On Dantzig-Wolfe Relaxation of Rank Constrained Optimization: Exactness, Rank Bounds, and Algorithms

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March 3, 2023

Linear and Non-Linear Mixed Integer Optimization Workshop

Motivation

Nonconvex ***X***

- Question: If we cut the tennis ball ***X***, is the convex hull of the intersection equal to the intersection of the convex hull?

$$\mathrm{conv}(\mathfrak{L} \cap X) = \mathfrak{L} \cap \mathrm{conv}(X)$$

  - How about cut twice, three times, …?

Rank Constrained Optimization Problem (RCOP)

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{X}} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

- where $\mathcal{X}$: rank$-k$ constrained domain set

$$\mathcal{X} := \{ \boldsymbol{X} \in \mathcal{Q} : \text{rank}(\boldsymbol{X}) \leq k, F_j(\boldsymbol{X}) \leq 0, \forall j \in [t] \}$$

- Matrix space $\mathcal{Q} := \mathcal{S}_+^n,\ \mathcal{S}^n$, or $\mathbb{R}^{n \times p}$

- $F_j(\cdot)$ can be nonconvex

Special Case I: QCQP

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{X}} \{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \le b_i^u, \forall i \in [m] \}$$

- Quadratically constrained quadratic program (QCQP)

$$\text{(QCQP)} \quad \min_{\boldsymbol{x} \in \mathbb{R}^n} \{ \boldsymbol{x}^\top \boldsymbol{Q}_0 \boldsymbol{x} + \boldsymbol{q}_0^\top \boldsymbol{x} : b_i^l \le \boldsymbol{x}^\top \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^\top \boldsymbol{x} \le b_i^u, \forall i \in [m] \}$$

- Introduce matrix variable $\boldsymbol{X} := \begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^\top \end{pmatrix}$

$$\text{(QCQP)} \quad \min_{\boldsymbol{X} \in \mathcal{X}} \{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \le b_i^u, \forall i \in [m], X_{11} = 1 \}$$

  - Domain set $\mathcal{X} := \{ \boldsymbol{X} \in \mathcal{S}_+^{n+1} : \text{rank}(\boldsymbol{X}) \le 1 \}$

Special Case II: Low-Rank Unsupervised Learning

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{X}} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

- RCOP can cover Fair PCA (Tantipongpipat et al. 2019)

$$\text{(Fair PCA)} \quad \max_{(z, \boldsymbol{X}) \in \mathbb{R} \times \mathcal{X}} \{ z : z \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle, \forall i \in [m] \}$$

  - Domain set $\mathcal{X} := \{ \boldsymbol{X} \in \mathcal{S}_+^n : \text{rank}(\boldsymbol{X}) \leq k, ||\boldsymbol{X}||_2 \leq 1 \}$

- Matrix completion, signal processing, experimental design…

Special Case III: Sparse Optimization

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{X}} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

- RCOP covers Sparse Optimization

$$\min_{(z,\boldsymbol{X}) \in \mathbb{R} \times \mathcal{X}} \{ z : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \}$$

- Domain set

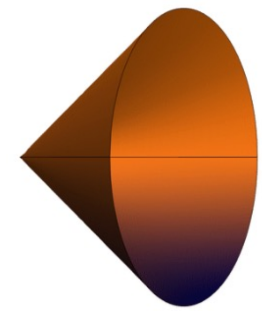
$$\mathcal{X} := \{ \boldsymbol{X} \in \mathcal{S}^n : \text{rank}(\boldsymbol{X}) \leq k, \boldsymbol{X} = \text{Diag}(\text{diag}(\boldsymbol{X})), \|\boldsymbol{y} - \boldsymbol{A}\,\text{diag}(\boldsymbol{X})\|_2 \leq z \}$$

⇕

$$||\boldsymbol{diag}(\boldsymbol{X})||_0 \leq k$$

Dantzig-Wolfe Relaxation (DWR)

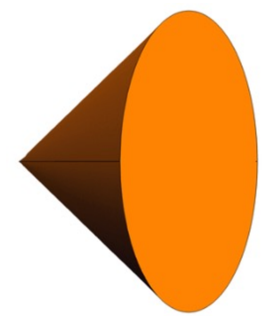
$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{X}} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$



$\mathcal{X}$

Relaxation

$$\text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \text{conv}(\mathcal{X})} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}.$$



conv($\mathcal{X}$)

- Replace domain set $\mathcal{X}$ by its convex hull conv($\mathcal{X}$)
- Feasible set of RCOP $\mathcal{C}$: intersecting domain set $\mathcal{X}$ with m two-sided LMIs
- Feasible set of DWR $\mathcal{C}_{\text{rel}}$: intersecting conv($\mathcal{X}$) with m two-sided LMIs

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{C}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle, \quad \text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \mathcal{C}_{\text{rel}}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle$$

RCOP and DWR

$$(\text{RCOP}) \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X}\in\mathcal{C}} \langle \boldsymbol{A}_0, \boldsymbol{X} \;\geq\; \text{DWR}) \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X}\in\mathcal{C}_{\text{rel}}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle$$

- Feasible set $\mathcal{C}$: intersecting domain set $\mathcal{X}$ with m two-sided LMIs
- Feasible set $\mathcal{C}_{\text{rel}}$: intersecting $\text{conv}(\mathcal{X})$ with m two-sided LMIs

Observation. $\text{V}_{\text{opt}} \geq \text{V}_{\text{rel}}$, $\text{conv}(\mathcal{C}) \subseteq \mathcal{C}_{\text{rel}}$.

