Explicit convex hull description of bivariate quadratic sets with indicator variables

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MIQCPs with indicator variables

- Consider a mixed-integer quadratically constrained optimization problem (MIQCP) with indicator variables of the form:

$$
\begin{aligned}
\min \quad & x^T Q_0 x + c_0^T x && \text{(QPI)} \\
\text{s.t.} \quad & x_i(1 - z_i) = 0, \quad \forall i \in \{1, \dots, n\} \\
& x^T Q_j x + c_j^T x \le b_j, \quad \forall j \in \{1, \dots, m\} \\
& Ax \le \alpha, \ Bz \le \beta \\
& x \in \mathbb{R}^n, \ z \in \{0,1\}^n.
\end{aligned}
$$

- The binary variables z are often referred to as indicator variables.
- Problem (QPI) subsumes several classes of difficult optimization problems such as best subset selection, constrained portfolio optimization, optimal power flow with transmission switching, among others.
- We are not assuming that Q_j is positive semi-definite (PSD).
- We are interested in the quality of convex relaxations for Problem (QPI).

Convexification of MIQCPs with indicator variables

- Introduce auxiliary variables $X_{ij} := x_i x_j$ for all $i, j \in [n]$ to obtain a reformulation of Problem (QPI):

$$\begin{aligned} \min \quad & \langle Q_0, X \rangle + c_0^T x \\ \text{s.t.} \quad & x_i(1 - z_i) = 0, \quad \forall i \in [n] \\ & X = xx^T \\ & \langle Q_j, X \rangle + c_j^T x \leq b_j, \quad \forall j \in [m] \\ & Ax \leq \alpha, \; Bz \leq \beta \\ & x \in \mathbb{R}^n, \; z \in \{0,1\}^n, \; X \in \mathbb{S}_+^n, \end{aligned} \tag{QPI'}$$

  where $\mathbb{S}_+^n$ denotes the set of $n \times n$ PSD matrices.

- Assumption: $Ax \leq \alpha$ implies $x \geq 0$.

- To construct strong convex relaxations for Problem (QPI′), we should effectively convexify the set:

$$\mathcal{S}_n = \left\{ (x, X, z) : \; X = xx^T, \; x(1 - z) = 0, \; x \geq 0, \; z \in \{0,1\}^n \right\}$$

Convexification of $\mathcal{S}_n$

- The closure of the convex hull of $\mathcal{S}_1 = \{(x_1, X_{11}, z_1) : X_{11} = x_1^2,\ x_1(1 - z_1) = 0,\ x_1 \geq 0,\ z_1 \in \{0, 1\}\}$, is (Akturk et al 2009, Gunluk and Linderoth 2010):

$$\overline{\text{conv}}(\mathcal{S}_1) = \Big\{(x, X, z) :\ X_{11} z_1 \geq x_1^2,\ x_1 \geq 0,\ X_{11} \geq 0,\ z_1 \in [0, 1]\Big\}.$$

- The Perspective relaxation of Problem (QPI′):

$$\begin{aligned} \min\quad & \langle Q_0, X\rangle + c_0^T x \qquad\qquad \text{(Persp)} \\ \text{s.t.}\quad & X \succeq xx^T \\ & X_{ii} z_i \geq x_i^2, \quad \forall i \in [n] \\ & \langle Q_j, X\rangle + c_j^T x \leq b_j, \quad \forall j \in [m] \\ & Ax \leq \alpha,\ Bz \leq \beta,\ x \in \mathbb{R}^n,\ z \in [0, 1]^n,\ X \in \mathbb{S}_+^n. \end{aligned}$$

- Very effective for problems with $m = 0$, $Q_0 \in \mathbb{S}_+^n$ such that Q_0 is diagonal or diagonally dominant (Frangioni and Gentile 2006, Gunluk and Linderoth 2010, Dong et al 2015).

