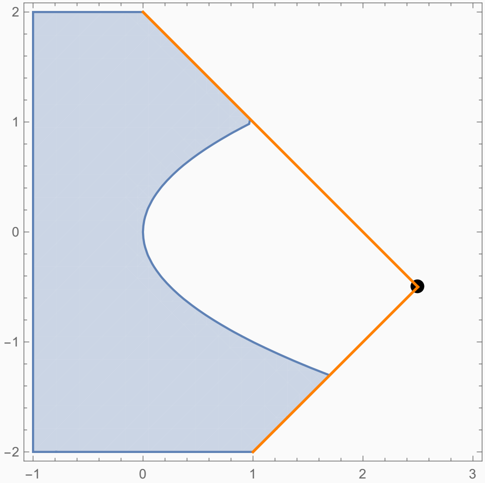
Maximal quadratic-free sets: basic constructions and steps towards a full characterization

Gonzalo Muñoz - Universidad de O'Higgins, Chile

February 28, 2023 @ ICERM

Intersection cuts in a nutshell

Intersection cuts in pictures



Feasible set, S (blue); $\bar{s}$ vertex of LP relaxation (black)

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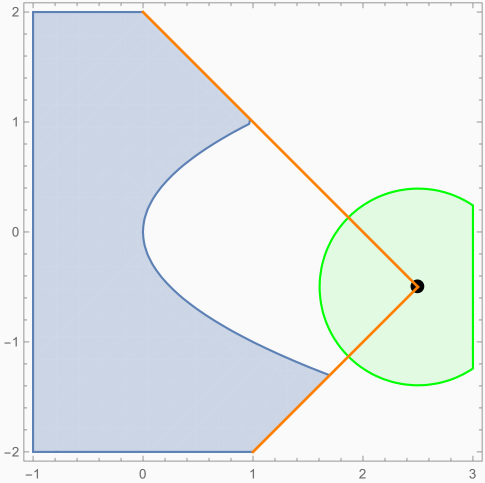
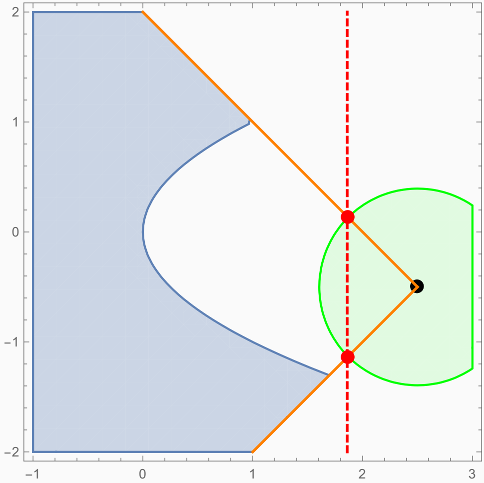


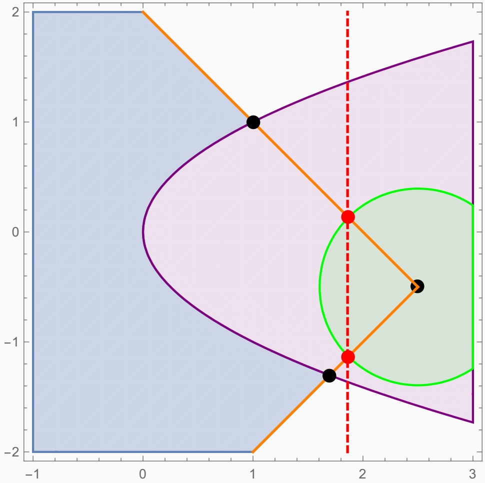
S-free set (green) (Dey and Wolsey 2010)

