Number of inequalities in integer-programming descriptions of a set

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INTRODUCTION
**High-level messages**

We could reflect more about:

- choices of the coefficient field for MIP,
High-level messages

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• different ways and limitations of modeling within a given modeling approach
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Furthermore:

- Some modeling-related problems might surpass NP-completeness in their hardness.
Real relaxation complexity $rc(X)$ of a finite set $X \subseteq \mathbb{Z}^d$ is the smallest number of facets in a relaxation $P$ of $X$, where a relaxation is a polyhedron $P$ satisfying $P \cap \mathbb{Z}^d = X$. 

Remark: $rc(X) < \infty \iff rc_Q(X) < \infty \iff \text{conv}(X) \cap \mathbb{Z}^d = X$ (we call such sets lattice-convex)

Generalization $\mathbb{Z}^d \Rightarrow Y$: $X \subseteq Y \Rightarrow rc(X, Y) := \min \{ |\text{facets}(P)| : P \cap Y = X \}$
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**Generalization** $\mathbb{Z}^d \leadsto Y$:

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Potential applications in AI/ML:

• separate $X$ (rabbits) from $Y \setminus X$ (non-rabbits) by a system $Ax \leq b$ of $k$ inequalities.
• $k = 1 \rightsquigarrow \text{support-vector machine}$
• Try using larger $k$ for better separation.
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Questions of Kaibel & Weltge (2015)

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2. \( \text{rc}(X) = \text{rc}_\mathbb{Q}(X) \)? Does the field matter for modeling?
3. A simple set to test our understanding:

\[
\Delta_d := \{0, e_1, \ldots, e_d\} \subseteq \mathbb{Z}^d,
\]

the vertex set of the standard simplex.

What is \( \text{rc}(\Delta_d) \) for every \( d \)?
Yes, we don’t know this either! For every \( d \geq 6 \).
Observation on Question 3:

\( rc \mathbb{Q}(\Delta_d) = d + 1, \)

because every rational relaxation of \( \Delta_d \) is bounded
and so has least \( d + 1 \) facets, and

\( \text{conv}(\Delta_d) \) has \( d + 1 \) facets.
Problem with the computability comes from $|\mathbb{Z}^d| = \infty$

Kaibel & Weltge 2015: $d = 2 \implies \text{rc}(X) = \text{rc}_{\mathbb{Q}}(X)$ and it is computable.

Idea: If $d = 2 = \dim(X)$, there is a finite set $Y \subseteq \mathbb{Z}^d$ that observes $X$ such that separating of $X$ from $Y$ is already enough to separate $\mathbb{Z}^d \setminus X$. 
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![Diagram of a grid with a shaded triangle]
\[ d = 2 \rightsquigarrow d = 3 \]
\( d = 2 \implies d = 3 \)

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\( d = 3 \implies \text{rc}(X) = \text{rc}_\mathbb{Q}(X) \) and it is computable.
\( d = 2 \Rightarrow d = 3 \)

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\[ d = 3 \Rightarrow rc(X) = rc_Q(X) \] and it is computable.

\( d = 2 \Rightarrow d = 3: \text{ small step for man, one giant leap for MIP } 😊 \)
\[ d = 3 \quad \leadsto \quad d = 4 \]
\[ d = 3 \rightarrow d = 4 \]

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\[ d = 4 \Rightarrow \text{rc}(X) = \text{rc}_\mathbb{Q}(X) \]

and so
\[ \text{rc}(\Delta_d) = \text{rc}_\mathbb{Q}(\Delta_d) = d + 1 \text{ for } d \leq 4. \]

But we don't know about computability, and there are good reasons for this:

**A. & Hojny & Schymura 2021:**
\[ \Delta_4 \text{ can be separated by 4 linear inequalities from any finite subset of } \mathbb{Z}^4 \setminus \Delta_4, \text{ but not from the whole } \mathbb{Z}^4 \setminus \Delta_4. \]

These kind of weird phenomena make deciding computability a hard problem.
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These kind of weird phenomena make deciding computability a hard problem.
$d = 4 \iff d = 5$ and higher.
$d = 4 \leadsto d = 5 \text{ and higher.}$

Aprile & A. & Di Summa & Hojny 2022+:

$rc(\Delta_d) < rc_{\mathbb{Q}}(\Delta_d)$ for every $d \geq 5$. 