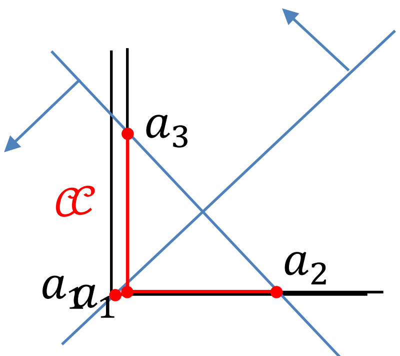
Goal: Show DWR ≡ RCOP

$$(\text{RCOP}) \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{C}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle, \equiv \quad (\text{DWR}) \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \mathcal{C}_{\text{rel}}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle$$

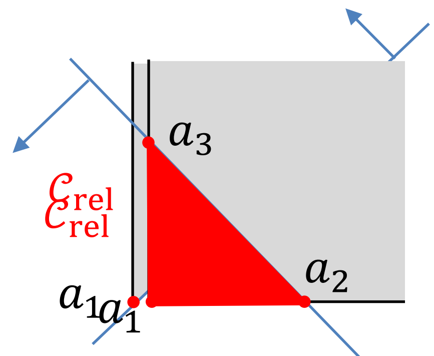
- We would like to understand when DWR ≡ RCOP
  - $\mathcal{C}_{\text{rel}} = \text{conv}(\mathcal{C})$ or
  - $\text{V}_{\text{opt}} = \text{V}_{\text{rel}}$

Exactness Notion of DWR — Geometric View

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{C}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle, \quad \text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \mathcal{C}_{\text{rel}}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle$$



$\mathcal{X}$: nonnegative axes

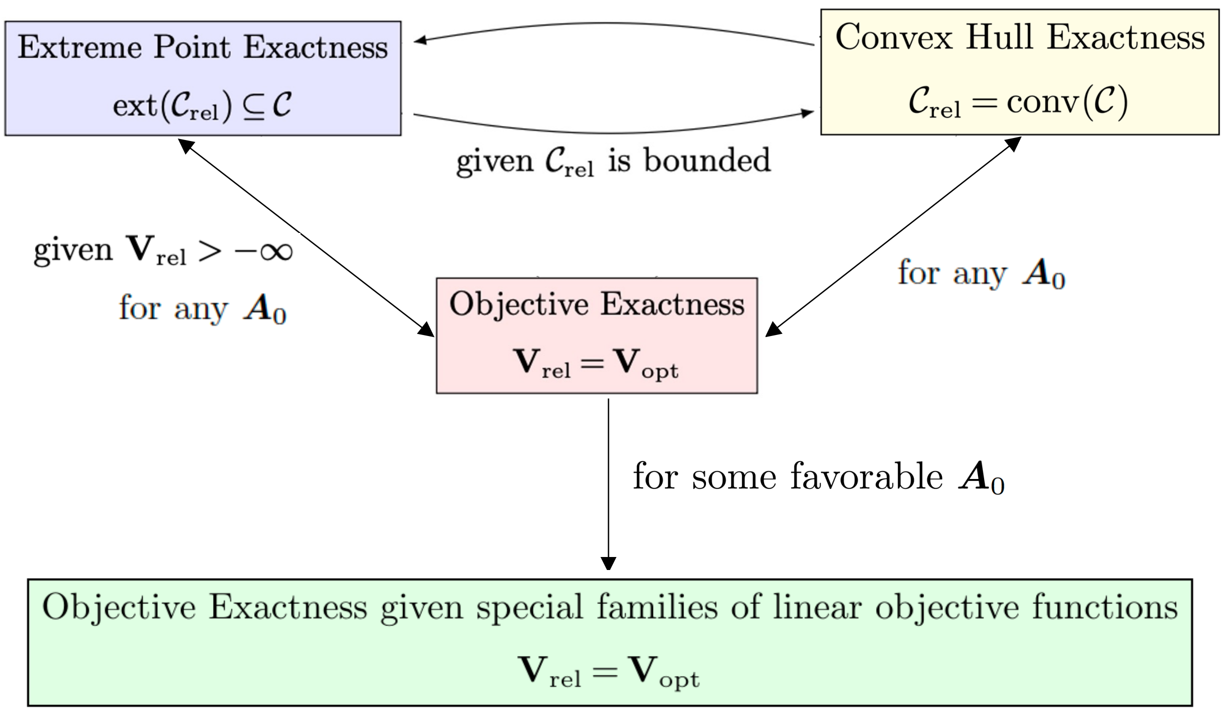


conv($\mathcal{X}$): nonnegative orthant

Example: Intersect $\mathcal{X}$ with $m = 1$ LMI

Exactness Notion of DWR — Optimality View

$$\text{(RCOP)} \quad \mathbf{V}_{\text{opt}} := \min_{\boldsymbol{X} \in \mathcal{C}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle, \quad \text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \mathcal{C}_{\text{rel}}} \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle$$