Convexification of $\mathcal{S}_n$

- To improve the quality of convex relaxations for Problem (QPI′), it is natural to study the facial structure of $\overline{\text{conv}}(\mathcal{S}_2)$:

$$\mathcal{S}_2 := \Big\{(x, X, z) : X_{11} = x_1^2,\ X_{12} = x_1x_2,\ X_{22} = x_2^2,\ x_1(1 - z_1) = 0,$$
$$x_2(1 - z_2) = 0, x_1, x_2 \geq 0,\ z_1, z_2 \in \{0, 1\}\Big\}.$$

- Anstreicher and Burer (Mathematical Programming, 2021): $\mathcal{S}_2$ with additional constraints $x_1, x_2 \leq 1$; an extended formulation, containing three additional variables, for the convex hull that is SDP representable.
- Convex hull of the epigraph of bivariate convex quadratic functions:
  - Atamturk et al (JMLR, 2021): $Z_2^- := \{(x, t, z) : t \geq d_1x_1^2 - 2x_1x_2 + d_2x_2^2,\ x_i(1 - z_i) = 0, x_i \geq 0, z_i \in \{0, 1\}, i \in \{1, 2\}\}$ with $d_1, d_2 > 0$, $d_1d_2 \geq 1$,
  - Han et al (Mathematical Programming, 2023): $Z_2^+ := \{(x, t, z) : t \geq d_1x_1^2 + 2x_1x_2 + d_2x_2^2,\ x_i(1 - z_i) = 0, x_i \geq 0, z_i \in \{0, 1\}, i \in \{1, 2\}\}$ with $d_1, d_2 > 0$, $d_1d_2 \geq 1$. Also propose an extended SDP relaxation.
- More interesting results on convex quadratic optimization with indicator variables (Atamturk and Gomez 2019, Wei et al 2021, Wei et al 2022).

The convex hull of $\mathcal{S}_2$

We give an explicit characterization for the closure of the convex hull of $\mathcal{S}_2$ in the space of original variables

- A De Rosa and A Khajavirad, Explicit convex hull description of bivariate quadratic sets with indicator variables, arXiv:2208.08703, 2022.

Constructing the convex hull of $\mathcal{S}_2$

- We can construct the convex hull of $\mathcal{S}_2$ using

$$\overline{\text{conv}}(\mathcal{S}_2) = \overline{\text{conv}}\Big(\overline{\text{conv}}(\mathcal{P}_1) \cup \overline{\text{conv}}(\mathcal{P}_2) \cup \overline{\text{conv}}(\mathcal{P}_3) \cup \overline{\text{conv}}(\mathcal{P}_4)\Big),$$

$$\begin{aligned}
\mathcal{P}_1 &:= \{(x, X, z) : z_1 = z_2 = 0,\ x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\},\\
\mathcal{P}_2 &:= \{(x, X, z) : z_1 = 1, z_2 = 0,\ X_{11} = x_1^2,\ x_2 = X_{12} = X_{22} = 0,\ x_1 \geq 0\},\\
\mathcal{P}_3 &:= \{(x, X, z) : z_1 = 0, z_2 = 1,\ x_1 = X_{11} = X_{12} = 0,\ X_{22} = x_2^2,\ x_2 \geq 0\},\\
\mathcal{P}_4 &:= \{(x, X, z) : z_1 = z_2 = 1,\ X_{11} = x_1^2,\ X_{12} = x_1x_2,\ X_{22} = x_2^2,\ x_1, x_2 \geq 0\}.
\end{aligned}$$

