Families of genus-2 curves with 5-torsion

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Overview

Context: $\exists$ lots of parametrizations of elliptic curves with torsion (by modular curves like $\mathsf{X}_1(p)$, $\mathsf{X}(p)$); very few for abelian varieties of dimension > 1.

We find "$\mathsf{X}_1(5)$" for genus-2 Jacobians, i.e. parametrize (C, T) with $g(C) = 2$ and 5-torsion $T \in \mathrm{Jac}(C)$. Also the involution [sic] $(C, T) \leftrightarrow (C, 2T)$ [because $(C, T) \cong (C, -T)$].

Then refine to (C, T) with some level-2 structure (e.g. "$\mathsf{X}_1(10)$"), or satisfying some extra geometric or RM condition. E.g. a 2-dim. family with $\mathrm{Jac}(C)$ generically simple (even $\mathrm{End}_{\mathbf{C}} = \mathbf{Z}$) and torsion $(\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/5\mathbf{Z})$.

Connections elsewhere in the LMFDB (modular forms; hypergeometric periods).

Classical elliptic modular curves vs. genus-2 modular threefolds

We've long known and used formulas for $X_0(p)$ for plenty of p (100+) and a good number of $X_1(p)$ and even $X(p)$, some with beautiful geometry.

Not so for curves of genus 2: for $p > 3$, only partial parametrizations: notably, split Jacobians (Jac(C) isogenous to $E \times E'$, Howe–Leprévost–Poonen 2000 and Howe 2015); also, $p = 5$ with $T = P - P_0$ for some Weierstrass point P_0 (Boxall–Grant–Leprévost 2001: $y^2 + (x^3 + a_1 x^2 + a_2 x)y = a_5 x$ with P_0, P at $x = 0, \infty$).

For $p = 2$ and $p = 3$, similar formulas are known for $g = 2$ . . .

$p = 2$:

One 2-torsion point on $g = 1$ curve $y^2 = P_3(x) \longleftrightarrow$ partial factorization, $P_3 = P_1 P_2$;

One 2-torsion point on $g = 2$ curve $y^2 = P_6(x) \longleftrightarrow$ partial factorization, $P_6 = P_2 P_4$.

In each case, full level-2 structure (i.e. all 2-torsion rational) $\longleftrightarrow$ full factorization of P_3 or P_6.

$p = 3$:

One 3-torsion point on $y^2 = P_3(x)$: choice of a_1, a_2, a_3 s.t. $P_3 = (x + a_2)^3 + (a_1 x + a_3)^2$. Full level-3 structure: X(3), tetrahedral symmetry over $\mathbf{Q}(\mu_3)$.

One 3-torsion point on $y^2 = P_6(x)$: quadratic β s.t. $P_6 = c\gamma^3 + \beta^2$. Full level-3 structure: Burkhardt quartic, $\mathrm{Sp}_4(\mathbf{Z}/3\mathbf{Z})$ symmetry (again needs $\mathbf{Q}(\mu_3)$).