 Choice of the field matters! How big is the discrepancy?
\[ d = 4 \leftrightarrow d = 5 \text{ and higher.} \]

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Aprile & A. & Di Summa & Hojny:
\[
\frac{\text{rc}(\Delta_d)}{\text{rc}_\mathbb{Q}(\Delta_d)} \rightarrow 0 \quad (d \to \infty)
\]
Recall: $rc_Q(\Delta_d) = d + 1$. We show:
Recall: \( rc_{\mathbb{Q}}(\Delta_d) = d + 1 \). We show:

**Aprile & A. & Di Summa & Hojny:**

\[
rc(\Delta_d) = \mathcal{O} \left( \frac{d}{\sqrt{\log d}} \right)
\]

Apart from \( \mathbb{Q} \), at most \( d \) irrational numbers are enough:

\[
rc_F(\Delta_d) \leq \mathcal{O} \left( \frac{d}{\sqrt{\log d}} \right)
\]

holds for the relaxation complexity with respect to a coefficient field \( F \) with \( \mathbb{Q} \subseteq F \subseteq \mathbb{R} \) and

\[
\dim_{\mathbb{Q}}(F) \leq d
\]
TOOL: MIP RELAXATIONS OF FINITE SETS
Consider the set $\mathbb{Z}^n \times \mathbb{R}$ of mixed-integer points.
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This is a union of vertical lines arranged in a grid according to $\mathbb{Z}^n$. 
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• This is a union of vertical lines arranged in a grid according to $\mathbb{Z}^n$.

\[
\begin{align*}
\{−1\} \times \mathbb{R} & \quad \{0\} \times \mathbb{R} & \quad \{1\} \times \mathbb{R} & \quad \{2\} \times \mathbb{R}
\end{align*}
\]
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**Remark:** Not every finite $S \subseteq \mathbb{Z}^n \times \mathbb{R}$ has a mixed-integer relaxation, but some do.
A full-dimensional polytope $Q \subseteq \mathbb{R}^{n+1}$ has upper, lower and lateral facets, corresponding to inequalities of the form

$$\begin{align*}
  x_{n+1} &\leq u_i(x_1, \ldots, x_n) & i \in I \\
  x_{n+1} &\geq l_j(x_1, \ldots, x_n) & j \in J \\
  0 &\leq a_k(x_1, \ldots, x_n) & k \in K
\end{align*}$$
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A full-dimensional polytope $Q$ is a mixed-integer relaxation of a finite set $S \subseteq \mathbb{Z}^n \times \mathbb{R}$ if and only if
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1. The projection of \( Q \) on the space of the \( n \) integer variables is an integer relaxation of the projection of \( S \).
   (That is, \( Q \) hits exactly the lines in \( \mathbb{Z}^d \times \mathbb{R} \) on which points of \( S \) sit.)
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2. Every point of $S$ is contained in an upper and a lower facet of $Q$. 

\[ \{z\} \times \mathbb{R}, \text{ with } z \in \mathbb{Z}^n \]
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Straightforward to check:

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**Straightforward to check:**

1. Fourier-Motzkin elimination for $Q = \{x \in \mathbb{R}^{n+1} : Ax \leq b\}$,
2. Verification of the equality cases on points of $S$. 
PROOF IDEA OF $rc(\Delta_5) \leq 5$. 
What can irrational numbers do?
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With the irrational numbers one can “fold” $\mathbb{Z}^2$ into $\mathbb{R}$:

$$(a, b) \in \mathbb{Z}^2 \mapsto a - \frac{1}{\sqrt{2}} b \in \mathbb{R} \quad \text{injective.}$$

That means: $a - \frac{1}{\sqrt{2}} b$ “remembers” the two values $a, b \in \mathbb{Z}$. 
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Our choice of the rational number $-\frac{1}{\sqrt{2}}$ here actually does not matter much (we use it for historical reasons).
Change $\Delta_5$ to an equivalent set $X$
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Write $\simeq$ for unimodular equivalence, equivalence up to unimodular transformations $x \mapsto Ux + v$, with $U \in \mathbb{Z}^{d \times d}, |\det(U)| = 1, v \in \mathbb{Z}^d$. 

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \Delta_5 
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \subseteq \mathbb{Z}_5
\]
Change $\Delta_5$ to an equivalent set $X$

Write $\simeq$ for unimodular equivalence, equivalence up to unimodular transformations $x \mapsto Ux + \nu$, with $U \in \mathbb{Z}^{d \times d}, |\det(U)| = 1, \nu \in \mathbb{Z}^d$.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \Delta_5 \simeq X := \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \subseteq \mathbb{Z}^5$$
Apply a skew irrational projection to $X$:

$$
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
\end{bmatrix} \mapsto
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 - \frac{1}{\sqrt{2}} x_5 \\
\end{bmatrix}
$$

injective as $\mathbb{Z}^5 \rightarrow \mathbb{Z}^3 \times \mathbb{R}$
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\[
\begin{bmatrix}
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  x_4 - \frac{1}{\sqrt{2}} x_5 
\end{bmatrix}
\]

injective as $\mathbb{Z}^5 \rightarrow \mathbb{Z}^3 \times \mathbb{R}$

This gives

$$X = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix} \rightarrow S = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}$$

$S \subseteq \mathbb{Z}^3 \times \mathbb{R}$
Project out the last component, $\mathbb{Z}^3 \times \mathbb{R} \rightarrow \mathbb{Z}^3$:

\[
S = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}}
\end{bmatrix} \rightarrow T := \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

