On Dantzig-Wolfe Relaxation of Rank Constrained Optimization: Exactness, Rank Bounds, and Algorithms

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Question: If we cut the tennis ball $X$, is the convex hull of the intersection equal to the intersection of the convex hull?

$$\text{conv}(\mathcal{L} \cap X) = \mathcal{L} \cap \text{conv}(X)$$

How about cut twice, three times, ...?
Rank Constrained Optimization Problem (RCOP)

\[
(V_{\text{opt}}) \quad V_{\text{opt}} := \min_{X \in \mathcal{X}} \{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \}
\]

- where \( \mathcal{X} : \text{rank} - k \) constrained domain set

\[
\mathcal{X} := \{ X \in Q : \text{rank}(X) \leq k, F_j(X) \leq 0, \forall j \in [t] \}
\]

- Matrix space \( Q := S^n_+, S^n, \) or \( \mathbb{R}^{n \times p} \)

- \( F_j(\cdot) \) can be nonconvex
Special Case I: QCQP

\[
\text{(RCOP)} \quad V_{\text{opt}} := \min_{X \in \mathcal{X}} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \right\}
\]

- Quadratically constrained quadratic program (QCQP)

\[
\text{(QCQP)} \quad \min_{x \in \mathbb{R}^n} \left\{ x^T Q_0 x + q_0^T x : b_i^l \leq x^T Q_i x + q_i^T x \leq b_i^u, \forall i \in [m] \right\}
\]

- Introduce matrix variable \(X := \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}\)

\[
\text{(QCQP)} \quad \min_{X \in \mathcal{X}} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m], X_{11} = 1 \right\}
\]

- Domain set \(\mathcal{X} := \left\{ X \in S_n^{n+1} : \text{rank}(X) \leq 1 \right\}\)
Special Case II: Low-Rank Unsupervised Learning

\[(\text{RCOP}) \quad V_{opt} := \min_{X \in \mathcal{X}} \{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \}\]

- RCOP can cover Fair PCA (Tantipongpipat et al. 2019)

\[(\text{Fair PCA}) \quad \max_{(z, X) \in \mathbb{R} \times \mathcal{X}} \{ z : z \leq \langle A_i, X \rangle, \forall i \in [m] \}\]

- Domain set \(\mathcal{X} := \{ X \in S^n_+ : \text{rank}(X) \leq k, \|X\|_2 \leq 1 \}\)

- Matrix completion, signal processing, experimental design…
Special Case III: Sparse Optimization

\[ (\text{RCOP}) \quad V_{\text{opt}} := \min_{X \in \mathcal{X}} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \right\} \]

- RCOP covers Sparse Optimization

\[ \min_{(z, X) \in \mathbb{R} \times \mathcal{X}} \left\{ z : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \right\} \]

- Domain set

\[ \mathcal{X} := \{ X \in S^n : \text{rank}(X) \leq k, X = \text{Diag}(\text{diag}(X)), \|y - A \text{diag}(X)\|_2 \leq z \} \]

\[ \|\text{diag}(X)\|_0 \leq k \]
Dantzig-Wolfe Relaxation (DWR)

\( \text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\mathbf{X} \in \mathcal{X}} \left\{ \langle \mathbf{A}_0, \mathbf{X} \rangle : b_i^l \leq \langle \mathbf{A}_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \right\} \)

\( \text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\mathbf{X} \in \text{conv}(\mathcal{X})} \left\{ \langle \mathbf{A}_0, \mathbf{X} \rangle : b_i^l \leq \langle \mathbf{A}_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \right\} . \)

- Replace domain set \( \mathcal{X} \) by its convex hull \( \text{conv}(\mathcal{X}) \)
- Feasible set of RCOP \( \mathcal{C} \): intersecting domain set \( \mathcal{X} \) with \( m \) two-sided LMIs
- Feasible set of DWR \( \mathcal{C}_{\text{rel}} \): intersecting \( \text{conv}(\mathcal{X}) \) with \( m \) two-sided LMIs
RCOP and DWR