Literature Review

- DWR exactness conditions for the (QCQP) $\mathcal{X} := \{\boldsymbol{X} \in \mathcal{S}_+^n : \operatorname{rank}(\boldsymbol{X}) \leq 1\}$

Method	Literature	Result	Assumption
S-lemma	Yakubovich (1971), Fradkov and Yakubovich (1979), Sturm and Zhang (2003) …	Sufficient condition for objective exactness	Slater condition
Graph Structure	Kim and Kojima (2003), Sojoudi and Lavaei (2014), Burer and Ye (2020), Azuma et al. (2022) …	Sufficient condition for objective exactness	Nonnegative coefficients; Slater condition; Bipartite graph
Convex Lagrange dual multipliers	Wang and Kılınc-Karzan (2020, 2021, 2022)…	Necessary and sufficient conditions for convex hull, objective exactness	Dual Slater condition; Polyhedral dual set

- Note:
  - Mainly from the dual space with Slater condition
  - Ours is primal perspective

Main Contributions to the DWR Exactness

- Existing results: recover them and remove their assumptions
- New Results: exactness for IQP-2 and Fair SVD

Application	Problem	Setting	Exactness result	Assumption
QCQP ($k=1$)	QCQP-1	single quadratic constraint	extreme point	–
	TRS	single ball constraint	convex hull	–
	GTRS	single quadratic inequality constraint	convex hull	–
	Two-sided GTRS	single two-sided quadratic constraint	extreme point	$\boldsymbol{Q}_1 \neq \boldsymbol{0}$; $-\infty < b_1^l \leq b_1^u < +\infty$
	HQP-2	homogeneous QCQP with two quadratic constraints	extreme point	
	IQP-2	inhomogeneous objective with two homogeneous quadratic constraints	extreme point	–
Fair Unsupervised Learning ($k \geq 1$)	Fair PCA	two groups	convex hull	–
	Fair SVD	three groups	convex hull	–

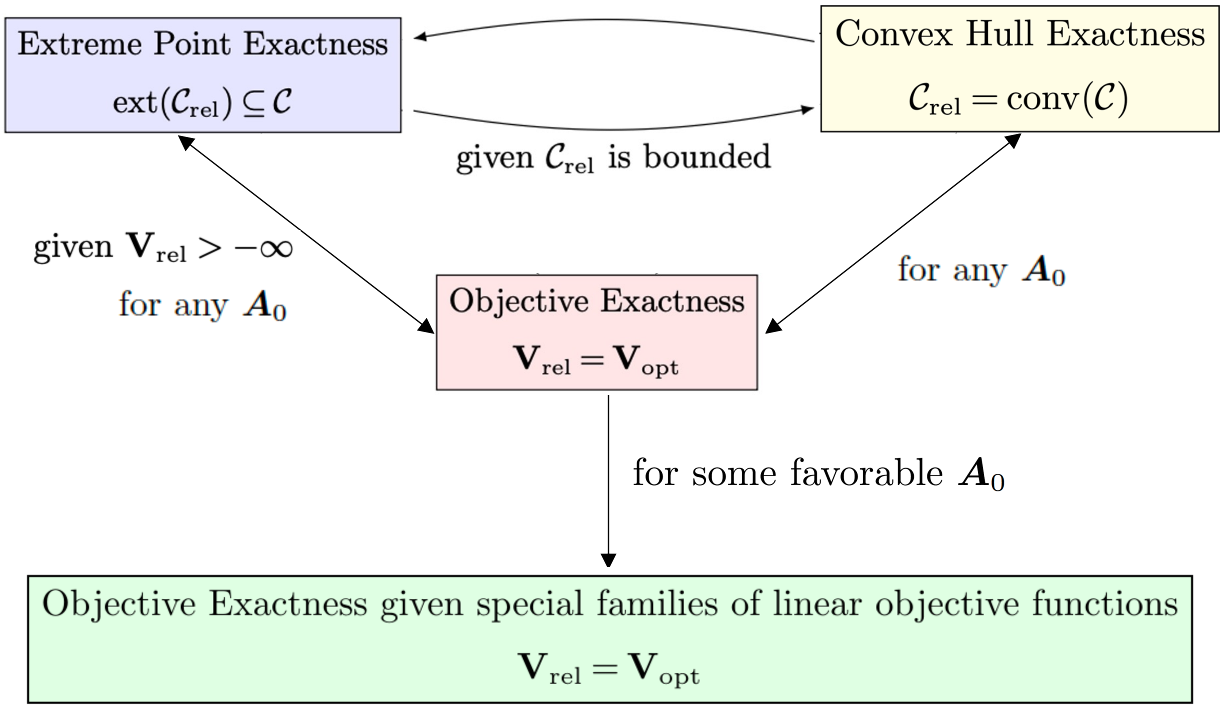
Main Contributions on DWR Exactness

- We derive the "if and only if" conditions for all the three exactness notions
  - Beyond the QCQP
  - Primal perspective
  - Remove many assumptions in the literature, e.g., Slater condition
  - Geometric interpretation
- Generalize and extend exactness results for applications problems in QCQP and fair unsupervised learning