Intersection cuts in pictures



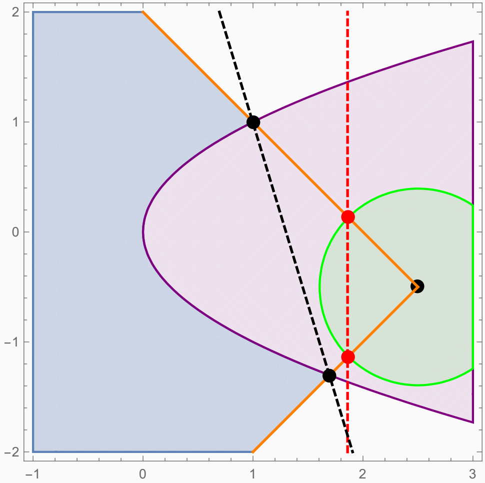
Intersection cut (red) (Balas 1971)

Intersection cuts in pictures



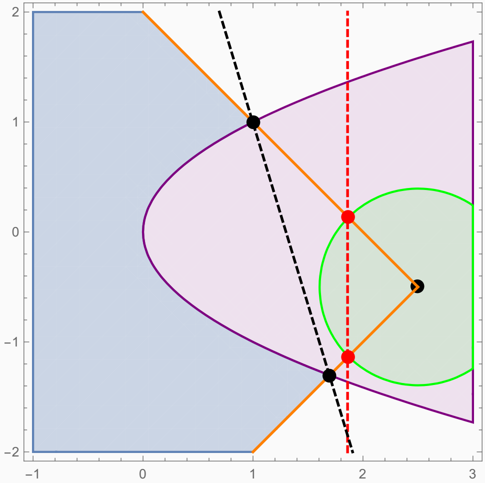
Larger S-free set (purple)

Intersection cuts in pictures



Deeper intersection cut (black)

Intersection cuts in pictures



C is *maximal S-free* if it is not contained in another S-free set

Our setting

An important case: quadratic set

$$S = \{s \in \mathbb{R}^p \,:\, s^\top A s + b^\top s + c \leq 0\}.$$

with $\bar{s} \notin S$.

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$$\bar{s} \notin \bigcap_{i=1}^{m} \{s \in \mathbb{R}^p \,:\, s^\top A_i s + b_i^\top s + c_i \leq 0\}$$

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- An LP relaxation of a QCQP carries info of all constraints, thus an intersection cut would do so too.

Related work

Intersection cuts in non-convex settings

- Fischetti, Ljubić, Monaci and Sinnl (2016)$\rightarrow$ **bilevel-free sets**
- Fischetti and Monaci (2019) $\rightarrow$ **bilinear-free sets**
- Serrano (2019) $\rightarrow$ **concave underestimators of factorable functs**
- Bienstock, Chen and M. (2019, 2020) $\rightarrow$ **maximal outer-product-free sets**

Beyond intersection cuts

- Kılınç-Karzan (2015) $\rightarrow$ **minimal inequalities for disjunctive conic sets**
- Burer and Kılınç-Karzan (2017) $\rightarrow$ **second-order cone intersected with quadratic**
- Santana and Dey (2018) $\rightarrow$ **convex hull of quadratic constraint $\cap$ polytope is SOC representable**

What we'll talk about today

The agenda for today: to show the basic step in the construction of **maximal** **quadratic-free sets** and (very) recent extensions

This will cover work with Antonia Chmiela[1], Joseph Paat[2] and Felipe Serrano[3].

[1]Zuse Institute Berlin, Germany
[2]University of British Columbia, Canada
[3]Cardinal Operations, Germany

Homogeneous quadratics

A canonical form for homogeneous quadratics

In this talk, we consider a set of the form

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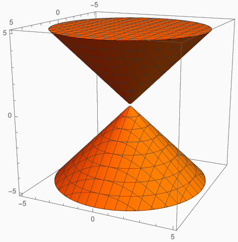
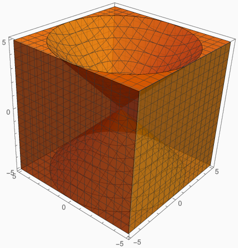