(RCOP) \( V_{\text{opt}} := \min_{X \in \mathcal{C}} \langle A_0, X \rangle \geq 0 \) (DWR) \( V_{\text{rel}} := \min_{X \in \mathcal{C}_{\text{rel}}} \langle A_0, X \rangle \)

- Feasible set \( \mathcal{C} \): intersecting domain set \( \mathcal{X} \) with \( m \) two-sided LMIs
- Feasible set \( \mathcal{C}_{\text{rel}} \): intersecting \( \text{conv}(\mathcal{X}) \) with \( m \) two-sided LMIs

**Observation.** \( V_{\text{opt}} \geq V_{\text{rel}}, \text{conv}(\mathcal{C}) \subseteq \mathcal{C}_{\text{rel}}. \)
Goal: Show $\text{DWR} \equiv \text{RCOP}$

\[
(V_{\text{opt}} := \min_{x \in \mathcal{C}} \langle A_0, x \rangle) \quad (V_{\text{rel}} := \min_{x \in \mathcal{C}_{\text{rel}}} \langle A_0, x \rangle)
\]

- We would like to understand when $\text{DWR} \equiv \text{RCOP}$
  - $\mathcal{C}_{\text{rel}} = \text{conv}(\mathcal{C})$ or
  - $V_{\text{opt}} = V_{\text{rel}}$
Exactness Notion of DWR — Geometric View

\[(\text{RCOP}) \quad \mathcal{V}_{opt} := \min_{\mathbf{x} \in \mathcal{C}} \langle \mathbf{A}_0, \mathbf{X} \rangle, \quad (\text{DWR}) \quad \mathcal{V}_{rel} := \min_{\mathbf{x} \in \mathcal{C}_{rel}} \langle \mathbf{A}_0, \mathbf{X} \rangle\]

Example: Intersect $\mathcal{X}$ with $m = 1$ LMI
**Exactness Notion of DWR — Optimality View**

\[
(RCOP) \quad V_{opt} := \min_{X \in \mathcal{C}} \langle A_0, X \rangle, \quad (DWR) \quad V_{rel} := \min_{X \in \mathcal{C}_{rel}} \langle A_0, X \rangle
\]

**Extreme Point Exactness**
\[\text{ext}(\mathcal{C}_{rel}) \subseteq \mathcal{C}\]

**Objective Exactness**
\[V_{rel} = V_{opt}\]

**Convex Hull Exactness**
\[\mathcal{C}_{rel} = \text{conv}(\mathcal{C})\]

**Objective Exactness** given special families of linear objective functions
\[V_{rel} = V_{opt}\]
Literature Review

- DWR exactness conditions for the (QCQP) $\mathcal{X} := \{ X \in S^n_+: \text{rank}(X) \leq 1 \}$

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<td>Slater condition</td>
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<td>Graph Structure</td>
<td>Kim and Kojima (2003), Sojoudi and Lavaei (2014), Burer and Ye (2020), Azuma et al. (2022)…</td>
<td>Sufficient condition for objective exactness</td>
<td>Nonnegative coefficients; Slater condition; Bipartite graph</td>
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<td>Convex Lagrange dual multipliers</td>
<td>Wang and Kılınç-Karzan (2020, 2021, 2022)…</td>
<td>Necessary and sufficient conditions for convex hull, objective exactness</td>
<td>Dual Slater condition; Polyhedral dual set</td>
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- Note:
  - Mainly from the dual space with Slater condition
  - Ours is primal perspective
Main Contributions to the DWR Exactness

- Existing results: recover them and remove their assumptions
- New Results: exactness for IQP-2 and Fair SVD

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<td>TRS</td>
<td>single ball constraint</td>
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<td>GTRS</td>
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<td>Two-sided GTRS</td>
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<td>(Q_1 \neq 0;) (-\infty &lt; b_1^l \leq b_1^u &lt; +\infty)</td>
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<td>HQP-2</td>
<td>homogeneous QCQP with two quadratic constraints</td>
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<td>IQP-2</td>
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<td>three groups</td>
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Main Contributions on DWR Exactness