Extreme Point Exactness

"iff" Condition of the Extreme Point Exactness

$\mathrm{ext}(\mathcal{C}_{\mathrm{rel}})$: all extreme points



"iff" Condition of the Extreme Point Exactness

Recall. $\mathcal{C} := \{\boldsymbol{X} \in \mathcal{X} : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m]\}$

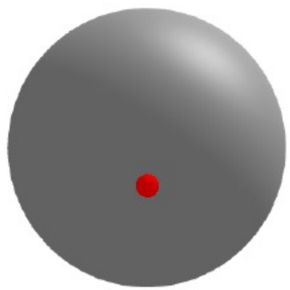
$$\mathcal{C}_{\mathrm{rel}} := \{\boldsymbol{X} \in \mathrm{conv}(\mathcal{X}) : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m]\}$$

- When $\mathrm{ext}(\mathcal{C}_{\mathrm{rel}}) \subseteq \mathcal{C}$? Depend on $\leq$ m-dim faces in $\mathrm{conv}(\mathcal{X})$

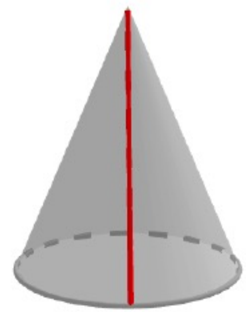
Theorem.

All extreme points in set $\mathcal{C}_{\mathrm{rel}}$ belong to $\mathcal{C}$ ⟷ "iff" ⟷ Any $\leq$ m-dimensional face in $\mathrm{conv}(\mathcal{X})$ is contained in $\mathcal{X}$

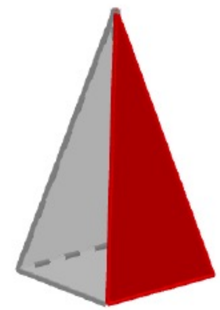
What are Faces?



0-dim face
(Point)



1-dim face
(Edge)



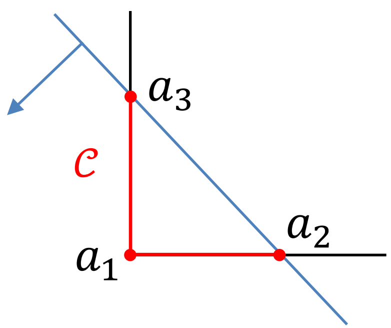
2-dim face
(Plane)

Definition. For a closed convex set D, a convex subset F of D is called a face if for any line segment $[a, b] \subseteq D$ such that $[a, b] \cap F \neq \emptyset$, we have $[a, b] \subseteq F$.

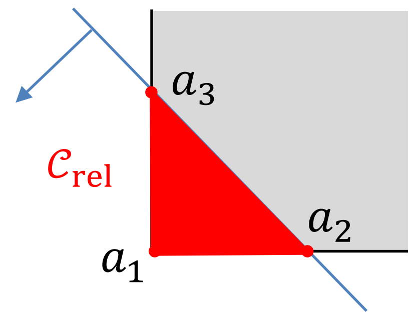
What Faces are Extreme Points of $\mathcal{C}_{\text{rel}}$ Located?

$$\mathcal{C}_{\text{rel}} := \left\{ \boldsymbol{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

- Extreme Points of $\mathcal{C}_{\text{rel}}$ lie on $\leq$ m-dim faces in $\text{conv}(\mathcal{X})$
  - Hold for any m LMIs
- Example: $\mathcal{X} \coloneqq \{\boldsymbol{X} \in \mathcal{S}_+^2 \colon \text{rank}(\boldsymbol{X}) \leq 1, X_{12} = 0\}$
  - Add $m = 1$ LMI: $X_{11} + X_{22} \leq 1$



$\mathcal{X}$: nonnegative axes



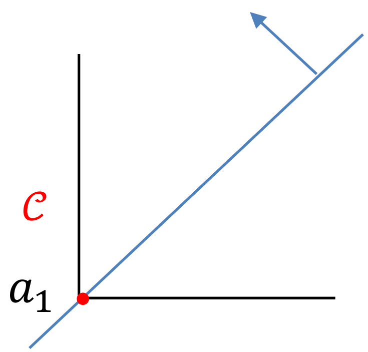
$\text{conv}(\mathcal{X})$: nonnegative orthant

- $\text{ext}(\mathcal{C}_{\text{rel}}) = \{a_1, a_2, a_3\} \subseteq \mathcal{C}$
- All the extreme points lie on the **Point and Edges** (i.e., $\leq$ 1-dim faces) in $\text{conv}(\mathcal{X})$

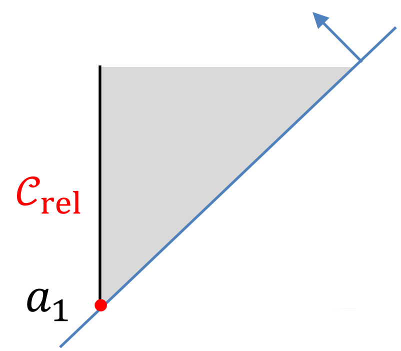
What Faces are Extreme Points of $\mathcal{C}_{\text{rel}}$ Located?