and

$$\begin{aligned}
\overline{\text{conv}}(\mathcal{P}_1) &= \{(x, X, z) : z_1 = z_2 = 0,\ x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\},\\
\overline{\text{conv}}(\mathcal{P}_2) &= \{(x, X, z) : z_1 = 1, z_2 = 0,\ X_{11} \geq x_1^2,\ x_2 = X_{12} = X_{22} = 0,\ x_1 \geq 0\},\\
\overline{\text{conv}}(\mathcal{P}_3) &= \{(x, X, z) : z_1 = 0, z_2 = 1,\ x_1 = X_{11} = X_{12} = 0,\ X_{22} \geq x_2^2,\ x_2 \geq 0\},\\
\overline{\text{conv}}(\mathcal{P}_4) &= \{(x, X, z) : z_1 = z_2 = 1,\ X_{11} \geq x_1^2,\ X_{22} \geq x_2^2,\\
&\qquad (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2, x_1 \geq 0,\ x_2 \geq 0,\ X_{12} \geq 0\}.
\end{aligned}$$

A convex extended formulation for $\overline{\mathrm{conv}}(\mathcal{S}_2)$

- Using disjunctive programming:

$$\overline{\mathrm{conv}}(\mathcal{S}_2) = \mathrm{cl}\Big\{(x, X, z) : \ \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma\Big\},$$

where $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^4)$, $\tilde{X} = (\tilde{X}^1, \ldots, \tilde{X}^4)$, with $\tilde{x}^i = (\tilde{x}^i_1, \tilde{x}^i_2) \in \mathbb{R}^2$ and $\tilde{X}^i = (\tilde{X}^i_{11}, \tilde{X}^i_{12}, \tilde{X}^i_{22}) \in \mathbb{R}^3$ for all $i \in \{1, \ldots, 4\}$, and

$$\Sigma := \Big\{(x, X, z, \tilde{x}, \tilde{X}, \lambda) :$$

$$x = \sum_{i=1}^{4} \tilde{x}^i, \ X = \sum_{i=1}^{4} \tilde{X}^i, \ z = \sum_{i=1}^{4} \lambda_i z^i, \ \sum_{i=1}^{4} \lambda_i = 1, \ \lambda_i \geq 0 \ \forall i$$

$$\left(\frac{\tilde{x}^i}{\lambda_i}, \frac{\tilde{X}^i}{\lambda_i}, z^i\right) \in \overline{\mathrm{conv}}(\mathcal{P}_i) \text{ if } \lambda_i > 0, \ \tilde{x}^i = \tilde{X}^i = 0 \text{ if } \lambda_i = 0\Big\}.$$

A convex extended formulation for $\overline{\text{conv}}(\mathcal{S}_2)$

- Using disjunctive programming:

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z) : \ \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma\Big\},$$

where

$$\Sigma = \Big\{(x, X, z, \tilde{x}, \tilde{X}, \lambda) :$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \ \lambda_i \geq 0 \ \forall i \in \{1, \ldots, 4\}, \ z_1 = \lambda_2 + \lambda_4, \ z_2 = \lambda_3 + \lambda_4,$$

$$x_1 = \tilde{x}_1^2 + \tilde{x}_1^4, \ X_{11} = \tilde{X}_{11}^2 + \tilde{X}_{11}^4, \ x_2 = \tilde{x}_2^3 + \tilde{x}_2^4, \ X_{22} = \tilde{X}_{22}^3 + \tilde{X}_{22}^4, \ X_{12} = \tilde{X}_{12}^4$$

$$\begin{cases} \lambda_2 \tilde{X}_{11}^2 \geq (\tilde{x}_1^2)^2, \ \tilde{x}_1^2 \geq 0, & \text{if } \lambda_2 \neq 0 \\ \tilde{x}_1^2 = \tilde{X}_{11}^2 = 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} \lambda_3 \tilde{X}_{22}^3 \geq (\tilde{x}_2^3)^2, \ \tilde{x}_2^3 \geq 0, & \text{if } \lambda_3 \neq 0 \\ \tilde{x}_2^3 = \tilde{X}_{22}^3 = 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} \lambda_4 \tilde{X}_{11}^4 \geq (\tilde{x}_1^4)^2, \ \lambda_4 \tilde{X}_{22}^4 \geq (\tilde{x}_2^4)^2, \ (\lambda_4 \tilde{X}_{11}^4 - (\tilde{x}_1^4)^2)(\lambda_4 \tilde{X}_{22}^4 - (\tilde{x}_2^4)^2) \geq (\lambda_4 \tilde{X}_{12}^4 - \tilde{x}_1^4 \tilde{x}_2^4)^2, \\ \tilde{x}_1^4 \geq 0, \ \tilde{x}_2^4 \geq 0, \ \tilde{X}_{12}^4 \geq 0, & \text{if } \lambda_4 \neq 0 \\ \tilde{x}_1^4 = \tilde{x}_2^4 = \tilde{X}_{11}^4 = \tilde{X}_{12}^4 = \tilde{X}_{22}^4 = 0, & \text{otherwise} \end{cases}$$

