On Constrained Mixed-Integer DR-Submodular Minimization

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Classical Submodularity

Ground set $N = \{1, 2, \ldots, n\}$

Definition (submodular set function)

A function $f : 2^N \to \mathbb{R}$ is **submodular** if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for any $X, Y \subseteq N$.

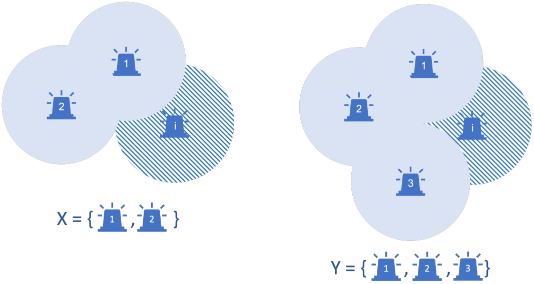
Intuition. Submodularity $\Leftrightarrow$ Diminishing (Marginal) Returns (DR).

Classical Submodularity

Marginal return $\rho_i(S) := f(S \cup \{i\}) - f(S)$.

Definition (submodular set function, alternative)

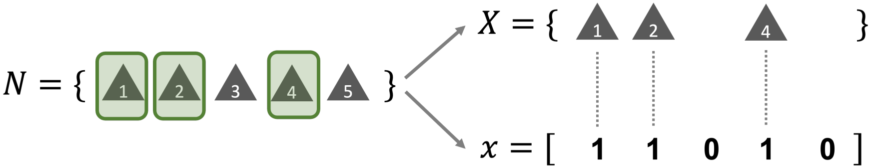
A function $f : 2^N \to \mathbb{R}$ is **submodular** if $\rho_i(X) \geq \rho_i(Y)$ for any $X \subseteq Y \subseteq N$, $i \in N \backslash Y$.



Applications: Sensor networks [Yu and Küçükyavuz, 2021a,b], social networks [Wu and Küçükyavuz, 2019], risk aversion [Wu and Küçükyavuz, 2018, 2020], portfolio optimization [Yu and Küçükyavuz, 2021c], etc.

Classical Submodular Set Function Optimization

Decision space: Selection from a **single** ground set. Modeled by **binary** variables.



Slight abuse of notation: Use $f(X)$ and $f(x)$ interchangeably.

Submodular Minimization

Unconstrained Submodular Set Function Minimization

- Strongly poly-time solvable [e.g., Iwata et al., 2001, Orlin, 2009]
- conv (epigraph of f) is given by extended polymatroid inequalities (EPI) [Atamtürk and Narayanan, 2021, Lovász, 1983]
- Separation of EPIs is easy (greedy does it) [Edmonds, 1970] $\rightarrow$ An equivalent (exponential) LP

Cardinality-constrained Submodular Set Function Minimization

- NP-hard and hard to approximate (polynomial factor lower bounds on the approximation factor) [Svitkina and Fleischer, 2011]
- Use the EPIs to solve the resulting problem as an MILP (delayed cut generation) [e.g., Atamtürk and Narayanan, 2008, for mean-risk optimization]

Research question:

Exact approaches for optimizing generalizations of submodular set functions to **mixed-integer** variables (i.e., choose **multiple (discrete or continuous)** copies of each item)?

Complexity is unclear. For example, certain finitely convergent convexification schemes for mixed-binary or pure integer programs become infinitely convergent for mixed-integer programs.

Diminishing Returns (DR)-Submodular Minimization

Diminishing Returns (DR)-Submodular Functions

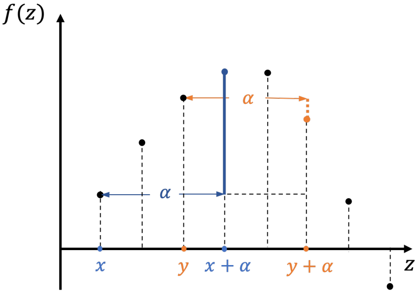
$\mathbf{e}^i$: a vector with 1 in the ith entry, 0 elsewhere.