$$\mathcal{C}_{\text{rel}} := \{\boldsymbol{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m]\}$$

- Extreme Points of $\mathcal{C}_{\text{rel}}$ lie on $\leq$ m-dim faces in $\text{conv}(\mathcal{X})$
  - Hold for any m LMIs
- Example: Add $m = 1$ LMI to set $\mathcal{X}$



$\mathcal{X}$: nonnegative axes



$\text{conv}(\mathcal{X})$: nonnegative orthant

- $\text{ext}(\mathcal{C}_{\text{rel}}) = \{a_1\} \subseteq \mathcal{C}$
- The extreme point is a **Point** in $\text{conv}(\mathcal{X})$

"iff" Condition of the Extreme Point Exactness

Recall.
$$\mathcal{C} := \{\boldsymbol{X} \in \mathcal{X} : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \le b_i^u, \forall i \in [m]\}$$
$$\mathcal{C}_{\text{rel}} := \{\boldsymbol{X} \in \text{conv}(\mathcal{X}) : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \le b_i^u, \forall i \in [m]\}$$

- Given $\text{ext}(\mathcal{C}_{\text{rel}})$ is contained in $\le$ m-dim faces in $\text{conv}(\mathcal{X})$, when $\text{ext}(\mathcal{C}_{\text{rel}}) \subseteq \mathcal{C}$ holds?

Theorem.

All extreme points in set $\mathcal{C}_{\text{rel}}$ belong to $\mathcal{C}$ ⟷ "iff" ⟷ Any $\le$ m-dimensional face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$

Proof.

Sufficiency. Any extreme point Y of $\mathcal{C}_{\text{rel}}$ belongs to $\le$ m-dim faces in $\text{conv}(\mathcal{X}) \subseteq \mathcal{X}$. And Y satisfies the m LMIs and thus $Y \in \mathcal{C}$.

Necessity. Prove by contradiction.

□

Geometric Interpretation of "iff" Condition

Recall. $$\mathcal{C} := \left\{ \boldsymbol{X} \in \mathcal{X} : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

$$\mathcal{C}_{\text{rel}} := \left\{ \boldsymbol{X} \in \text{conv}(\mathcal{X}) : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}$$

Theorem.

All extreme points in set $\mathcal{C}_{\text{rel}}$ belong to $\mathcal{C}$ ⟷ "iff" ⟷ Any $\leq$ m-dimensional face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$

Step I: where are extreme points in $\mathcal{C}_{\text{rel}}$ located for any m LMIs?

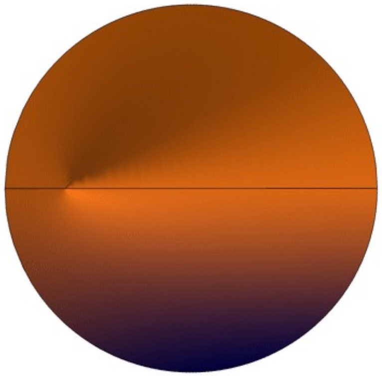
- On $\leq m$-dim faces of $\text{conv}(\mathcal{X})$!

Step II: when set $\mathcal{C}$ contains these extreme point locations?

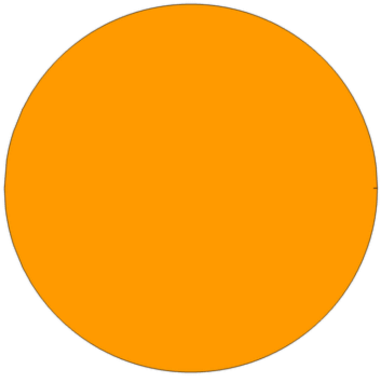
- Any $\leq m$-dim face in $\text{conv}(\mathcal{X})$ belongs to $\mathcal{X}$

Application: QCQP

- For QCQP, $\mathcal{X} := \{\boldsymbol{X} \in \mathcal{S}_+^n\colon \operatorname{rank}(\boldsymbol{X}) \le 1\}$ and $\operatorname{conv}(\mathcal{X}) := \mathcal{S}_+^n$
- Any ≤ 2-dim face of $\operatorname{conv}(\mathcal{X})$ is contained in $\mathcal{X}$



$\mathcal{X}$



$\operatorname{conv}(\mathcal{X})$

- **Point**: 0-dim face; **Edge**: 1-dim face; **Plane**: 2-dim face

Lemma. For QCQP, any ≤ 2-dim face of $\operatorname{conv}(\mathcal{X})$ is contained in $\mathcal{X}$.

Extending Many Interesting Results in QCQP

Lemma. For QCQP, any ≤ 2-dim face of $\mathrm{conv}(\mathcal{X})$ is contained in $\mathcal{X}$.

