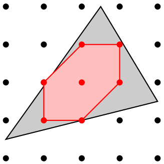
Number of inequalities in integer-programming descriptions of a set



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Marco Di Summa – University of Padua
Christopher Hojny – University of Eindhoven
Matthias Schymura – Brandenburg Technical University

INTRODUCTION

High-level messages

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Furthermore:

- Some modeling-related problems might surpass NP-completeness in their hardness.

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Remark:
$\mathrm{rc}(X) < \infty \Leftrightarrow \mathrm{rc}_{\mathbb{Q}}(X) < \infty \Leftrightarrow \mathrm{conv}(X) \cap \mathbb{Z}^d = X$
(we call such sets **lattice-convex**)

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Generalization $\mathbb{Z}^d \rightsquigarrow Y$**:**

$X \subseteq Y \Rightarrow$
$\mathrm{rc}(X, Y) := \min\big\{|\mathrm{facets}(P)| : P \cap Y = X\big\}$

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   - Try using larger k for better separation.

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2. $\mathrm{rc}(X) = \mathrm{rc}_{\mathbb{Q}}(X)$? Does the field matter for modeling?
3. A simple set to test our understanding:

$$\Delta_d := \{0, e_1, \ldots, e_d\} \subseteq \mathbb{Z}^d,$$

the vertex set of the standard simplex.
What is $\mathrm{rc}(\Delta_d)$ for every d?
Yes, we don't know this either! For every $d \geq 6$.

Observation on Question 3:
$\text{rc}_{\mathbb{Q}}(\Delta_d) = d + 1$,
because every rational relaxation of Δ_d is bounded
and so has least $d + 1$ facets, and
$\text{conv}(\Delta_d)$ has $d + 1$ facets.

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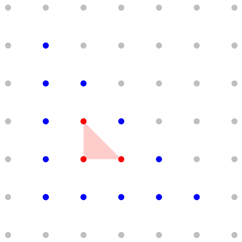
Idea: If $d = 2 = \dim(X)$, there is a finite set $Y \subseteq \mathbb{Z}^d$ that observes X such that separating of X from Y is already enough to separate $\mathbb{Z}^d \setminus X$.



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$d = 2 \rightsquigarrow d = 3$: small step for man, one giant leap for MIP ☺

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These kind of weird phenomena make deciding computability a hard problem.

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Aprile & A. & Di Summa & Hojny:

$$\frac{\mathrm{rc}(\Delta_d)}{\mathrm{rc}_{\mathbb{Q}}(\Delta_d)} \xrightarrow[d\to\infty]{} 0$$

Recall: $\mathrm{rc}_{\mathbb{Q}}(\Delta_d) = d + 1$. We show:

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Aprile & A. & Di Summa & Hojny:

$$\mathrm{rc}(\Delta_d) = \mathcal{O}\left(\frac{d}{\sqrt{\log d}}\right)$$

Apart from $\mathbb{Q}$, at most d irrational numbers are enough:

$$\mathrm{rc}_{\mathbb{F}}(\Delta_d) \leq \mathcal{O}\left(\frac{d}{\sqrt{\log d}}\right)$$

holds for the relaxation complexity with respect to a coefficient field $\mathbb{F}$ with $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$ and

$$\dim_{\mathbb{Q}}(\mathbb{F}) \leq d$$

TOOL: MIP RELAXATIONS OF FINITE SETS

- Consider the set $\mathbb{Z}^n \times \mathbb{R}$ of mixed-integer points.

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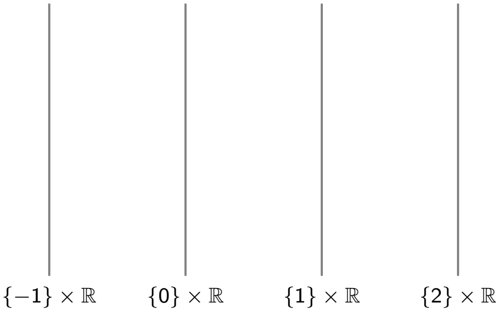

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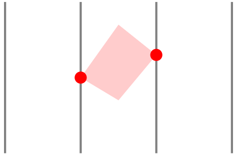

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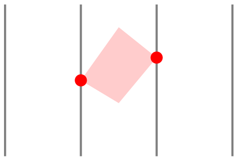
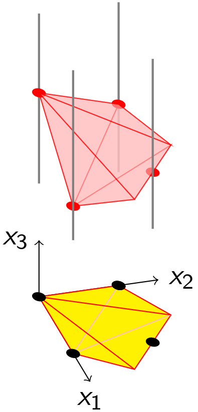
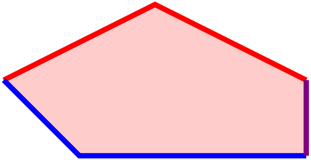




Remark: Not every finite $S \subseteq \mathbb{Z}^n \times \mathbb{R}$ has a mixed-integer relaxation, but some do.

A full-dimensional polytope $Q \subseteq \mathbb{R}^{n+1}$ has **upper, lower and lateral facets**, corresponding to inequalities of the form

$$x_{n+1} \leq u_i(x_1, \ldots, x_n) \qquad i \in I$$
$$x_{n+1} \geq l_j(x_1, \ldots, x_n) \qquad j \in J$$
$$0 \leq a_k(x_1, \ldots, x_n) \qquad k \in K$$



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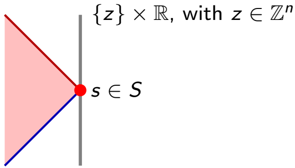

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Straightforward to check:

1. Fourier-Motzkin elimination for $Q = \{x \in \mathbb{R}^{n+1} : Ax \leq b\}$,
2. Verification of the equality cases on points of S.