$$\Big\}.$$

A partial projection - first step

- Projecting out $\lambda_1, \lambda_2, \lambda_3, \tilde{x}_1^2, \tilde{x}_2^3, \tilde{X}_{12}^4, \tilde{X}_{11}^2, \tilde{X}_{22}^3$, we obtain:

$$\max\{0, z_1 + z_2 - 1\} \leq \lambda_4 \leq \min\{z_1, z_2\}, \; X_{12} \geq 0,$$

$$\begin{cases} (z_1 - \lambda_4)(X_{11} - \tilde{X}_{11}^4) \geq (x_1 - \tilde{x}_1^4)^2, \; \tilde{x}_1^4 \leq x_1, & \text{if } \lambda_4 \neq z_1 \\ \tilde{x}_1^4 = x_1, \; \tilde{X}_{11}^4 = X_{11}, & \text{otherwise} \end{cases}$$

$$\begin{cases} (z_2 - \lambda_4)(X_{22} - \tilde{X}_{22}^4) \geq (x_2 - \tilde{x}_2^4)^2, \; \tilde{x}_2^4 \leq x_2, & \text{if } \lambda_4 \neq z_2 \\ \tilde{x}_2^4 = x_2, \; \tilde{X}_{22}^4 = X_{22}, & \text{otherwise} \end{cases}$$

$$\begin{cases} \lambda_4 \tilde{X}_{11}^4 \geq (\tilde{x}_1^4)^2, \; \lambda_4 \tilde{X}_{22}^4 \geq (\tilde{x}_2^4)^2, \; (\lambda_4 \tilde{X}_{11}^4 - (\tilde{x}_1^4)^2)(\lambda_4 \tilde{X}_{22}^4 - (\tilde{x}_2^4)^2) \geq (\lambda_4 \tilde{X}_{12}^4 - \tilde{x}_1^4 \tilde{x}_2^4)^2, \\ \tilde{x}_1^4 \geq 0, \; \tilde{x}_2^4 \geq 0, \; \tilde{X}_{12}^4 \geq 0, \quad \text{if } \lambda_4 \neq 0 \\ \tilde{x}_1^4 = \tilde{x}_2^4 = \tilde{X}_{11}^4 = \tilde{X}_{12}^4 = \tilde{X}_{22}^4 = 0, \quad \text{otherwise} \end{cases}$$

- Replacing, $\tilde{X}_{11}^4 = X_{11}$ by $\tilde{X}_{11}^4 \leq X_{11}$, $\tilde{X}_{22}^4 = X_{22}$ by $\tilde{X}_{22}^4 \leq X_{22}$, and removing $X_{12} \geq 0$, we obtain the OptPairs relaxation of Han et al (MPA 2023):

$$\begin{pmatrix} z_1 - \lambda_4 & x_1 - \tilde{x}_1^4 \\ x_1 - \tilde{x}_1^4 & X_{11} - \tilde{X}_{11}^4 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} z_2 - \lambda_4 & x_2 - \tilde{x}_2^4 \\ x_2 - \tilde{x}_2^4 & X_{22} - \tilde{X}_{22}^4 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} \lambda_4 & \tilde{x}_1^4 & \tilde{x}_2^4 \\ \tilde{x}_1^4 & \tilde{X}_{11}^4 & X_{12} \\ \tilde{x}_1^4 & X_{12} & \tilde{X}_{22}^4 \end{pmatrix} \succeq 0$$

$$z_1 + z_2 - 1 \leq \lambda_4, \; 0 \leq \tilde{x}_1^4 \leq x_1, \; 0 \leq \tilde{x}_2^4 \leq x_2$$