$T$ is a vertex set of a triangular prism:

\[
\text{Crucial property: } \text{conv}(T) \cap \mathbb{Z}^3 = \text{vert(\text{conv}(T))}.
\]
Summary. From dimension 5 to dimension 4 to dimension 3:

\[
\Delta_5 \cong X \to S \to T
\]

\[
\mathbb{Z}^5 = \mathbb{Z}^5 \to \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z}^3
\]
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We will also follow this path backwards in our arguments:

- \( T \) – vertices of the prism
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- \( S \) – lifted vertices of the prism
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We will also follow this path backwards in our arguments:

- \( T \) – vertices of the prism
- \( S \) – lifted vertices of the prism
- \( X \) – lifting of \( S \), unimodular copy of \( \Delta_5 \)
$Q := \text{conv}(S) = \text{conv} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{bmatrix}$

This polytope has three upper facets, three lower facets and three lateral facets.
That is, the inequality description has the form:

\[ x_4 \leq u_i(x_1, x_2, x_3) \quad i = 1, 2, 3 \]
\[ x_4 \geq l_j(x_1, x_2, x_3) \quad j = 1, 2, 3 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1 + x_2 \leq 1 \]

You may use SageMath to calculate this description: it can do exact polyhedral computations over fields like $\mathbb{Q}[^2]$. 
In the ambient set $\mathbb{Z}^3 \times \mathbb{R}$, the $Q$ is a mixed-integer relaxation of $S$:

$$Q \cap (\mathbb{Z}^3 \times \mathbb{R}) = S = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
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\end{bmatrix}.$$
In the ambient set $\mathbb{Z}^3 \times \mathbb{R}$, the $Q$ is a mixed-integer relaxation of $S$:

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Verifiable directly (see the discussion of MIP relaxations of finite sets).
$Q$ is lifted to the unbounded relaxation of $X \simeq \Delta_5$

$$X = \left\{ x \in \mathbb{Z}^5 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix} \in Q \right\}$$

given by 9 inequalities.
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given by 9 inequalities. **Justification:**

$Q$ is a mixed-integer relaxation of $S \Rightarrow p \in S \iff$
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given by 9 inequalities. **Justification:**

$Q$ is a mixed-integer relaxation of $S \Rightarrow p \in S \iff$

$$p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix} \in S = \begin{cases} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \end{cases}$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in X = \begin{cases} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{cases}$$
From 9 to 5 inequalities:
Move from

\[ Q = \{ x \in \mathbb{R}^4 : Ax \leq b \} \]

with \( Ax \leq b \) being a system of 9 inequalities to

\[ Q' = \{ x \in \mathbb{R}^4 : A'x \leq b' \} \]

where \( Q \subseteq Q' \) and \( A' \leq b' \) is a system of 5 inequalities only with:
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- We borrow one upper-facet and two lower-facet inequalities from \( Ax \leq b \).
- Each of the two remaining inequalities is a combination of two facet inequalities for \( Q \).
- We need to make sure that \( Q' \) is a mixed-integer relaxation of \( S \). Once \( Q' \) is found, that can be done in a straightforward manner (tedious computation for mixed-integer relaxation of the finite set \( S \)).
$Q'$ is lifted to the relaxation of $X \simeq \Delta_5$:

$$X = \left\{ x \in \mathbb{Z}^5 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \frac{1}{\sqrt{2}} x_5 \end{bmatrix} \in Q' \right\}$$

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by 5 inequalities.

$\Rightarrow \text{rc}(\Delta_5) \leq 5.$
“As a mathematical subject, it is a rich combination of aspects of geometry, algebra, number theory, and combinatorics. The interplay between the mathematics, modeling, and algorithmics makes it a deep and fascinating subject of applied mathematics.”
Introduction

MIP relaxations of finite sets

Proof idea of $\text{rc}(\Delta_5) \leq 5$

Asymptotic estimate for $\text{rc}(\Delta_d)$
Thanks!

May the MIP be with you!
BONUS MATERIAL:
ASYMPTOTIC ESTIMATE FOR $rc(\Delta_d)$
• Use the same process $\Delta_d \simeq X \rightarrow S \rightarrow T$, where $T = \{0, 1\}^k$ and $k = \mathcal{O}(\log d)$, which we turn around and do backwards.
• Use the same process $\Delta_d \simeq X \to S \to T$, where $T = \{0, 1\}^k$ and $k = \mathcal{O}(\log d)$, which we turn around and do backwards.

• Lift $T = \{0, 1\}^k$ to $S \subseteq \mathbb{Z}^k \times \mathbb{R}$ by supplying heights such that $Q := \text{conv}(S)$ has the following properties:
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• $S$ can be covered by $O\left(\frac{d}{\sqrt{\log d}}\right)$ simplicial upper facets of $Q$ and also
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  • by $O\left(\frac{d}{\sqrt{\log d}}\right)$ simplicial lower facets of $Q$. 

The arguments rely on the combinatorics of regular subdivisions of $\text{conv}(T) = [0, 1]^k$. We use the probabilistic method to show the existence of the above coverings.
• Use the same process $\Delta_d \simeq X \to S \to T$, where $T = \{0, 1\}^k$ and $k = \mathcal{O}(\log d)$, which we turn around and do backwards.

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