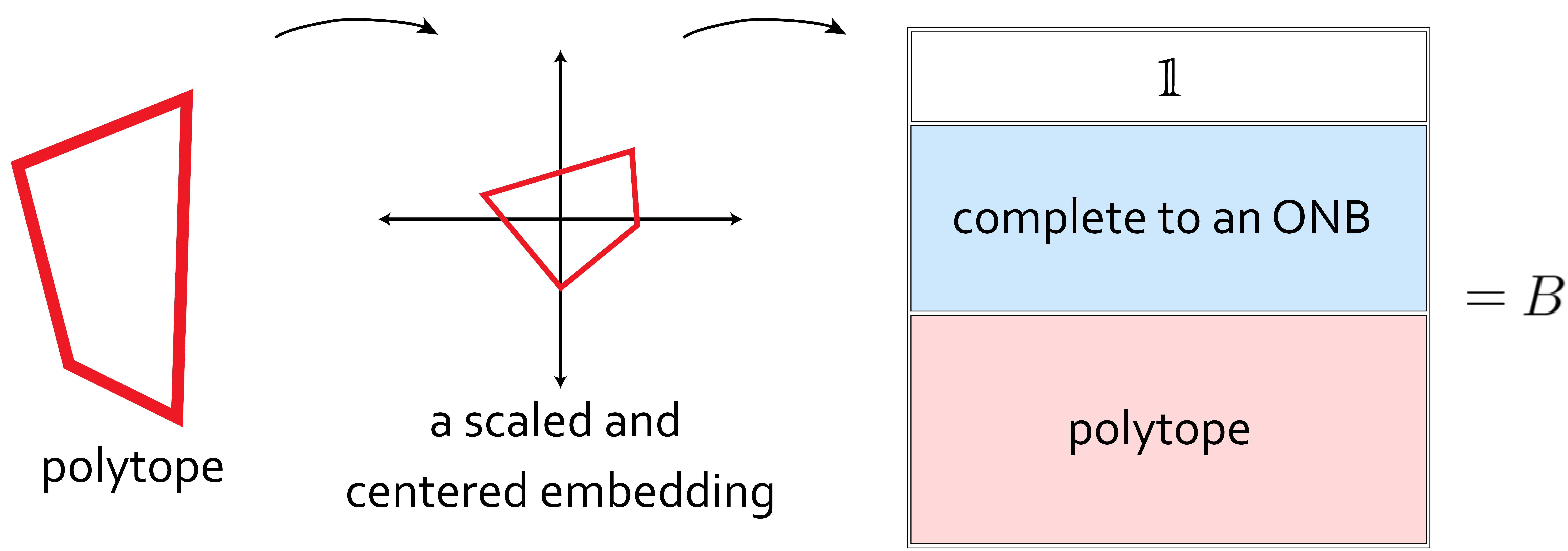
Embedding Polytopes in Graphs

Theorem (B. and Shiroma): The algorithm

Input: A polytope P described by its n vertices

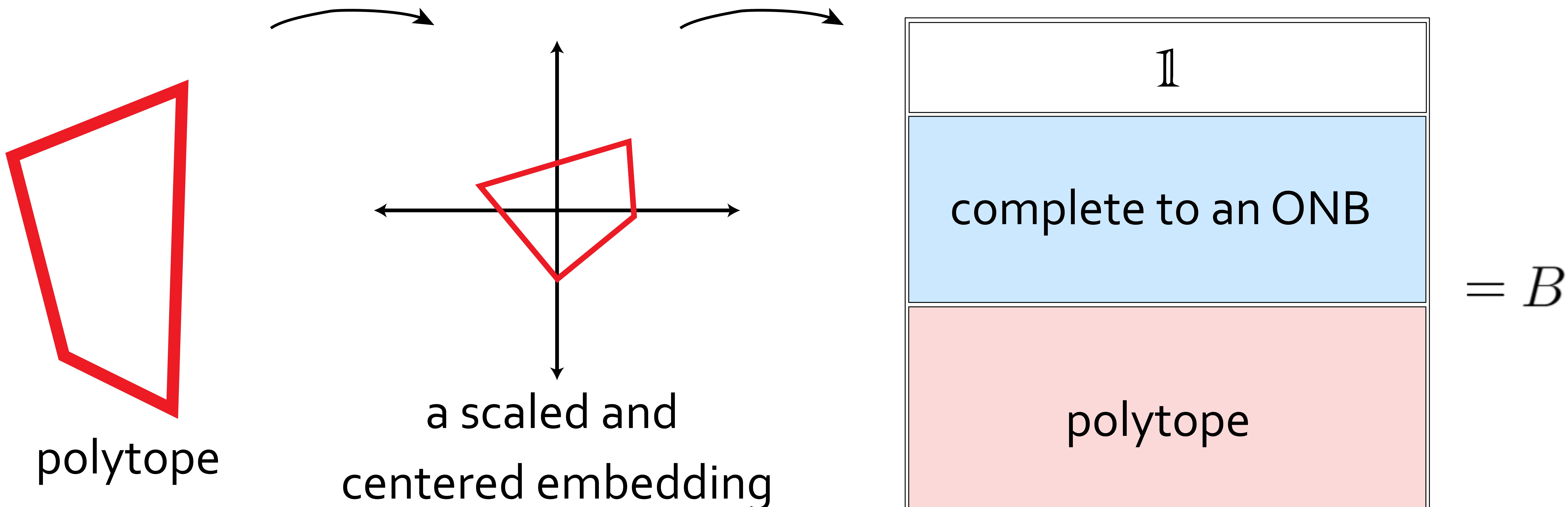
Output: $G = ([n], E, w)$ which has an eigenpolytope combinatorially equivalent to P.

can be done in $\mathcal{O}(n^3)$ time.



Universality of Eigenpolytopes

Corollary (B. and Shiroma): Every combinatorial polytope appears as the eigenpolytope of a positively weighted graph.



Complexity Results

Theorem (Chandrasekaran-Kabadi-Murty, Dyer, Fukuda-Liebling-Margot)**:**
The following decision problem is strongly NP-complete.

Instance: A polytope P described by its vertices

Question: Is P simplicial?

simplicial polytope = every facet is a simplex

.

Complexity Results

Theorem (Chandrasekaran-Kabadi-Murty, Dyer, Fukuda-Liebling-Margot)**:**
The following decision problem is strongly NP-complete.
Instance: A polytope P described by its vertices
Question: Is P simplicial?

simplicial polytope = every facet is a simplex

Theorem (B. and Shiroma)**:**
The following decision problem is strongly NP-complete.
Instance: $G = ([n], E, w)$ with an eigenspace ordering and $k \in \{2, \ldots, m-1\}$
Question: Does G have a k-graphical design with cardinality smaller than the “facet bound”?

Theorem: For each $k = 1, \ldots, m-1$ there is a positively weighted k-design of size at most $\sum_{i=1}^{k} \dim \Lambda_i$.

Complexity Results

Theorem (Dyer, Linial)**:**

The following counting problem is #P-complete.

Instance: A polytope P described by its vertices

Question: How many facets does P have?

Complexity Results

Theorem (Dyer, Linial)**:**

The following counting problem is #P-complete.

Instance: A polytope P described by its vertices

Question: How many facets does P have?

Theorem (B. and Shiroma)**:**

The following counting problem is #P-complete.

Instance: $G = ([n], E, w)$ with an eigenspace ordering and $k \in \{2, \dots, m-1\}$

Question: How many positively weighted k-graphical designs does G have?