A partial projection - second step

- Projecting out $\lambda_1, \lambda_2, \lambda_3,\ \tilde{x}_1^2, \tilde{x}_2^3, \tilde{X}_{12}^4, \tilde{X}_{11}^2, \tilde{X}_{22}^3, \tilde{X}_{11}^4, \tilde{X}_{22}^4$, we obtain:
$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z):\ \exists (x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) \in \tilde{\Sigma}\Big\},$$

$$\tilde{\Sigma} := \Big\{(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : X_{12} > 0,\ z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\},\ \lambda_4 > 0,$$
$$X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} \geq 0,\ X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0,$$
$$\Big(X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4}\Big)\Big(X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{(z_2 - \lambda_4)}\Big) \geq \Big(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\Big)^2,$$
$$0 \leq \tilde{x}_1^4 \leq x_1,\ 0 \leq \tilde{x}_2^4 \leq x_2\Big\}$$

- Whenever we write a function of the form $f = \frac{u^2}{v}$, $v > 0$, we refer to it closure defined as:

$$\hat{f}(u, v) := \begin{cases} \frac{u^2}{v}, & \text{if } v > 0 \\ 0, & \text{if } u = v = 0 \\ +\infty & \text{if } u \neq 0,\ v = 0. \end{cases}$$

A projection strategy

- Consider a function $f : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ and two sets $\mathcal{G} \subset \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$. Define

$$\mathcal{Q} := \{x \in \mathcal{G} \,:\, \exists y \in \mathbb{R}^m \text{ s.t. } f(x, y) \le 0,\ (x, y) \in \mathcal{D}\}.$$

For every $x \in \mathcal{G}$, consider the following optimization problem:

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & (x, y) \in \mathcal{D} \end{aligned} \tag{Proj}$$

and consider a set

$$\mathcal{E} \subseteq \{x \in \mathcal{G} \,:\, \text{Problem (Proj) admits a minimizer, denoted by } y_x\}.$$

Then

$$\mathcal{Q} \cap \mathcal{E} = \mathcal{Q}_{\mathcal{E}} := \{x \in \mathcal{E} \,:\, f(x, y_x) \le 0,\ (x, y_x) \in \mathcal{D}\}.$$

Projection by convex optimization

- To characterize $\overline{\mathrm{conv}}(\mathcal{S}_2)$, it suffices to solve the following parametric convex optimization problem for all $(x, X, z) \in \mathcal{G} \supset \overline{\mathrm{conv}}(\mathcal{S}_2)$:

$$\begin{aligned} \min_{\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4} \quad & \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{\left(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\right)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}}, \qquad (\mathsf{P}_{x,X,z}) \\ \text{s.t.} \quad & X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, \\ & z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0 \\ & 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2. \end{aligned}$$

The Convex hull of $\mathcal{S}_2$

- Theorem: There exists a convex set $\tilde{\mathcal{S}}$ such that $\mathrm{ri}(\mathrm{conv}(\mathcal{S}_2)) \subseteq \tilde{\mathcal{S}} \subseteq \overline{\mathrm{conv}}(\mathcal{S}_2)$ and:

$$\overline{\mathrm{conv}}(\mathcal{S}_2) = \bigcup_{i=1}^{8} \mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i),$$

  where the sets $\mathcal{R}_i$, $i \in \{1, \ldots, 8\}$ satisfy $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{8} \mathcal{R}_i \supseteq \overline{\mathrm{conv}}(\mathcal{S}_2)$, and