$n = 2,\ m = 1$



$n = 1,\ m = 2$

Constructing Q-free sets

Since $\lambda^\mathsf{T} x \leq \|x\|$ when $\|\lambda\| = 1$, we can show that

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} : \|y\| \leq \lambda^\mathsf{T} x\} \quad \text{is } Q\text{-free.}$$

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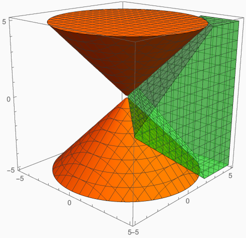
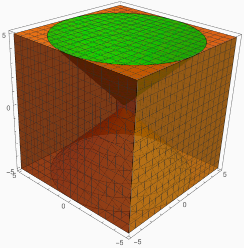
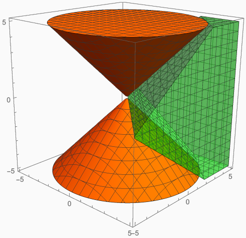
Theorem (M. and Serrano '21)

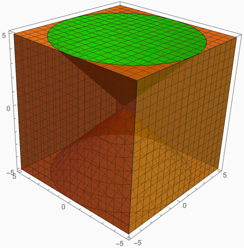
C_λ is maximal Q-free.





Maximality proof





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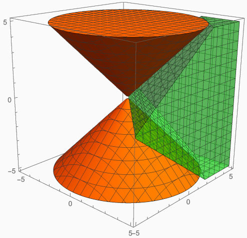

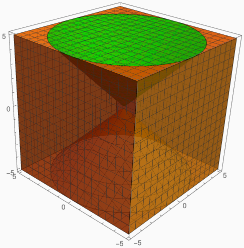

Proof sketch.
We use an outer-description of C_λ:

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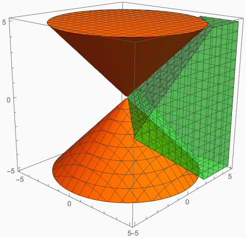 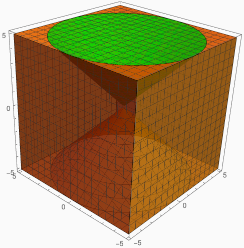

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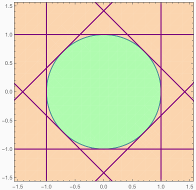
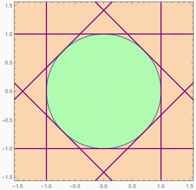
The point (λ, β) is in $Q \cap C_\lambda$ (because $\|\lambda\| = \|\beta\|$) and "exposes" the inequality $-\lambda^{\mathsf{T}} x + \beta^{\mathsf{T}} y \le 0$. $\square$

Having exposing points suffices for maximality





Theorem (M. and Serrano '21)

Let S be a closed set and $C = \{x \in \mathbb{R}^n : \alpha^T x \leq \beta, \forall(\alpha, \beta) \in \Gamma\}$ an S-free set. Suppose for every $\alpha^T x \leq \beta$ there is an $\bar{x} \in S \cap C$ such that

$$\underbrace{\alpha^{\mathsf{T}}\bar{x} = \beta \quad \wedge \quad \tilde{\alpha}^{\mathsf{T}}\bar{x} < \tilde{\beta} \quad (\tilde{\alpha}, \tilde{\beta}) \neq (\alpha, \beta)}_{\bar{x} \text{ exposes } (\alpha, \beta)}$$

Then, C is maximal S-free.

This generalizes the sufficient part of the criterion of Dey and Wolsey (2010) for lattice-free sets.