- Using "iff" condition of extreme point exactness

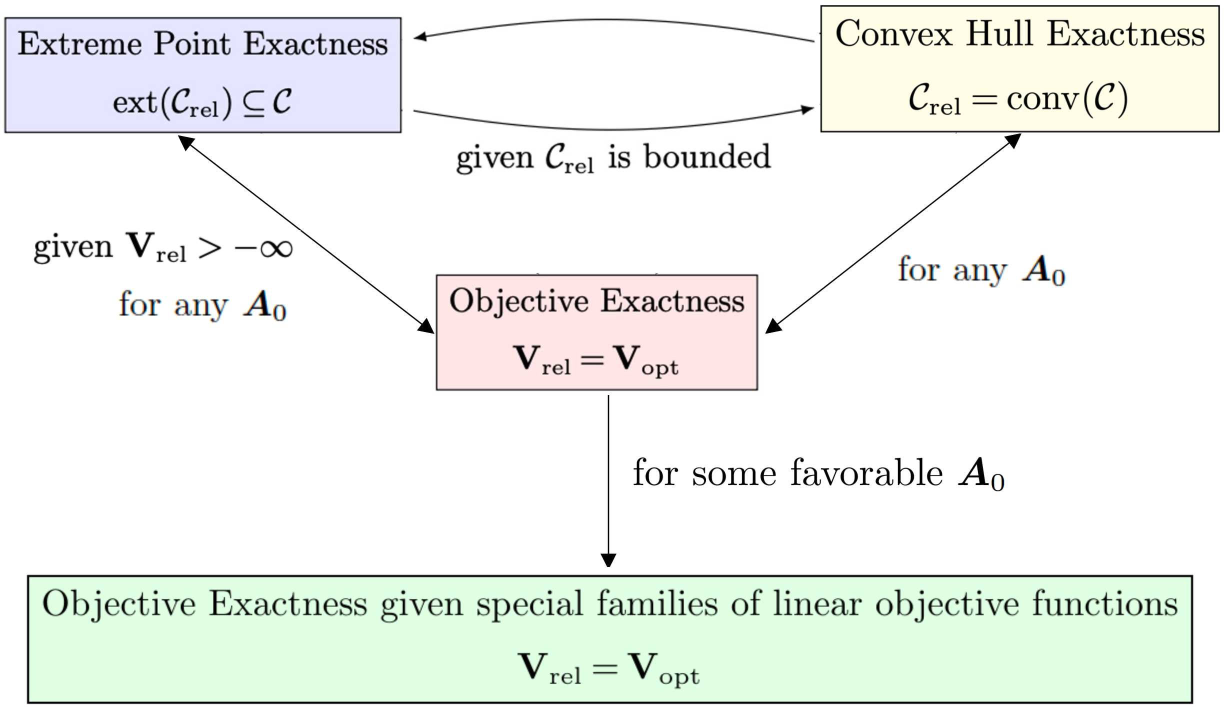
Theorem. For QCQP, its DWR attains extreme point exactness whenever there are any ≤ 2 LMIs.

- Trust region subproblem (TRS)
- Generalized TRS
- Two-sided generalized TRS
- Homogeneous QCQP with 2 quadratic constraints
- Inhomogeneous QCQP with 2 homogeneous quadratic constraints

Convex Hull Exactness

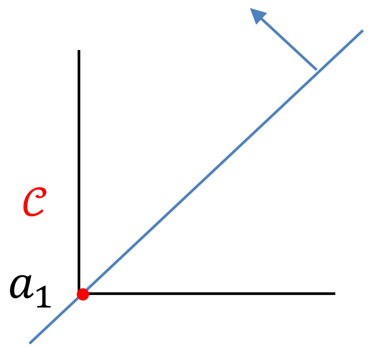
“iff” Condition of the Convex Hull Exactness

$\text{ext}(\mathcal{C}_{\text{rel}})$: all extreme points

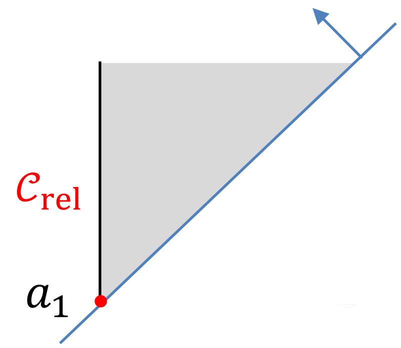


Convex Hull Exactness $\geq$ Extreme Point Exactness

- Example: $\mathcal{X} := \{\boldsymbol{X} \in \mathcal{S}_+^2 \colon \operatorname{rank}(\boldsymbol{X}) \leq 1, X_{12} = 0\}$
  - Add $m = 1$ LMI: $X_{11} \leq X_{22}$



$\mathcal{X}$: nonnegative axes



conv($\mathcal{X}$): nonnegative orthant

- Extreme point exactness holds, while convex hull exactness does not
- One-dim faces F_1, F_2 lie on the Edge and Plane (i.e., $\leq$ 2-dim faces) in conv($\mathcal{X}$)

What Faces are Extreme Directions of $\mathcal{C}_{\text{rel}}$ Located?

$$\mathcal{C}_{\text{rel}} := \{\boldsymbol{X} \in \text{conv}(\mathcal{X}) : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X}\rangle \le b_i^u, \forall i \in [m]\}$$

- Where are the one-dim faces of the recession cone of $\mathcal{C}_{\text{rel}}$ located?
  - On $\le (m+1)$-dim face of the recession cone of $\text{conv}(\mathcal{X})$

Lemma. For any m LMIs, each one-dim face of the intersection set $\mathcal{C}_{\text{rel}}$ is contained in a $\le (m+1)$-dim face of the recession cone of $\text{conv}(\mathcal{X})$.