- if $i \in \{1, 2, 6\}$, then

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i) = \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ X_{22} \geq \frac{x_2^2}{z_2},\ x_1, x_2 \geq 0,\ X_{12} \geq 0, \\ z_1, z_2 \in [0, 1]\Big\} \cap \mathrm{cl}(\mathcal{R}_i),$$

  where

$$\mathcal{R}_1 := \Big\{(x, X, z) :\ x_1 x_2 (z_1 + z_2 - 1) \leq X_{12} z_1 z_2,\ X_{12} \max\{z_1, z_2\} \leq x_1 x_2,\ X_{12} > 0,\ z_1, z_2 > 0\Big\} \\ \cup \Big\{(x, X, z) : X_{12} = 0\Big\},$$

The Convex hull of $\mathcal{S}_2$

$$\mathcal{R}_2 := \Big\{(x,X,z):\ z_1 \le z_2,\ X_{12}z_2 > x_1x_2,\ X_{12}z_1 \le x_1x_2,\ X_{12} > 0,\ z_1 > 0,$$

$$x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2) \ge z_1(X_{12}z_2 - x_1x_2)^2\Big\},$$

$$\mathcal{R}_6 := \Big\{(x,X,z):\ X_{12}z_1z_2 < x_1x_2(z_1+z_2-1),\ X_{12} > 0,\ z_1, z_2 > 0,$$

$$(1-z_1)(z_1+z_2-1)x_1^2(X_{22}z_2 - x_2^2) \ge \Big(X_{12}z_1z_2 - x_1x_2(z_1+z_2-1)\Big)^2\Big\}.$$

- if $i \in \{3,4\}$, then

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i) = \Big\{(x,X,z) : X_{22} \ge \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \ge \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,$$

$$x_1, x_2 \ge 0,\ X_{12} \ge 0,\ z_1, z_2 \in [0,1] \le 1\Big\} \cap \mathrm{cl}(\mathcal{R}_i),$$

$$\mathcal{R}_3 := \Big\{(x,X,z) : z_1 < z_2, X_{12}x_2 > X_{22}x_1, z_1(X_{12}z_2 - x_1x_2)^2 > x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2),$$

$$x_1 \ge 0,\ X_{12} > 0,\ z_1 > 0\Big\},$$

$$\mathcal{R}_4 := \{(x,X,z) : z_2 \le z_1,\ X_{12}x_2 > X_{22}x_1,\ x_1 \ge 0,\ X_{12} > 0,\ z_2 > 0\}.$$

The Convex hull of $\mathcal{S}_2$

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_5) = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \geq \Big(X_{12} - \frac{x_1 x_2}{z_1}\Big)^2,$$
$$x_1, x_2 \geq 0,\ X_{12} \geq 0,\ z_1, z_2 \in [0,1]\Big\} \cap \mathrm{cl}(\mathcal{R}_i),$$

where

$$\mathcal{R}_5 = \{(x, X, z) :\ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \geq X_{12}x_2,\ x_2 \geq 0,\ X_{12} > 0,\ z_1, z_2 > 0\},$$

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_7) = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2,$$
$$x_1, x_2 \geq 0,\ X_{12} \geq 0,\ z_1, z_2 \in [0,1]\Big\} \cap \mathrm{cl}(\mathcal{R}_i),$$

where

$$\mathcal{R}_7 := \{(x, X, z) : X_{12}z_1z_2 < x_1x_2(z_1 + z_2 - 1), X_{12} > 0, z_1, z_2 > 0, x_1^2(x_2^2 - X_{22}(1 - z_1))(X_{22}z_2 - x_2^2)$$
$$> 2x_1x_2X_{12}z_1(X_{22}z_2 - x_2^2) - X_{12}^2(X_{22}(z_1 + z_2 - 1) + x_2^2(1 - 2z_1 - z_2(1 - z_1)))\}.$$