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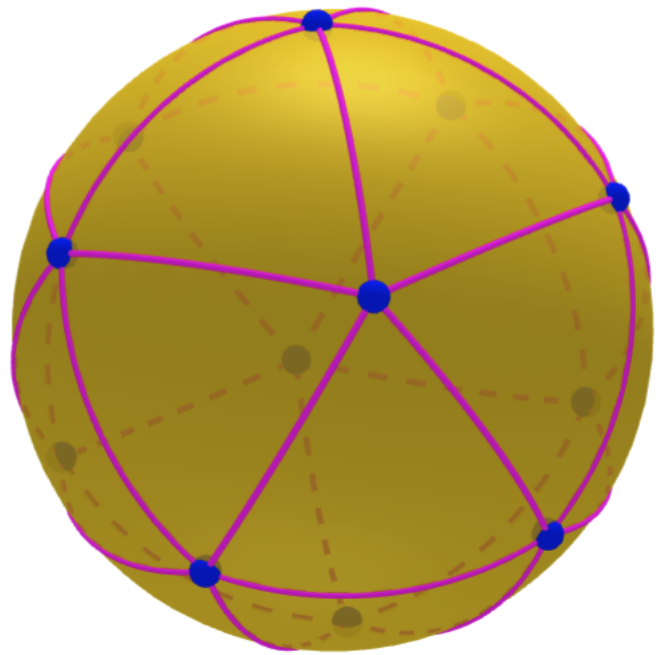
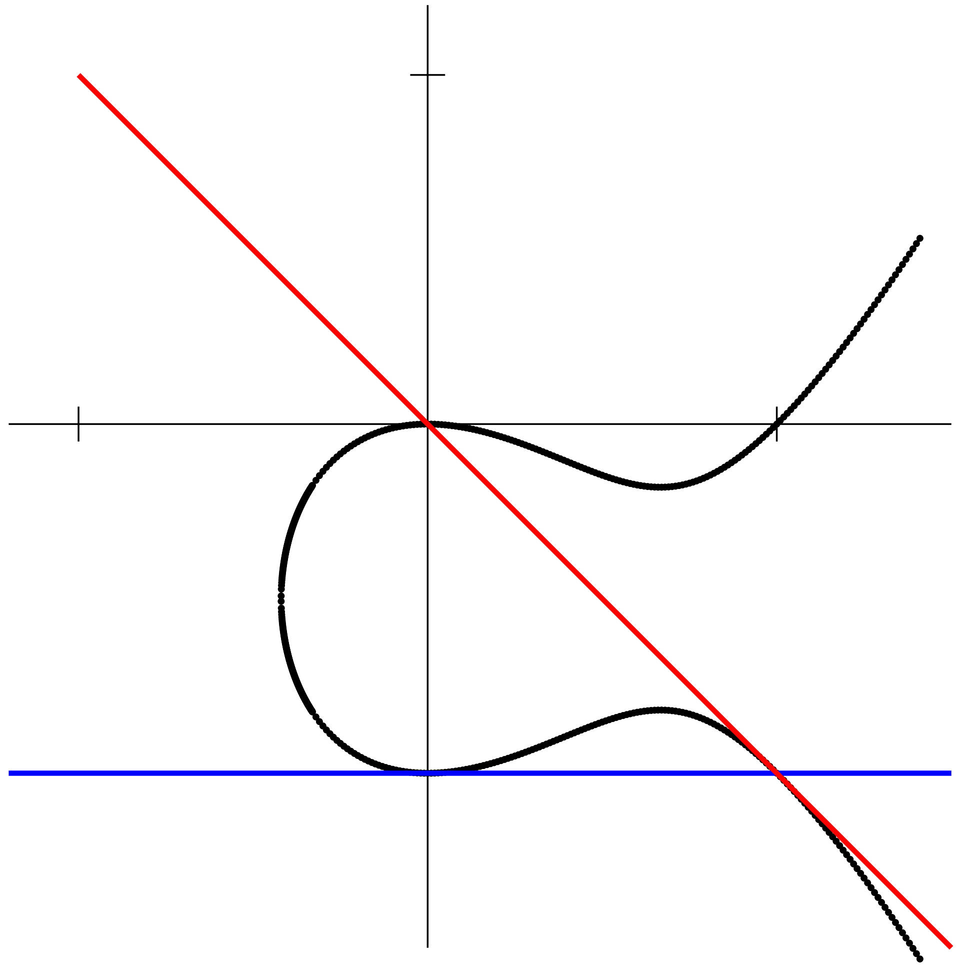
Next case is $p = 5$. For elliptic curves, $\mathrm{X}_0(5)$ and $\mathrm{X}_1(5)$ are well known, and even $\mathrm{X}(5)$ is famously rational with icosahedral symmetry over $\mathbf{Q}(\boldsymbol{\mu}_5)$.

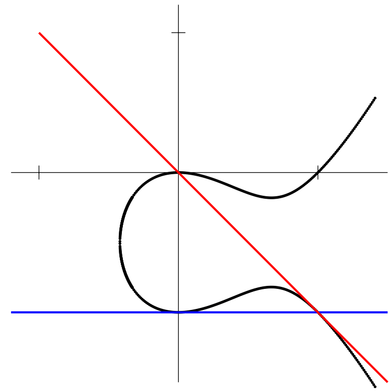


In genus 2, the moduli threefold "$\mathrm{X}(5)$" is of general type. But "$\mathrm{X}_1(5)$" turns out to be still rational — and still parallel to the modular curve.