- We derive the “if and only if” conditions for all the three exactness notions
  - Beyond the QCQP
  - Primal perspective
  - Remove many assumptions in the literature, e.g., Slater condition
  - Geometric interpretation

- Generalize and extend exactness results for applications problems in QCQP and fair unsupervised learning
Extreme Point Exactness
“iff” Condition of the Extreme Point Exactness

\[ \text{ext}(\mathcal{C}_{rel}) \leq \mathcal{C} \]

Given \( \mathcal{C}_{rel} \) is bounded

Objective Exactness
\[ V_{rel} = V_{opt} \]

for any \( A_0 \)

for some favorable \( A_0 \)

Objective Exactness given special families of linear objective functions
\[ V_{rel} = V_{opt} \]
“iff” Condition of the Extreme Point Exactness

Recall. \( \mathcal{C} := \{ \mathbf{X} \in \mathcal{X} : b_i^l \leq \langle A_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \} \)

\( \mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle A_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \} \)

○ When \( \text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C} \)? Depend on \( \leq m\text{-dim faces} \) in \( \text{conv}(\mathcal{X}) \)

Theorem.

All extreme points in set \( \mathcal{C}_{\text{rel}} \) belong to \( \mathcal{C} \) \hspace{2cm} \text{“iff”} \hspace{2cm} \text{Any } \leq m\text{-dimensional face in } \text{conv}(\mathcal{X}) \text{ is contained in } \mathcal{X} \)
Definition. For a closed convex set $D$, a convex subset $F$ of $D$ is called a face if for any line segment $[a, b] \subseteq D$ such that $[a, b] \cap F \neq \emptyset$, we have $[a, b] \subseteq F$. 
What Faces are Extreme Points of $C_{rel}$ Located?

$C_{rel} := \{ X \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \}$

- Extreme Points of $C_{rel}$ lie on $\leq m$-dim faces in $\text{conv}(\mathcal{X})$
  - Hold for any $m$ LMIs
- Example: $\mathcal{X} := \{ X \in S^2_+: \text{rank}(X) \leq 1, X_{12} = 0 \}$
  - Add $m = 1$ LMI: $X_{11} + X_{22} \leq 1$

$\mathcal{X}$: nonnegative axes
$\text{conv}(\mathcal{X})$: nonnegative orthant

- $\text{ext}(C_{rel}) = \{ a_1, a_2, a_3 \} \subseteq C$
- All the extreme points lie on the Point and Edges (i.e., $\leq 1$-dim faces) in $\text{conv}(\mathcal{X})$
What Faces are Extreme Points of $\mathcal{C}_{\text{rel}}$ Located?

$$\mathcal{C}_{\text{rel}} := \{ \mathbf{X} \in \text{conv}(\mathcal{X}) : b^l_i \leq \langle \mathbf{A}_i, \mathbf{X} \rangle \leq b^u_i, \forall i \in [m] \}$$

- Extreme Points of $\mathcal{C}_{\text{rel}}$ lie on $\leq m$-dim faces in $\text{conv}(\mathcal{X})$
  - Hold for any $m$ LMIs
- Example: Add $m = 1$ LMI to set $\mathcal{X}$

- $\text{ext}(\mathcal{C}_{\text{rel}}) = \{a_1\} \subseteq \mathcal{C}$
- The extreme point is a Point in $\text{conv}(\mathcal{X})$
“iff” Condition of the Extreme Point Exactness

Recall. \[ C := \{ X \in \mathcal{X} : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \} \]

\[ C_{rel} := \{ X \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \} \]

- Given \( \text{ext}(C_{rel}) \) is contained in \( \leq m\)-dim faces in \( \text{conv}(\mathcal{X}) \), when \( \text{ext}(C_{rel}) \subseteq C \) holds?

Theorem.