Definition (DR-submodular)

A function $f : \mathcal{X} \subseteq \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is **DR-submodular** if

$$f(\mathbf{x} + \alpha\mathbf{e}^i) - f(\mathbf{x}) \geq f(\mathbf{y} + \alpha\mathbf{e}^i) - f(\mathbf{y})$$

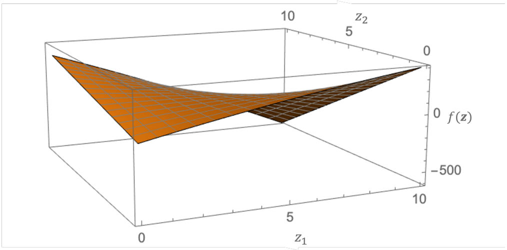
for every $i \in \{1, 2, \ldots, n+m\}$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $\mathbf{x} \leq \mathbf{y}$ component-wise, and for all $\alpha \in \mathbb{R}_+$ such that $\mathbf{x} + \alpha\mathbf{e}^i, \mathbf{y} + \alpha\mathbf{e}^i \in \mathcal{X}$.



Note: DR-submodular functions are not necessarily concave.

DR-Submodular Functions

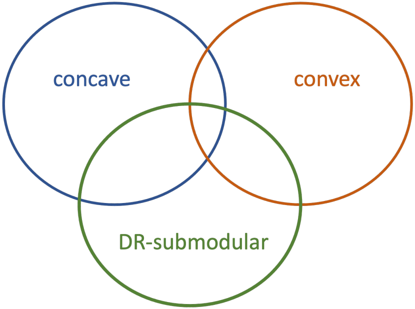
Example. Quadratic functions with non-positive Hessian entries (can be non-convex and non-concave).



A continuous DR-submodular function $f(\mathbf{z}) = -z_1^2 - 13z_1z_2 + 50z_1 + 30z_2$.

Example. Submodular set functions when $\mathcal{X} = \{0, 1\}^n$.

Mixed-Integer DR-Submodular Optimization



Existing Literature

DR-submodular maximization

$$\max_{\mathbf{z}\in\mathcal{Z}} f(\mathbf{z})$$

- Pure integer: Ene and Nguyen [2016], Soma and Yoshida [2017], Soma and Yoshida [2018].
- Continuous: Bian et al. [2017], Ene and Nguyen [2020], Medal and Ahanor [2022], Niazadeh et al. [2018], Sadeghi and Fazel [2020].

DR-submodular minimization

$$\min_{\mathbf{z}\in\mathcal{Z}} f(\mathbf{z})$$

- Bach [2019], Ene and Nguyen [2016].
- Pseudo-polynomial algorithms, pure integer variables, box constraints.
- Questions: Mixed-integer variables? Beyond box constraints? Polynomial-time solvable?

Problem Description

$$\min_{\mathbf{z} \in \mathcal{Z}(\mathcal{G},\mathbf{u})} f(\mathbf{z}),$$

where f is DR-submodular and $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ is a DAG representing the monotonicity relations among variables in $\mathcal{V} = \{1, 2, \ldots, n+m\}$.

$$\mathcal{Z}(\mathcal{G}, \mathbf{u}) := \{\mathbf{z} \in \mathbb{Z}^n \times \mathbb{R}^m : \mathbf{0} \leq \mathbf{z} \leq \mathbf{u}, z_i \leq z_j, \ \forall (i,j) \in \mathcal{A}\}.$$

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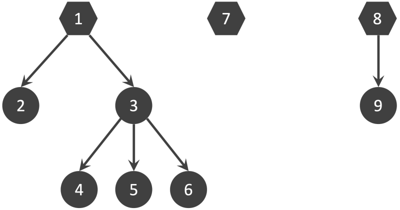
$\mathcal{G} = (\mathcal{V}, \mathcal{A})$: a **directed rooted forest**.

Definition (Directed rooted forest)

A disjoint union of directed rooted tree(s) with arcs pointing away from the root(s).



Example A

Example B

Equivalent Formulation

Recall

$$\min_{\mathbf{z}\in\mathcal{Z}(\mathcal{G},\mathbf{u})} f(\mathbf{z}),$$

$$\mathcal{Z}(\mathcal{G},\mathbf{u}) := \{\mathbf{z}\in\mathbb{Z}^n\times\mathbb{R}^m : \mathbf{0}\le\mathbf{z}\le\mathbf{u}, z_i\le z_j,\ \forall\,(i,j)\in\mathcal{A}\}.$$

Equivalently,

$$\min\left\{w : (\mathbf{z},w)\in \text{conv}\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)\right\},$$