Example of 5-torsion on elliptic curve

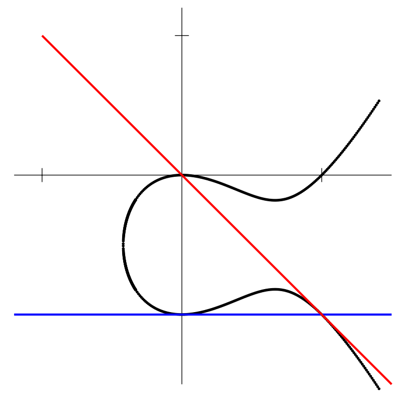
5-torsion on 11.a3 (a.k.a. $X_1(11)$) : $y^2 + y = x^3 - x^2$



[diagram repeat] 

We reconstruct any elliptic curve E with 5-torsion point P from such a picture: translate y so the tangent to P is $y = 0$, getting $y^2+(a_1x+a_3)y = x^2(x-x_0)$ with $-2P$ at $(x_0, 0)$; then line through $-2P$ and $-P$: $(0, -a_3)$ must be tangent at $2P$, "etc."

For 5-torsion on a genus-2 Jacobian, we typically get analogous picture from the effective divisors that represent $K + T$ and $K-2T$. Since $2(K+T)+(K-2T) \sim \iota^*(K+T)+2(K-2T) \sim 3K$, we get divisors of functions $y-\text{cubic}(x)$:

[diagram repeat] 

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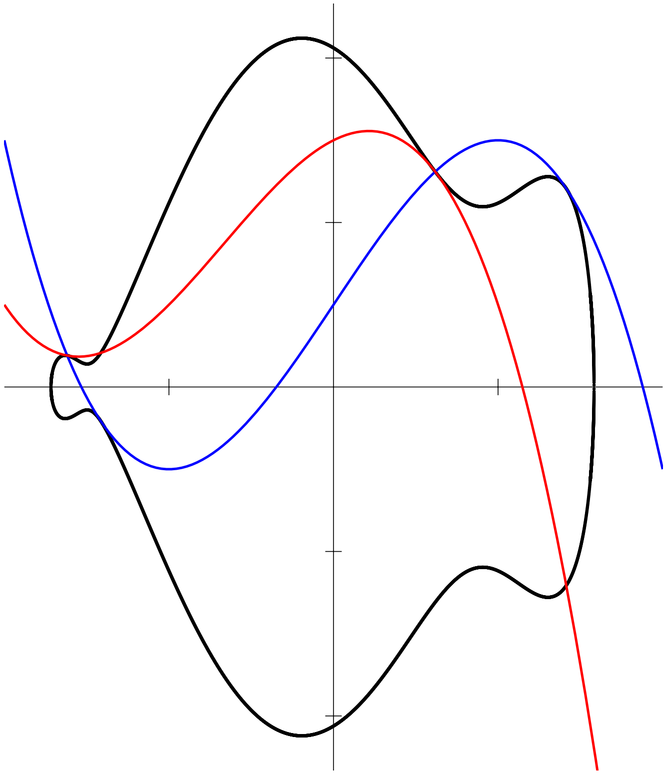
Example of 5-torsion on $g = 2$ curve

5-torsion on Jacobian of genus-2 curve 18055.b.18055.1:
$y^2 = \frac{1}{4}(x^3 + x + 1)^2 - x^6 - x^5 + 3x^4 + 3x^3 - 6x^2 - 3x + 4$