All extreme points in set \( C_{rel} \) belong to \( C \) \hspace{1cm} “iff” \hspace{1cm} Any \( \leq m \)-dimensional face in \( \text{conv}(\mathcal{X}) \) is contained in \( \mathcal{X} \)

Proof.

Sufficiency. Any extreme point \( Y \) of \( C_{rel} \) belongs to \( \leq m\)-dim faces in \( \text{conv}(\mathcal{X}) \) \( \subseteq \mathcal{X} \). And \( Y \) satisfies the \( m \) LMIs and thus \( Y \in C \).

Necessity. Prove by contradiction.
Geometric Interpretation of “iff” Condition

Recall. \[ \mathcal{C} := \{ \mathbf{X} \in \mathcal{X} : b_i^l \leq \langle \mathbf{A}_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \} \]
\[ \mathcal{C}_{rel} := \{ \mathbf{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle \mathbf{A}_i, \mathbf{X} \rangle \leq b_i^u, \forall i \in [m] \} \]

Theorem.

All extreme points in set \( \mathcal{C}_{rel} \) belong to \( \mathcal{C} \) \hspace{2cm} “iff” \hspace{2cm} Any \( \leq m \)-dimensional face in \( \text{conv}(\mathcal{X}) \) is contained in \( \mathcal{X} \)

Step I: where are extreme points in \( \mathcal{C}_{rel} \) located for any \( m \) LMIs?

- On \( \leq m \)-dim faces of \( \text{conv}(\mathcal{X}) \)!

Step II: when set \( \mathcal{C} \) contains these extreme point locations?

- Any \( \leq m \)-dim face in \( \text{conv}(\mathcal{X}) \) belongs to \( \mathcal{X} \)
Application: QCQP

- For QCQP, \( \mathcal{X} := \{ \mathbf{X} \in S_+^n : \text{rank}(\mathbf{X}) \leq 1 \} \) and \( \text{conv}(\mathcal{X}) := S_+^n \)
- Any \( \leq 2 \)-dim face of \( \text{conv}(\mathcal{X}) \) is contained in \( \mathcal{X} \)

- **Point**: 0-dim face; **Edge**: 1-dim face; **Plane**: 2-dim face

**Lemma.** For QCQP, any \( \leq 2 \)-dim face of \( \text{conv}(\mathcal{X}) \) is contained in \( \mathcal{X} \).
Extending Many Interesting Results in QCQP

**Lemma.** For QCQP, any $\leq 2$-dim face of $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$.

- Using “iff” condition of extreme point exactness

**Theorem.** For QCQP, its DWR attains extreme point exactness whenever there are any $\leq 2$ LMIs.

- Trust region subproblem (TRS)
- Generalized TRS
- Two-sided generalized TRS
- Homogeneous QCQP with 2 quadratic constraints
- Inhomogeneous QCQP with 2 homogeneous quadratic constraints
Convex Hull Exactness
“iff” Condition of the Convex Hull Exactness

$\text{ext}(C_{\text{rel}})$: all extreme points

$\text{ext}(C_{\text{rel}}) \subseteq C$

Convex Hull Exactness

$C_{\text{rel}} = \text{conv}(C)$

given $C_{\text{rel}}$ is bounded

Objective Exactness

$V_{\text{rel}} = V_{\text{opt}}$

for any $A_0$

for some favorable $A_0$

Objective Exactness given special families of linear objective functions

$V_{\text{rel}} = V_{\text{opt}}$
Convex Hull Exactness $\geq$ Extreme Point Exactness

- Example: $\mathcal{X} := \{X \in S_+^2: \text{rank}(X) \leq 1, X_{12} = 0\}$
  - Add $m = 1$ LMI: $X_{11} \leq X_{22}$

- Extreme point exactness holds, while convex hull exactness does not
- One-dim faces $F_1, F_2$ lie on the Edge and Plane (i.e., $\leq 2$-dim faces) in $\text{conv}(\mathcal{X})$
What Faces are Extreme Directions of $\mathcal{C}_{\text{rel}}$ Located?

\[ \mathcal{C}_{\text{rel}} := \{ X \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \} \]

- Where are the one-dim faces of the recession cone of $\mathcal{C}_{\text{rel}}$ located?
  - On $\leq (m + 1)$-dim face of the recession cone of $\text{conv}(\mathcal{X})$

**Lemma.** For any $m$ LMIs, each one-dim face of the intersection set $\mathcal{C}_{\text{rel}}$ is contained in a $\leq (m + 1)$-dim face of the recession cone of $\text{conv}(\mathcal{X})$. 