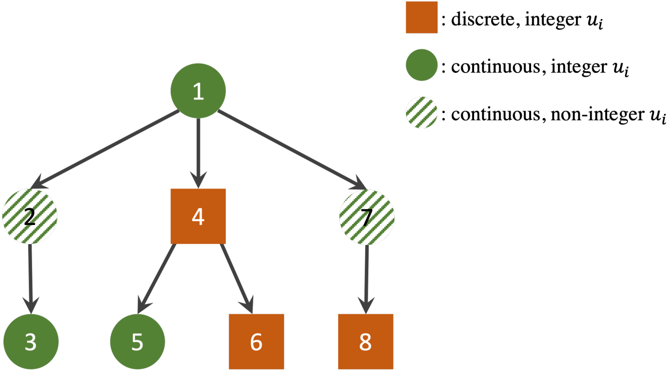
where

$$\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})} := \{(\mathbf{z},w)\in\mathcal{Z}(\mathcal{G},\mathbf{u})\times\mathbb{R} : w\ge f(\mathbf{z})\}.$$

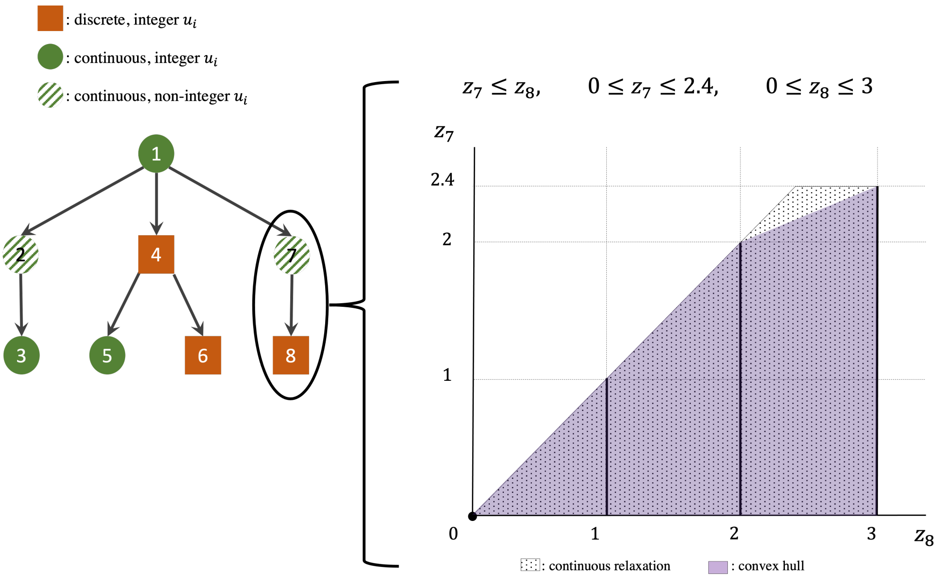
Understand $\text{conv}\left(\mathcal{Z}(\mathcal{G}, \mathbf{u})\right)$

Observation. Continuous relaxation of $\mathcal{Z}(\mathcal{G}, \mathbf{u})$ is **not** necessarily $\text{conv}\left(\mathcal{Z}(\mathcal{G}, \mathbf{u})\right)$.

Example.



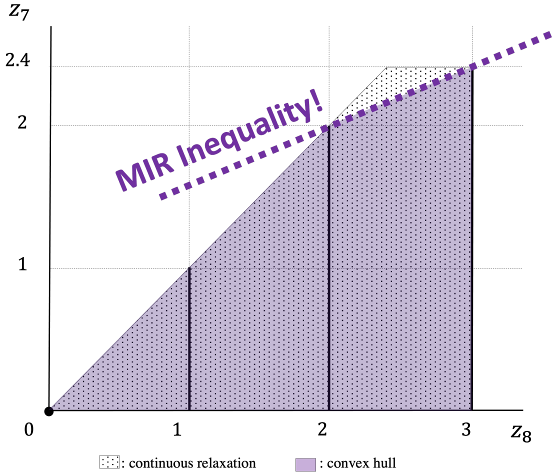
Understand conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$



Understand conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$

Mixed-Integer Rounding (MIR) inequality [Nemhauser and Wolsey, 1990]:

$$-z_8 + \frac{z_7}{u_7 - \lfloor u_7 \rfloor} \le \frac{\lfloor u_7 \rfloor (\lceil u_7 \rceil - u_7)}{u_7 - \lfloor u_7 \rfloor}.$$



Understand conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$

Ψ = The set of fractionally upper-bounded continuous variables with discrete descendant(s).