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we can tranform it via homogenization and diagonalization onto

$$\{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \|y\|, \, a^\top x + d^\top y = 1\}$$

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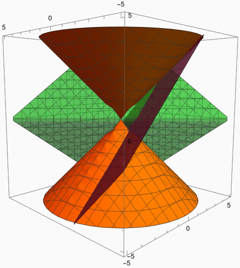

Thus $C_\lambda \cap \{(x, y) \, : \, a^\mathsf{T} x + d^\mathsf{T} y = 1\}$ is S-free

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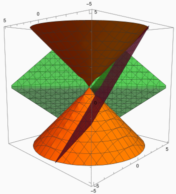
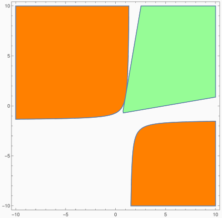
However, maximality is not preserved when taking slices





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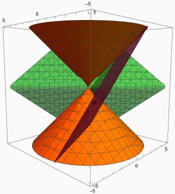
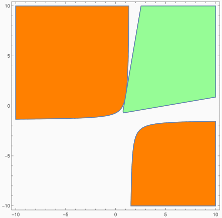
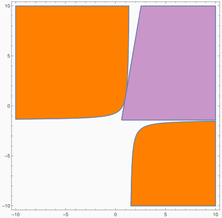




In **M. and Serrano (2021)** and **Chmiela, M. and Serrano (2022)** we show how to grow the slice of C_λ onto a maximal S-free set:



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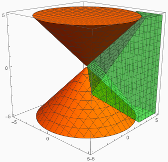
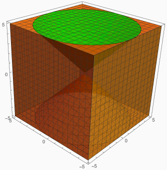
Option 2: Is C_λ all there is for Q?





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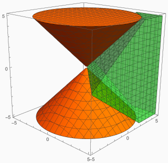
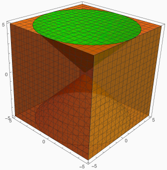
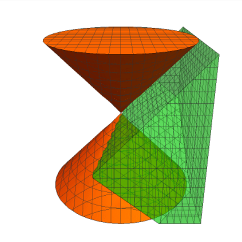




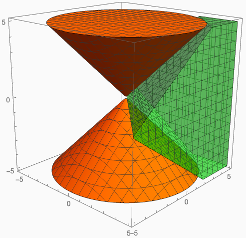
NO. The following "twisted wedge" C is also maximal:

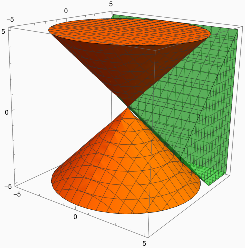


All maximal Q-free sets in 3D

We can prove that each twisted wedge C can be obtained from a C_λ through a linear, invertible transformation that

leaves Q invariant!





Recap before we dive into more technical stuff

So far, we have seen that

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} : \|y\| \leq \lambda^\mathsf{T} x\}$$

is a simple maximal Q-free set and in a sense, is all there is in 3 dimensions.

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Why do care? More Q-free sets mean more alternatives for cutting planes.

Beyond C_λ

Rewriting Q

Recall that

$$Q = \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \|y\|\}$$

Since $\|y\| = \max\{\beta^\mathsf{T} y \, : \, \|\beta\| = 1\}$, we have

$$Q = \bigcup_{\|\beta\|=1} \{(x, y) \in \mathbb{R}^{n+m} \, : \, \|x\| \leq \beta^\mathsf{T} y\}$$

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$\rightarrow$ Q is the union of convex sets.

Separation of convex sets $\Rightarrow$ any Q-free set can be separated from each S_β:

$$S_\beta := \{(x, y) \in \mathbb{R}^{n+m} \,:\, \|x\| \leq \beta^\mathsf{T} y\}$$

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Note that for any unit vector $\Gamma(\beta)$

$$\Gamma(\beta)^\mathsf{T} x \leq \beta^\mathsf{T} y$$

is a valid inequality for S_β. This motivates the definition of

$$C_\Gamma = \{(x, y) \in \mathbb{R}^{n+m} : \beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x \;\; \forall \, \beta \in D^m\}$$ which is always Q-free.

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We can push this idea to show

Theorem (M., Paat and Serrano '23)

Let C be a full-dimensional maximal Q-free set. There exists a function $\Gamma : D^m \to D^n$ such that

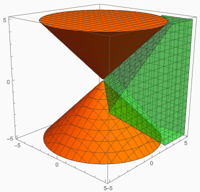
$$C = C_\Gamma.$$

Examples

$$C_\Gamma = \{(x, y) \in \mathbb{R}^{n+m} : \beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x \;\; \forall\, \beta \in D^m\}$$

In the following 3D examples y only has one dimension $\rightarrow$ $\beta = \pm 1$.