Convex Hull Exactness
“iff” Condition of the Convex Hull Exactness

- When the domain set $\mathcal{X}$ is conic, the sufficient condition
  - Reduces to “Any $\leq (m + 1)$-dim face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$”
  - becomes necessary

**Theorem.** When the domain set $\mathcal{X}$ is conic and pointed.

Set $\mathcal{C}_{\text{rel}}$ is identical to the convex hull of set $\mathcal{C}$

“iff”

Any $\leq (m + 1)$-dim face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$
Lemma. For QCQP, any $\leq 2$-dim face of $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$.

- Using “iff” condition of convex hull exactness

Theorem. For homogeneous QCQP with 1 quadratic constraint, convex hull exactness holds.
For Fair PCA (Tantipongpipat et al. 2019)

\[
(Fair \ PCA) \quad \max_{(z, \mathbf{X}) \in \mathbb{R} \times \mathcal{X}} \{ z : z \leq \langle \mathbf{A}_i, \mathbf{X} \rangle, \forall i \in [m] \}
\]

- Domain set \( \mathcal{X} := \{ \mathbf{X} \in \mathbb{S}_+^n : \text{rank}(\mathbf{X}) \leq k, \|\mathbf{X}\|_2 \leq 1 \} \)

**Theorem.**
For Fair PCA with two groups \( m = 2 \), the convex hull exactness holds.
Solution Algorithms
Column Generation Algorithm for Solving DWR

(DWR) \[ V_{\text{rel}} := \min_{X \in \text{conv}(\mathcal{X})} \left\{ \langle A_0, X \rangle : b_i^l \leq \langle A_i, X \rangle \leq b_i^u, \forall i \in [m] \right\}. \]

- Given the spectral domain set \( \mathcal{X} \), we explicitly described \( \text{conv}(\mathcal{X}) \)

- Proposition [Kim et al., 2021] When \( Q := S^n_+ \) denotes the positive semidefinite matrix space, we show \( \text{conv}(\mathcal{X}) = \text{proj}_X(\mathcal{Y}) \), where
  \[ \mathcal{Y} := \left\{ (X, x) \in Q \times \mathbb{R}^n_+ : f_j(x) \leq 0, \forall j \in [t], x_1 \geq \cdots \geq x_n, x_{k+1} = 0, x \geq \lambda(X) \right\}. \]

- Computationally expensive to formulate \( \text{conv}(\mathcal{X}) \)
  - Extended space
  - Majorization constraint
Given the explicit characterization of \( \text{conv}(\mathcal{X}) \),

- Directly use off-the-shelf solvers (Mosek) to solve DWR
  - Computationally expensive

- Column generation algorithm: at each iteration, directly solve the **pricing problem** over \( \text{conv}(\mathcal{X}) \)

\[
\text{(Pricing)} \quad \min_{\mathbf{X} \in \text{conv}(\mathcal{X})} \langle \mathbf{C}_t, \mathbf{X} \rangle = \min_{\mathbf{X} \in \mathcal{X}} \langle \mathbf{C}_t, \mathbf{X} \rangle
\]
Pricing Problem = A Simple Convex Program

\[ \min_{\mathcal{X} \subseteq \text{conv}(\mathcal{X})} \langle C_t, X \rangle = \min_{X \in \mathcal{X}} \langle C_t, X \rangle \]