"iff" Condition of the Convex Hull Exactness

- When the domain set $\mathcal{X}$ is conic, the sufficient condition
  - Reduces to "Any $\leq (m+1)$-dim face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$"
  - becomes necessary

Theorem. When the domain set $\mathcal{X}$ is conic and pointed.

Set $\mathcal{C}_{\text{rel}}$ is identical to the convex hull of set $\mathcal{C}$ ⟷ "iff" ⟷ Any $\leq (m+1)$ -dim face in $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$

QCQP Revisited

Lemma. For QCQP, any ≤ 2-dim face of $\text{conv}(\mathcal{X})$ is contained in $\mathcal{X}$.

- Using "iff" condition of convex hull exactness

Theorem. For homogeneous QCQP with 1 quadratic constraint, convex hull exactness holds.

Fair PCA

- For Fair PCA (Tantipongpipat et al. 2019)

$$\text{(Fair PCA)} \qquad \max_{(z,\boldsymbol{X})\in\mathbb{R}\times\mathcal{X}} \{z : z \le \langle \boldsymbol{A}_i, \boldsymbol{X}\rangle, \forall i \in [m]\}$$

  - Domain set $\mathcal{X} := \{\boldsymbol{X} \in \mathcal{S}_+^n : \text{rank}(\boldsymbol{X}) \le k, ||\boldsymbol{X}||_2 \le 1\}$

Theorem.
For Fair PCA with two groups $m = 2$, the convex hull exactness holds.

Solution Algorithms

Column Generation Algorithm for Solving DWR

$$\text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \text{conv}(\mathcal{X})} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}.$$

- Given the spectral domain set $\mathcal{X}$, we explicitly described $\text{conv}(\mathcal{X})$

Proposition[Kim et al., 2021] When $\mathcal{Q} := \mathcal{S}_+^n$ denotes the positive semidefinite matrix space, we show $\text{conv}(\mathcal{X}) = \text{proj}_X(\mathcal{Y})$, where

$$\mathcal{Y} := \left\{ (\boldsymbol{X}, \boldsymbol{x}) \in \mathcal{Q} \times \mathbb{R}_+^n : f_j(\boldsymbol{x}) \leq 0, \forall j \in [t], x_1 \geq \cdots \geq x_n, x_{k+1} = 0, \boldsymbol{x} \succeq \boldsymbol{\lambda}(\boldsymbol{X}) \right\}.$$

- Computationally expensive to formulate $\text{conv}(\mathcal{X})$
  - Extended space
  - Majorization constraint

Column Generation Algorithm for Solving DWR

$$\text{(DWR)} \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \text{conv}(\mathcal{X})} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \leq \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \leq b_i^u, \forall i \in [m] \right\}.$$

Given the explicit characterization of conv($\mathcal{X}$),

- Directly use off-the-shelf solvers (Mosek) to solve DWR
  - Computationally expensive

- Column generation algorithm: at each iteration, directly solve the pricing problem over conv($\mathcal{X}$)

$$\text{(Pricing)} \quad \min_{\boldsymbol{X} \in \text{conv}(\mathcal{X})} \langle \boldsymbol{C}_t, \boldsymbol{X} \rangle = \min_{\boldsymbol{X} \in \mathcal{X}} \langle \boldsymbol{C}_t, \boldsymbol{X} \rangle$$

Pricing Problem = A Simple Convex Program

$$\text{(Pricing)} \quad \min_{\boldsymbol{X} \in \operatorname{conv}(\mathcal{X})} \langle \boldsymbol{C}_t, \boldsymbol{X} \rangle = \min_{\boldsymbol{X} \in \mathcal{X}} \langle \boldsymbol{C}_t, \boldsymbol{X} \rangle$$

Theorem. For the spectral domain set,

$$\mathcal{X} := \{\boldsymbol{X} \in \mathcal{Q} : \operatorname{rank}(\boldsymbol{X}) \le k, F_j(\boldsymbol{X}) := f_j(\boldsymbol{\lambda}(\boldsymbol{X})) \le 0, \forall j \in [t]\}$$

the pricing problem reduced the following vector-based convex optimization:

$$\boldsymbol{\lambda}^* := \arg\max_{\boldsymbol{\lambda} \in \mathbb{R}_+^n} \left\{ \boldsymbol{\lambda}^\top \boldsymbol{\beta} : \lambda_i = 0, \forall i \in [k+1, n], f_j(\boldsymbol{\lambda}) \le 0, \forall j \in [t] \right\}.$$

Numerical Study: Compare Three Methods

$$(\text{DWR}) \quad \mathbf{V}_{\text{rel}} := \min_{\boldsymbol{X} \in \text{conv}(\mathcal{X})} \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : b_i^l \le \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle \le b_i^u, \forall i \in [m] \right\}.$$