Thus, $\Gamma(\beta)$ is part of the slopes of the two hyperplanes



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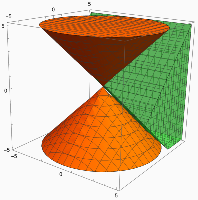

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A maximality condition

Recall that C_Γ is always Q-free. It turns out that, if Γ satisfies that

$$\underbrace{\|\Gamma(\beta) - \Gamma(\beta')\| < \|\beta - \beta'\| \quad \beta \neq \beta'}_{\text{“strict non-expansiveness”}}$$

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Proof sketch.
For each β, consider the point $(x, y) = (\Gamma(\beta), \beta)$. Under the above condition

- $(x, y) \in Q \cap C_\Gamma$
- The only inequality of C_Γ which is tight at (x, y) is $\beta^\top y \leq \Gamma(\beta)^\top x$

In other words, every inequality has an exposing point

□

Γ strictly non-expansive

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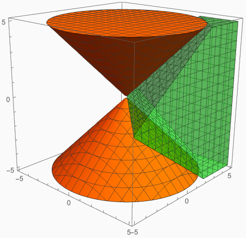
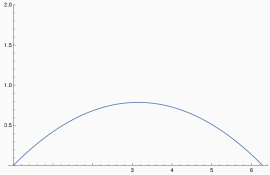


But we are not restricted to constant functions!

Γ strictly non-expansive

For example, for $n = m = 2$ we can construct a Γ function (from a circle to a circle) using polar coordinates:



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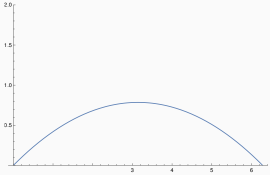
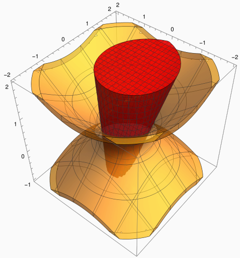


A 3D slice of the resulting 4D maximal Q-free set is:



A conjecture

We believe that the "strictly non-expansive" condition can be relaxed.

Conjecture

Consider the Q-free set

$$C_\Gamma = \{(x, y) \in \mathbb{R}^{n+m} : \beta^\mathsf{T} y \leq \Gamma(\beta)^\mathsf{T} x \ \ \forall\, \beta \in D^m\}.$$

with $\Gamma : D^m \to D^n$. If Γ is non-expansive, then C_Γ is maximal Q-free.

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with $\Gamma : D^m \to D^n$. *If* Γ *is non-expansive, then* C_Γ *is maximal Q-free.*

So far, we have the following partial result

Theorem (M., Paat and Serrano ’23)
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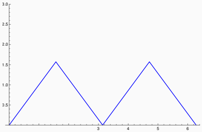
with $\Gamma : D^m \to D^n$. *If* Γ *is non-expansive and* C_Γ *is a polyhedron then* C_Γ *is maximal Q-free.*

A polyehdral example

For $n = m$ we can consider a $\Gamma(\beta) = |\beta|$. This function is non-expansive and it can be shown that it yields a polyhedral C_Γ. In polar coordinates for $n = m = 2$:



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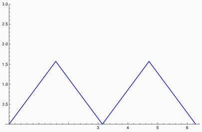
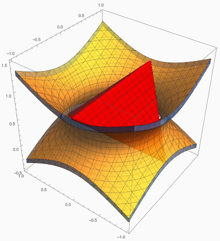


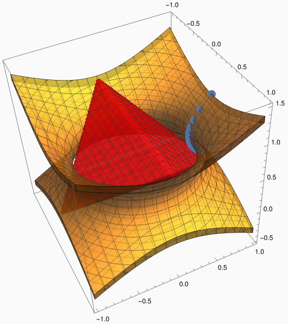
A 3D slice of the case $n = m = 2$ is:



Here there's no exposing point!