A (very silly) LP Relaxation

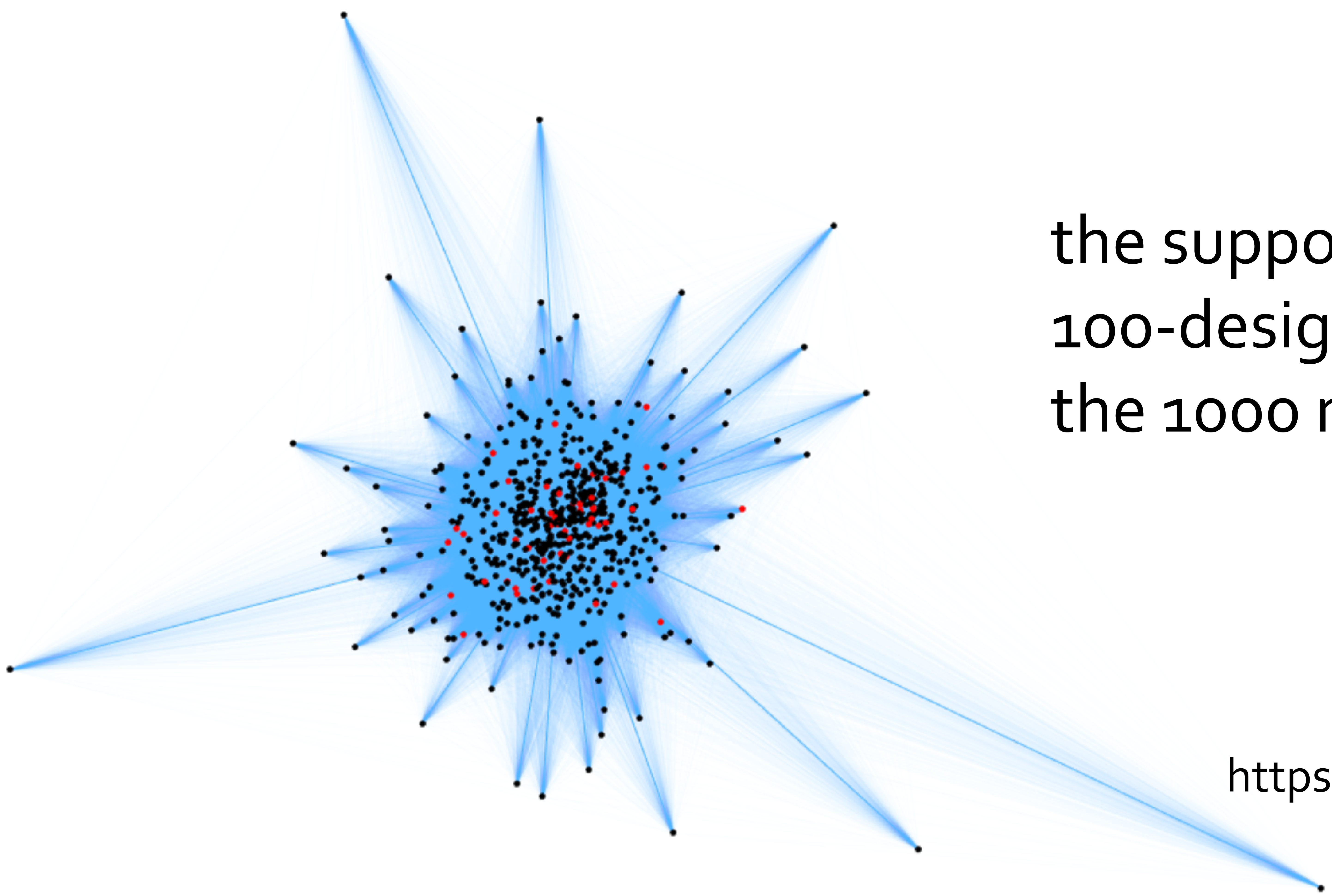
The following LP finds a (not necessarily minimal or minimum) positively weighted k-graphical design.

$$\min \|x\|_1 \qquad \text{s.t. } U_{\boldsymbol{k}} x = 0, \mathbb{1}^\top x = 1, x \geq 0$$

A (very silly) LP Relaxation

The following LP finds a (not necessarily minimal or minimum) positively weighted k-graphical design.

$$\min \|x\|_1 \qquad \text{s.t. } U_{\boldsymbol{k}} x = 0, \mathbb{1}^\top x = 1, x \geq 0$$



the support of a positively weighted 100-design on the similarity graph of the 1000 most common English words.

https://code.google.com/archive/p/word2vec/

Thank you!