PROOF IDEA OF $\mathrm{rc}(\Delta_5) \leq 5$**.**

What can irrational numbers do?

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With the irrational numbers one can "fold" $\mathbb{Z}^2$ into $\mathbb{R}$:

$$(a, b) \in \mathbb{Z}^2 \mapsto a - \frac{1}{\sqrt{2}} b \in \mathbb{R} \qquad \text{injective.}$$

That means: $a - \frac{1}{\sqrt{2}} b$ "remembers" the two values $a, b \in \mathbb{Z}$.

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Our choice of the rational number $-\frac{1}{\sqrt{2}}$ here actually does not matter much (we use it for historical reasons).

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$$\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\} = \Delta_5 \simeq X := \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\} \subseteq \mathbb{Z}^5$$

Apply a skew irrational projection to X:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix} \qquad \text{injective as } \mathbb{Z}^5 \to \mathbb{Z}^3 \times \mathbb{R}$$

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This gives

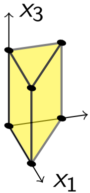
$$X = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\} \to S = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{matrix} \right\}$$

$S \subseteq \mathbb{Z}^3 \times \mathbb{R}$

Project out the last component, $\mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z}^3$:

$$S = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{matrix} \right\} \to T := \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix} \right\}$$

T is a a vertex set of a triangular prism:



Crucial property: $\mathrm{conv}(T) \cap \mathbb{Z}^3 = \mathrm{vert}(\mathrm{conv}(T))$.

Summary. From dimension 5 to dimension 4 to dimension 3:

$$\begin{array}{ccccccc} \Delta_5 & \simeq & X & \to & S & \to & T \\ \mathbb{Z}^5 & = & \mathbb{Z}^5 & \to & \mathbb{Z}^3 \times \mathbb{R} & \to & \mathbb{Z}^3 \end{array}$$

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- X – lifting of S, unimodular copy of Δ_5

$$Q := \mathrm{conv}(S) = \mathrm{conv}\left\{\begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{matrix}\right\}$$

This polytope has three upper facets, three lower facets and three lateral facets.
That is, the inequality description has the form:

$$\begin{aligned} x_4 &\le u_i(x_1, x_2, x_3) & i &= 1,2,3 \\ x_4 &\ge l_j(x_1, x_2, x_3) & j &= 1,2,3 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_1 + x_2 &\le 1 \end{aligned}$$

You may use SageMath to calculate this description: it can do exact polyhedral computations over fields like $\mathbb{Q}[\sqrt{2}]$.

In the ambient set $\mathbb{Z}^3 \times \mathbb{R}$, the Q is a mixed-integer relaxation of S:

$$Q \cap (\mathbb{Z}^3 \times \mathbb{R}) = S = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{matrix} \right\}.$$

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Verifiable directly (see the discussion of MIP relaxations of finite sets).

Q is lifted to the unbounded relaxation of $X \simeq \Delta_5$

$$X = \left\{ x \in \mathbb{Z}^5 : \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix}}_{=p} \in Q \right\}$$

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given by 9 inequalities. **Justification:**

Q is a mixed-integer relaxation of $S \Rightarrow p \in S \Leftrightarrow$

Q is lifted to the unbounded relaxation of $X \simeq \Delta_5$

$$X = \left\{ x \in \mathbb{Z}^5 : \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix}}_{=p} \in Q \right\}$$

given by 9 inequalities. **Justification:**
Q is a mixed-integer relaxation of $S \Rightarrow p \in S \Leftrightarrow$

$$p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix} \in S = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{matrix} \right\}$$

$$\Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in X = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\}$$

From 9 to 5 inequalities:
Move from

$$Q = \{x \in \mathbb{R}^4 \,:\, Ax \leq b\}$$

with $Ax \leq b$ being a system of 9 inequalities to

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- We need to make sure that Q' is a mixed-integer relaxation of S. Once Q' is found, that can be done in a straightforward manner (tedious computation for mixed-integer relaxation of the finite set S).

Q' is lifted to the relaxation of $X \simeq \Delta_5$:

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$\Rightarrow \text{rc}(\Delta_5) \leq 5$.

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algebra,
number theory, and
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The interplay between the
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Introduction

MIP relaxations of finite sets

Proof idea of $\mathrm{rc}(\Delta_5) \leq 5$

Asymptotic estimate for $\mathrm{rc}(\Delta_d)$

Thanks!

May the MIP be with you!

BONUS MATERIAL: ASYMPTOTIC ESTIMATE FOR $\mathrm{rc}(\Delta_d)$

- Use the same process $\Delta_d \simeq X \to S \to T$, where $T = \{0,1\}^k$ and $k = \mathcal{O}(\log d)$, which we turn around and do backwards.

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- The arguments rely on the combinatorics of regular subdivisions of $\mathrm{conv}(T) = [0,1]^k$.

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- The arguments rely on the combinatorics of regular subdivisions of $\mathrm{conv}(T) = [0,1]^k$.
- We use the probabilistic method to show the existence of the above coverings.