The Convex hull of $\mathcal{S}_2$

- $\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_8) =$

$$\Big\{(x, X, z) : \; X_{11} \geq \frac{x_1^2}{z_1}, \; X_{22} \geq \frac{x_2^2}{z_2}, \; x_1, x_2 \geq 0, \; X_{12} \geq 0, \; z_1, z_2 \in [0, 1],$$

$$z_1(1 - z_2)\Big(X_{11} - \frac{x_1^2}{z_1}\Big)x_2^2 \geq (z_1 + z_2 - 1)\Big(X_{12}\frac{z_1 z_2}{W} - x_1 x_2\Big)^2\Big\} \cap \text{cl}(\mathcal{R}_i),$$

where

$$W := (z_1 + z_2 - 1) - \frac{1}{x_2}\sqrt{(X_{22}z_2 - x_2^2)(1 - z_1)(z_1 + z_2 - 1)}.$$

and

$$\mathcal{R}_8 := \Big\{(x, X, z) : \; X_{12}z_1z_2 < x_1x_2(z_1 + z_2 - 1), \; X_{12} > 0, \; z_1, z_2 > 0\Big\} \setminus (\mathcal{R}_6 \cup \mathcal{R}_7),$$

How to generate supporting hyperplanes of $\overline{\mathrm{conv}}(\mathcal{S}_2)$?

- Consider an open set $\mathcal{A} \subset \mathbb{R}^n$, four continuously differentiable functions $f_1, f_2, h_1, h_2 : \mathcal{A} \to \mathbb{R}$, and two sets $\mathcal{R}, \mathcal{C} \subseteq \mathcal{A}$ such that $\mathcal{C}$ is an n-dimensional convex set and

$$\mathcal{R} = \Big\{x : f_1(x) > 0, f_2(x) \geq 0\Big\},$$

$$\mathcal{C} \cap \mathcal{R} = \Big\{x : h_1(x) \geq 0, h_2(x) \geq 0, f_1(x) > 0, f_2(x) \geq 0\Big\}.$$

- Consider a point $\hat{x} \in \mathcal{R}$ such that $h_1(\hat{x}) = 0$, $\nabla h_1(\hat{x}) \neq 0$, $h_2(\hat{x}) > 0$, $f_1(\hat{x}) > 0$, $f_2(\hat{x}) > 0$, and for which there exists $\hat{r} > 0$ such that $\mathcal{B}_{\hat{r}}(\hat{x}) \subset \mathcal{A}$ and:

  $\forall x \in \{x \in \mathcal{C} \cap \mathcal{R} \cap \mathcal{B}_{\hat{r}}(\hat{x}) : h_1(x) = 0\}$ and $\forall s > 0$ it holds $\mathcal{B}_s(x) \setminus (\mathcal{C} \cap \mathcal{R}) \neq \emptyset$.

  Then there exists $r > 0$ such that

$$\{x \in \mathcal{B}_r(\hat{x}) : h_1(x) = 0\} = \partial\mathcal{C} \cap \mathcal{B}_r(\hat{x}).$$

A simple separation algorithm

- Given a convex set $\mathcal{C}$, we say that the inequality $q(x) \geq 0$ supports $\mathcal{C}$ at $\hat{x}$, or is a supporting inequality for $\mathcal{C}$ at $\hat{x}$, if $q(x) \geq 0$ for all $x \in \mathcal{C}$ and $\{x : q(x) = 0\}$ is a supporting hyperplane of $\mathcal{C}$ at $\hat{x}$.