Theorem (informal; full description of conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$) [Yu and Küçükyavuz, 2022]

Under some conditions, conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$ is fully described by the trivial inequalities and the MIR inequalities for all $\psi \in \Psi$ and their children:

$$-z_{\mathrm{ch}(\psi)} + \frac{z_\psi}{u_\psi - \lfloor u_\psi \rfloor} \leq \frac{\lfloor u_\psi \rfloor (\lceil u_\psi \rceil - u_\psi)}{u_\psi - \lfloor u_\psi \rfloor}.$$

Characterize conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$

Proposition (validity of DR inequalities) [Yu and Küçükyavuz, 2022]

For *certain* permutations $\delta = (\delta(1), \delta(2), \ldots, \delta(|\mathcal{V}|))$, a **DR inequality associated with** δ

$$w \geq \sum_{k=0}^{|\mathcal{V}|} [t(\delta, \mathbf{z})_k - t(\delta, \mathbf{z})_{k+1}] \, f(P(\mathcal{T}^{\delta,k})).$$

is valid for $\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}$.

Characterize conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$

DR inequality [Yu and Küçükyavuz, 2022]

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Permutation

$\delta = (1,3,2)$

Permuta

$\delta = (1,$

Valid inequalities for $\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}$

Proposition [Yu and Küçükyavuz, 2022]

For *certain* permutations $\delta = (\delta(1), \delta(2), \ldots, \delta(|\mathcal{V}|))$, a **DR inequality associated with** δ is

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valid for $\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}$.

$t(\delta, \mathbf{z})_k$: linear expression of $\mathbf{z}$, with explicit form.

DR inequalities are **linear** and **homogeneous**.

They subsume the well-known extended polymatroid inequalities [Atamtürk and Narayanan, 2021, Edmonds, 1970, Lovász, 1983].

Note the equivalent DR-inequality

$$w \geq \sum_{k=1}^{|\mathcal{V}|} t(\delta, \mathbf{z})_k [f(P(\mathcal{T}^{\delta,k})) - f(P(\mathcal{T}^{\delta,k-1}))].$$

Extreme points of conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$

Proposition [Yu and Küçükyavuz, 2022]

The extreme points of conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$ are

$$P(\mathcal{S}) \in \mathbb{R}^{|\mathcal{V}|},\ \forall\, \mathcal{S} \subseteq \mathcal{V}.$$

For $i \in \mathcal{V}$,

$$P(\mathcal{S})_i = \begin{cases} \lfloor u_\psi \rfloor, & \text{if } i^\triangle(\mathcal{S}) = \psi \in \Psi, \text{ch}\,(\psi) \notin \mathcal{S}, \\ u_{i^\triangle(\mathcal{S})}, & \text{otherwise,} \end{cases}$$

$$i^\triangle(\mathcal{S}) := \begin{cases} \arg\max\{\text{depth}\,(j) : j \in \mathcal{S} \cap R^-(i)\}, & \text{if } \mathcal{S} \cap R^-(i) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

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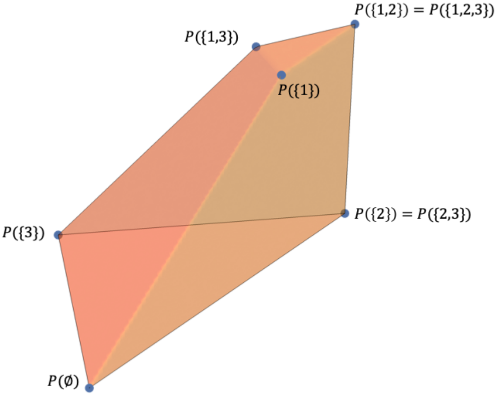

Nested subsets of $\mathcal{V}$ associated with permutation δ

$\delta = (\delta(1), \delta(2), \ldots, \delta(|\mathcal{V}|))$: a **permutation** of $\mathcal{V}$.

Nested subsets of $\mathcal{V}$ associated with δ:

$$\begin{aligned}
\mathcal{T}^{\delta,0} &:= \emptyset, \\
\mathcal{T}^{\delta,1} &:= \{\delta(1)\}, \\
&\ldots \\
\mathcal{T}^{\delta,k} &:= \{\delta(1), \delta(2), \ldots, \delta(k)\}, \\
&\ldots \\
\mathcal{T}^{\delta,|\mathcal{V}|} &:= \{\delta(1), \delta(2), \ldots, \delta(|\mathcal{V}|)\} = \mathcal{V}.
\end{aligned}$$

Extreme points of $\operatorname{conv}(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$ **associated with** δ:

$$\left\{ P(\mathcal{T}^{\delta,0}) = \mathbf{0},\ \ldots,\ P(\mathcal{T}^{\delta,k}),\ \ldots,\ P(\mathcal{T}^{\delta,|\mathcal{V}|}) = \mathbf{u} \right\}.$$

δ is **valid** when all the extreme points associated with δ are distinct *(technical details omitted)*.