**Theorem.** For the spectral domain set,

\[ \mathcal{X} := \{X \in \mathbb{Q} : \text{rank}(X) \leq k, F_j(X) := f_j(\lambda(X)) \leq 0, \forall j \in [t] \} \]

the pricing problem reduced the following vector-based convex optimization:

\[ \lambda^* := \arg\max_{\lambda \in \mathbb{R}^n_+} \{ \lambda^T \beta : \lambda_i = 0, \forall i \in [k+1,n], f_j(\lambda) \leq 0, \forall j \in [t] \}. \]
# Numerical Study: Compare Three Methods

\[(\text{DWR}) \quad V_{\text{rel}} := \min_{X \in \text{conv}(\mathcal{X})} \left\{ \langle A_0, X \rangle : b^l_i \leq \langle A_i, X \rangle \leq b^u_i, \forall i \in [m] \right\} \]

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<th>Need conv(\mathcal{X})?</th>
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<td>Mosek</td>
<td>Plug conv(\mathcal{X}) and directly solve the DWR</td>
<td>Yes</td>
</tr>
<tr>
<td>Naïve CG</td>
<td>Solve pricing problem over conv(\mathcal{X}) formulation</td>
<td>Yes</td>
</tr>
<tr>
<td>Proposed CG</td>
<td>Use vector-based reduction</td>
<td>No</td>
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- CG: Column Generation
- conv(\mathcal{X}) is an SDP formulation
Numerical Study: MIMO Network with $k \geq 1$

- Multiple-input and multiple-output (MIMO) radio network
  - The data streams at a transmitter $\leq$ the number of transmit antennas
  - Rank-$k$ constraint on the covariance matrix
  - Find the low rank data streams to minimize the total interference power

- Yu and Lau (2010) proposed a RCOP-type model with
  - $\mathcal{X} := \{ X \in \mathcal{S}_+^n : \text{rank}(X) \leq k, \logdet(I + X) \geq r, \text{tr}(X) \leq R \}$
  - $I$: identity matrix
  - Spectral domain set $\mathcal{X}$
Numerical Study: MIMO Network with $k \geq 1$

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</table>

- “--”: cannot be solved within 3600 seconds
- “*”: infeasible solution
Numerical Study: QCQP with $k = 1$

- Optimal Power Flow (OPF) problem is a classic QCQP (Eltved and Burer 2020)

  \[
  \text{(OPF)} \quad \min_{x \in \mathbb{R}^n} \{ x^\top Q_0 x + q_0^\top x : r \leq x^\top x \leq R, \ b_i^l \leq x^\top Q_i x + q_i^\top x \leq b_i^u, \forall i \in [m] \} 
  \]

- Introduce matrix variable $X := \begin{pmatrix} 1 & x^\top \\ x & xx^\top \end{pmatrix}$

- Move $r \leq \text{tr}(X) - 1 \leq R$ into the domain set

- Reformulate OPF as a RCOP-type model with
  - $\mathcal{X} := \{X \in S^{n+1}_+ : \text{rank}(X) \leq 1, r \leq \text{tr}(X) - 1 \leq R \}$
  - Spectral domain set $\mathcal{X}$
  - $\text{conv}(\mathcal{X}) := \{X \in S^{n+1}_+ : r \leq \text{tr}(X) - 1 \leq R \}$
## Numerical Study: QCQP with $k = 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Rank-$k$</th>
<th>$m$ LMIs</th>
<th>Mosek</th>
<th>Naïve CG</th>
<th>Our CG</th>
<th>Theory Rank Bound</th>
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</tbody>
</table>

- “--”: cannot be solved within 3600 seconds
- “*”: infeasible solution
Summary

- Study a rank-constrained optimization problem (RCOP)
  - General framework
  - Dantzig-Wolfe Relaxation (DWR)

- Derive “if and only if” conditions for the three DWR exactness
  - Only depend on the faces of the convex hull of domain $\mathcal{X}$
  - Geometric interpretation

- Beyond exactness, we derive rank bounds

- Column generation algorithm works well
Thank You!

Preprint is available at https://arxiv.org/pdf/2210.16191