- Let $\tilde{\mathcal{C}}$ be a convex relaxation of $\mathcal{S}_2$ defined by:

$$\tilde{\mathcal{C}} := \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ X_{22} \geq \frac{x_2^2}{z_2},\ (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1 x_2)^2,$$
$$X_{12} \geq 0,\ x_1, x_2 \geq 0,\ z_1, z_2 \in [0,1]\Big\}.$$

- The separation problem: Given a point $(\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{\mathcal{C}}$, decide whether $(\tilde{x}, \tilde{X}, \tilde{z})$ is in $\overline{\mathrm{conv}}(\mathcal{S}_2)$ or not, and in the latter case, find a supporting inequality for $\overline{\mathrm{conv}}(\mathcal{S}_2)$ that is violated by $(\tilde{x}, \tilde{X}, \tilde{z})$.

A simple separation algorithm over $\overline{\text{conv}}(\mathcal{S}_2)$

Input: A point $(\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{\mathcal{C}}$

Output: A Boolean `Inside` together with a supporting inequality $h(x, X, z) \geq 0$ for $\overline{\text{conv}}(\mathcal{S}_2)$, violated by $(\tilde{x}, \tilde{X}, \tilde{z})$ if `Inside = false`.

Find the unique $k \in \{1, \ldots, 8\}$ such that $(\tilde{x}, \tilde{X}, \tilde{z}) \in \mathcal{R}_k$

if *all inequalities defining* $\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_k)$ *are satisfied by* $(\tilde{x}, \tilde{X}, \tilde{z})$, **then**

- `Inside = true`
- **return**

else

- `Inside = false`
- **if** $k \in \{3, 4\}$, **then**
  - Let $q(x, X, z) = \left(X_{11} - \frac{x_1^2}{z_2}\right)\left(X_{22} - \frac{x_2^2}{z_2}\right) - \left(X_{12} - \frac{x_1 x_2}{z_2}\right)^2$
- **else if** $k = 5$, **then**
  - Let $q(x, X, z) = \left(X_{11} - \frac{x_1^2}{z_1}\right)\left(X_{22} - \frac{x_2^2}{z_1}\right) - \left(X_{12} - \frac{x_1 x_2}{z_1}\right)^2$
- **else if** $k = 8$, **then**
  - Let $q(x, X, z) = z_1(1 - z_2)\left(X_{11} - \frac{x_1^2}{z_1}\right)x_2^2 - (z_1 + z_2 - 1)\left(X_{12}\frac{z_1 z_2}{W} - x_1 x_2\right)^2$
- Define the point $(\hat{x}, \hat{X}, \hat{z})$ with components equal to $(\tilde{x}, \tilde{X}, \tilde{z})$ except for $\hat{X}_{11}$ which is chosen so that $q(\hat{x}, \hat{X}, \hat{z}) = 0$. Let $h(x, X, z)$ be the first-order Taylor expansion of $q(x, X, z)$ at $(\hat{x}, \hat{X}, \hat{z})$.
- **return** $h(x, X, z) \geq 0$

A corollary of our convex hull characterization

- For any $i \neq j \in [n]$, any supporting inequality of

$$\left\{ (x_i, x_j, X_{ii}, X_{ij}, X_{jj}, z_i) : \begin{pmatrix} z_i & x_i & x_j \\ x_i & X_{ii} & X_{ij} \\ x_j & X_{ij} & X_{jj} \end{pmatrix} \succeq 0 \right\}, \qquad (1)$$

  at a boundary point of (1) satisfying

$$X_{ij} z_i > x_i x_j, \quad X_{jj} x_i > X_{ij} x_j, \quad x_i, x_j > 0, \quad z_i, z_j \in (0,1),$$

  is a supporting inequality for $\overline{\text{conv}}(\mathcal{S}_2)$, and hence is valid inequality for $\overline{\text{conv}}(\mathcal{S}_n)$.

- Atamturk and Gomez 2019 propose the following rank-one inequalities

$$\begin{pmatrix} z_i + z_j & x_i & x_j \\ x_i & X_{ii} & X_{ij} \\ x_j & X_{ij} & X_{jj} \end{pmatrix} \succeq 0,$$

  and show the impact of the addition of such constraints to the perspective relaxation.