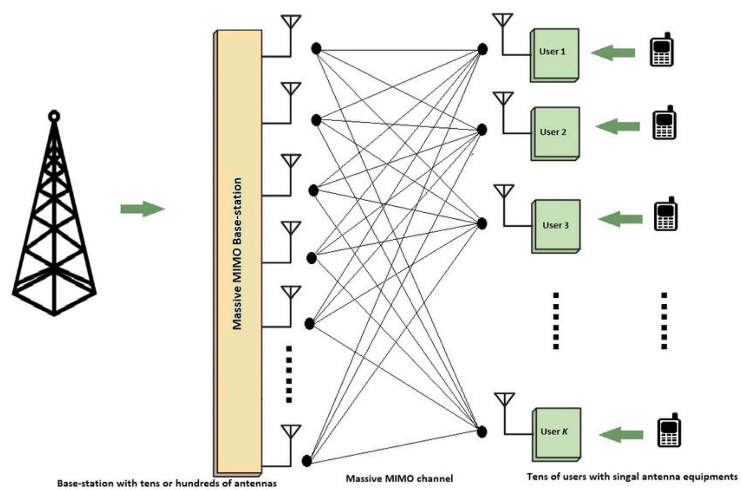
Method	Setting	Need conv($\mathcal{X}$)?
Mosek	Plug conv($\mathcal{X}$) and directly solve the DWR	Yes
Naïve CG	Solve pricing problem over conv($\mathcal{X}$) formulation	Yes
Proposed CG	Use vector-based reduction	No

- CG: Column Generation
- conv($\mathcal{X}$) is an SDP formulation

Numerical Study: MIMO Network with $k \geq 1$

- Multiple-input and multiple-output (MIMO) radio network
  - The data streams at a transmitter $\leq$ the number of transmit antennas
  - Rank-k constraint on the covariance matrix
  - Find the low rank data streams to minimize the total interference power

- Yu and Lau (2010) proposed a RCOP-type model with
  - $\mathcal{X} := \{X \in \mathcal{S}_+^n\colon \operatorname{rank}(\boldsymbol{X}) \leq k, \operatorname{logdet}(I + \boldsymbol{X}) \geq r, \operatorname{tr}(\boldsymbol{X}) \leq R\}$
  - I: identity matrix
  - Spectral domain set $\mathcal{X}$



Numerical Study: MIMO Network with $k \geq 1$

Parameters			Mosek		Naïve CG		Our CG		Theory Rank Bound
n	Rank-k	m LMIs	time(s)	rank	time(s)	rank	time(s)	rank	
50	5	5	43	2*	223	3	1	3	7
50	5	10	24	3*	1261	5	1	5	8
50	10	10	329	3*	--	--	1	4	13
100	10	10	--	--	--	--	2	5	13
100	10	15	--	--	--	--	2	5	14
100	15	15	--	--	--	--	3	7	19
500	25	25	--	--	--	--	24	8	31
500	25	50	--	--	--	--	179	9	34
500	50	50	--	--	--	--	181	27	59

- "--": cannot be solved within 3600 seconds
- "*": infeasible solution

Numerical Study: QCQP with $k = 1$

- Optimal Power Flow (OPF) problem is a classic QCQP (Eltved and Burer 2020)

$$\text{(OPF)} \quad \min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \boldsymbol{x}^\top \boldsymbol{Q}_0 \boldsymbol{x} + \boldsymbol{q}_0^\top \boldsymbol{x} : r \le \boldsymbol{x}^\top \boldsymbol{x} \le R,\ b_i^l \le \boldsymbol{x}^\top \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^\top \boldsymbol{x} \le b_i^u, \forall i \in [m] \right\}$$

- Introduce matrix variable $\boldsymbol{X} \coloneqq \begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^\top \end{pmatrix}$
- Move $r \le \operatorname{tr}(\boldsymbol{X}) - 1 \le R$ into the domain set
- Reformulate OPF as a RCOP-type model with
  - $\mathcal{X} \coloneqq \{X \in \mathcal{S}_+^{n+1} :\ \operatorname{rank}(\boldsymbol{X}) \le 1, r \le \operatorname{tr}(\boldsymbol{X}) - 1 \le R\}$
  - Spectral domain set $\mathcal{X}$
  - $\operatorname{conv}(\mathcal{X}) \coloneqq \{X \in \mathcal{S}_+^{n+1} : r \le \operatorname{tr}(\boldsymbol{X}) - 1 \le R\}$

Numerical Study: QCQP with $k = 1$

2Parameters			Mosek		Naïve CG		Our CG		Theory Rank Bound
n	Rank-k	m LMIs	time(s)	rank	time(s)	rank	time(s)	rank	
1500	1	60	642	1*	--	--	145	2	11
1500	1	75	844	1*	--	--	178	2	12
2000	1	75	--	--	--	--	308	2	12
2000	1	90	--	--	--	--	352	2	13
2500	1	90	--	--	--	--	448	3	13
2500	1	100	--	--	--	--	756	2	14

- "--": cannot be solved within 3600 seconds
- "*": infeasible solution

Summary

- Study a rank-constrained optimization problem (RCOP)
  - General framework
  - Dantzig-Wolfe Relaxation (DWR)
- Derive "if and only if" conditions for the three DWR exactness
  - Only depend on the faces of the convex hull of domain $\mathcal{X}$
  - Geometric interpretation
- Beyond exactness, we derive rank bounds
- Column generation algorithm works well

Thank You !

Preprint is available at https://arxiv.org/pdf/2210.16191