Maximality proof sketch

The idea of the proof is, for each facet, to construct an exposing sequence



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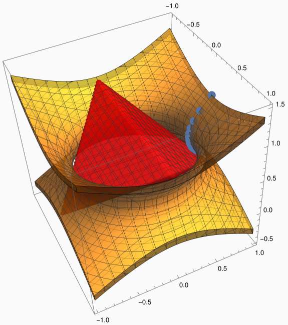
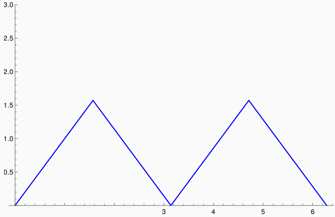


The sequence is such that every separating hyperplane sequence converges to the desired facet.

Why is this last example polyhedral?

The fact that $\Gamma(\beta) = |\beta|$ yields a polyhedral Q-free set may not be obvious. As seen before, in polar coordinates, the Γ function is



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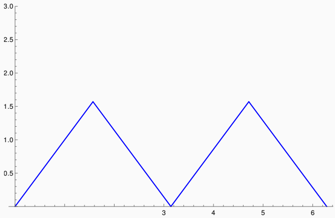

It can be shown that each break-point is a facet. Moreover, two consecutive breaking points are always isometries:

$$\|\Gamma(\beta) - \Gamma(\beta')\| = \|\beta - \beta'\|$$

and inequalities that lie "between" isometries are redundant.

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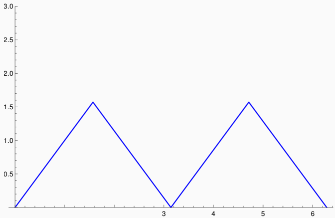

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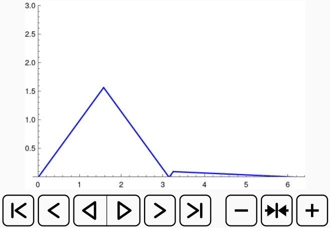
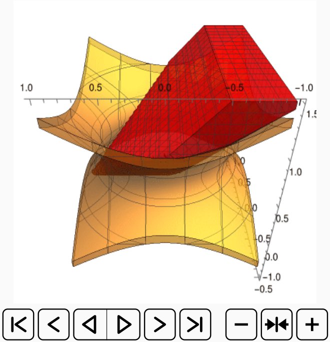
In **M., Paat and Serrano (2023)** we have a full characterization of when C_Γ is a polyhedron.

What if we consider the following family of Γ functions?
(in polar coordinates)





Some last fun observations

What if we consider the following family of Γ functions?
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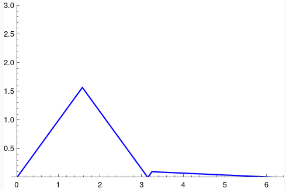

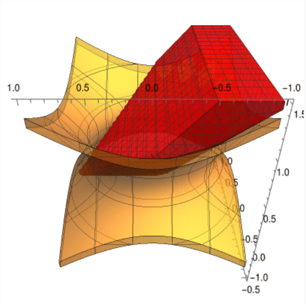

They all produce maximal Q-free sets, and only the last one is polyhedral!

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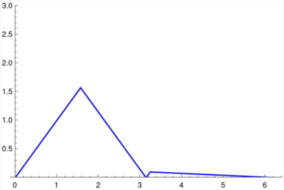

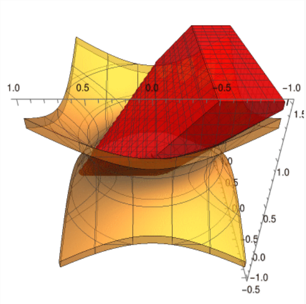

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Maximality of the non-polyhedral sets cannot be shown with the results of this talk

Summary

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Thank you!