An Important Property of conv $(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$

Proposition [Yu and Küçükyavuz, 2022]

For any $\bar{\mathbf{z}} \in \text{conv}(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$, let $\delta \leftarrow \texttt{Permutation\_Finder}(\bar{\mathbf{z}}, \mathcal{G}, \mathbf{u})$. Then $\bar{\mathbf{z}}$ can be written as the convex combination:

$$\bar{\mathbf{z}} = \sum_{k=0}^{|\mathcal{V}|} [t(\delta, \bar{\mathbf{z}})_k - t(\delta, \bar{\mathbf{z}})_{k+1}]\, P(\mathcal{T}^{\delta,k}).$$

Algorithm 1: `Permutation_Finder`

Input $\mathcal{G}$, $\mathbf{u}$, $\bar{\mathbf{z}} \in \text{conv}(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$;
for $k = 1, 2, \ldots, |\mathcal{V}|$ **do**
  $\Delta \leftarrow$ *legal* candidates for $\delta(k)$;
  $i^* \leftarrow \arg\max_{i \in \Delta} t((\delta(1), \ldots, \delta(k-1), \delta(\mathbf{k}) = \mathbf{i}), \bar{\mathbf{z}})_k$;
  `// Break ties arbitrarily in case of multiple maximizers.`
  $\delta \leftarrow (\delta(1), \ldots, \delta(k-1), i^*)$;
end
Output A valid δ.

An Important Property of conv$(\mathcal{Z}(\mathcal{G}, \mathbf{u}))$

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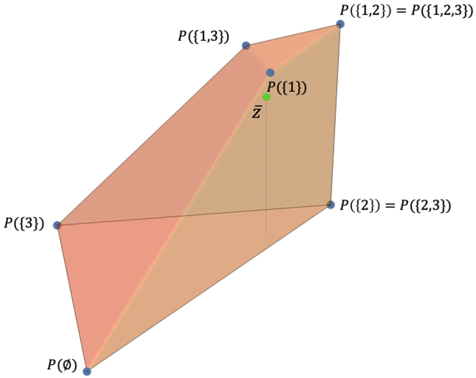

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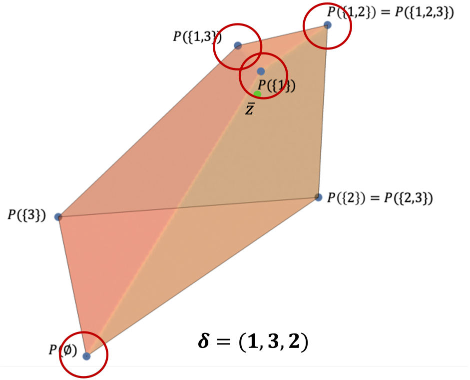

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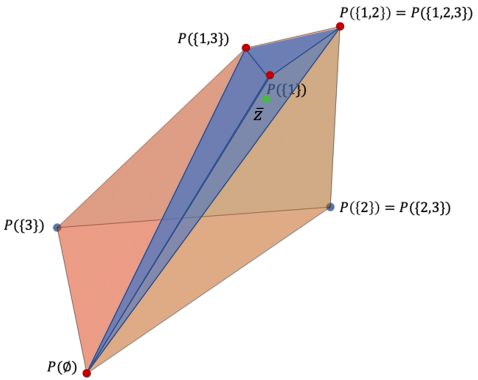

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$$\bar{\mathbf{z}} = \sum_{k=0}^{|\mathcal{V}|} \left[t(\delta, \bar{\mathbf{z}})_k - t(\delta, \bar{\mathbf{z}})_{k+1} \right] P(\mathcal{T}^{\delta,k}).$$

For $k \in \{1, \dots, |\mathcal{V}|\}$, let $i = \delta(k)$,

$$t(\delta, \mathbf{z})_k := \begin{cases} \dfrac{\eta_\psi(\mathbf{z}) - z_{i^\triangle(\mathcal{T}^{\delta,k-1})}}{\lfloor u_\psi \rfloor - u_{i^\triangle(\mathcal{T}^{\delta,k-1})}}, & \text{if } i = \psi \in \Psi \text{ and } \operatorname{ch}(\psi) \notin \mathcal{T}^{\delta,k-1}, \\ \dfrac{z_i - \eta_\psi(\mathbf{z})}{u_i - \lfloor u_\psi \rfloor}, & \text{else if } i^\triangle(\mathcal{T}^{\delta,k-1}) = \psi \in \Psi, \\ \dfrac{z_i - z_{i^\triangle(\mathcal{T}^{\delta,k-1})}}{u_i - u_{i^\triangle(\mathcal{T}^{\delta,k-1})}}, & \text{otherwise.} \end{cases}$$

$t(\delta, \mathbf{z})_0 = 1,\ t(\delta, \mathbf{z})_{|\mathcal{V}|+1} = 0.$

For any $\psi \in \Psi$, $\eta_\psi(\mathbf{z}) = \frac{z_\psi - (u_\psi - \lfloor u_\psi \rfloor) z_{\operatorname{ch}(\psi)}}{u_{\operatorname{ch}(\psi)} - u_\psi}$.

Gist: $t(\delta, \mathbf{z})_k$ is a linear expression of $\mathbf{z}$ that we can explicitly state.

Characterize conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$

Theorem (full description of conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$)[Yu and Küçükyavuz, 2022]

The DR inequalities, MIR inequalities, along with the box and monotonicity constraints, fully describe conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$.

Proof idea. $CP :=$ the set constructed by the DR inequalities, MIR inequalities, the box and monotonicity constraints.

- $CP \supseteq \text{conv}\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$. Validity of the inequalities.
- $CP \subseteq \text{conv}\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$. For any $[\mathbf{z}, w] \in CP$, find
  $\delta \leftarrow \texttt{Permutation\_Finder}(\mathbf{z}, \mathcal{G}, \mathbf{u})$.
  $[\mathbf{z}, w] =$ a convex combination of $[P(\mathcal{T}^{\delta,k}), f(P(\mathcal{T}^{\delta,k}))]_{k=0}^{|\mathcal{V}|}$
  $+$ a nonnegative multiple of $[\mathbf{0}, 1]$.

Therefore, $CP = \text{conv}\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$.

Characterize conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$

Theorem (full description of conv $\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$) [Yu and Küçükyavuz, 2022]

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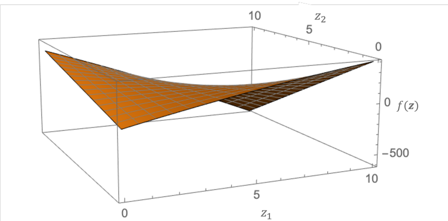
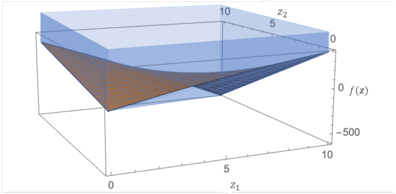


Mixed-integer nonlinear program.



Continuous linear program (with exponentially many constraints).

Exact Separation of DR Inequalities

Algorithm 2: `DR_Inequality_Separation`

Input $[\bar{\mathbf{z}}, \overline{w}] \notin \operatorname{conv}\left(\mathcal{P}_f^{\mathcal{Z}(\mathcal{G},\mathbf{u})}\right)$;

$\delta \leftarrow \texttt{Permutation\_Finder}(\mathcal{G}, \mathbf{u}, \bar{\mathbf{z}})$;

Output DR inequality associated with δ.

Proposition (exact separation of DR inequalities) [Yu and Küçükyavuz, 2022]

Algorithm 2 is an exact separation algorithm.

Remark. Algorithm 2 is an $\mathcal{O}(|\mathcal{V}|^2 \log |\mathcal{V}|)$ algorithm.

Corollary. The constrained mixed-integer DR-submodular minimization problem $\min_{\mathbf{z}\in\mathcal{Z}(\mathcal{G},\mathbf{u})} f(\mathbf{z})$ is polynomial-time solvable.

Takeaways

- DR-submodular optimization is a **mixed-integer extension** of classical submodular optimization.
- A polyhedral study on DR-submodular minimization
  - under **box** and possibly additional **monotonicity** constraints,
  - with **mixed-integer** variables.
- Propose **valid linear inequalities** and the complete **convex hull description** for the epigraph.

  Nonlinear program $\rightarrow$ Linear program.
- Provide an **exact separation** algorithm.
- Establish **polynomial** time complexity of this class of constrained mixed-integer DR-submodular minimization problems.

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