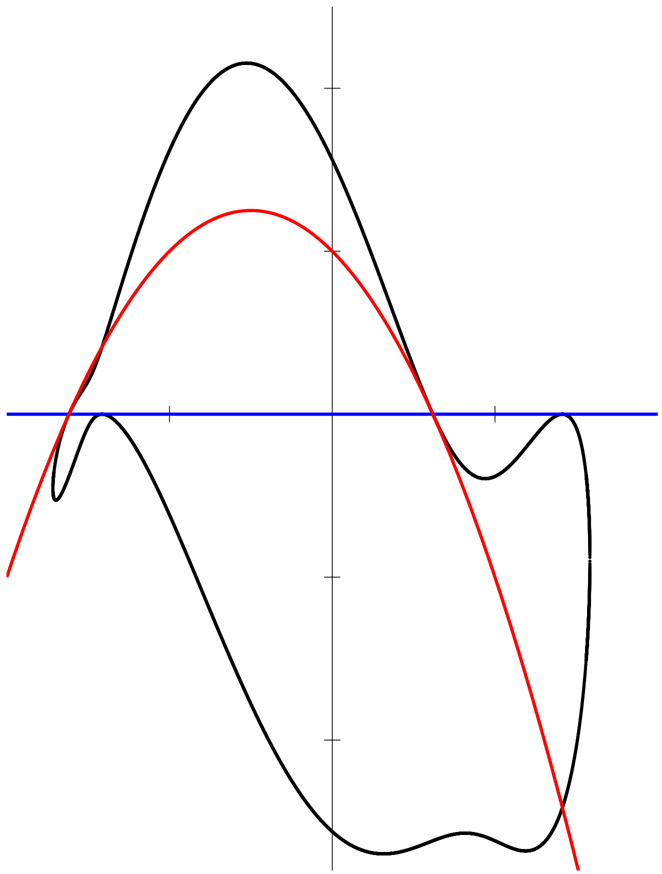
(blue curve: $2y = -x^3 + 3x + 1$; red: $2y = -x^3 + 2x^2 + x + 3$;
x-coordinates satisfy $x^2 = 2$ and $x^2 + x = 1$)

Translate y so blue is $y = 0$:

Example of 5-torsion on $g = 2$ curve, cont'd

5-torsion on Jacobian of genus-2 curve 18055.b.18055.1:

$$y^2 - (x^3 - 3x - 1)y = -x^6 - x^5 + 5x^4 + 4x^3 - 8x^2 - 4x + 4$$



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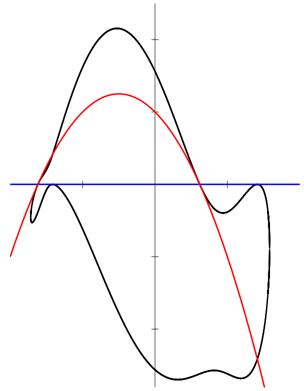

Then the RHS is a multiple of $Q^2(x)Q'(x)$ for quadratics Q, Q' vanishing on x-coordinates of the $K + T$ and $K - 2T$ divisor — indeed in our exmaple it's $-(x^2 - 2)^2(x^2 + x - 1)$; and substituting the red cubic for y then yields a multiple of QQ'^2. This gives an identity in Q, Q', and the linear coefficient of y, which we soon parametrize up to scaling Q, Q': the curve is

$$y^2 + (L'Q' - LQ)y = Q^2Q'$$

where L, L' are linear polynomials with $LL' = Q - Q'$; the divisors $\sim K + T$ and $K - 2T$ are $\{Q = 0, y = 0\}$ and $\{Q' = 0, y = LQ\}$.

The equation

$$y^2 + (L'Q' - LQ)y = Q^2Q'$$

has seven parameters, the 2+2+3 coefficients of L, L', Q; but generically L, L' are not proportional, and then using GL_2 we find an equivalent coordinate on the x-line for which

$$(L, L', Q) = (x, 1, q_2x^2 + q_1x + q_0).$$

so (q_0, q_1, q_2) are birational coordinates on the (C, T) moduli space. Changing T to $2T$ takes (q_0, q_1, q_2) to $(q_2, 1 - q_1, q_0)$; the quotient "$\mathrm{X}_0(5)$" by this involution is also rational.

To recover the modular elliptic curve $\mathrm{X}_1(5)$, regard the equation $y^2 + (L'Q' - LQ)y = Q^2Q'$ as an elliptic curve in the (Q, y) plane, with L, L' scalars and $Q' = Q - LL'$ as before. The 5-torsion points are again at $(Q, y) = (0, 0)$ and $(Q, y) = (LL', L^2L')$, and L'/L is a rational coordinate on $\mathrm{X}_1(5)$.

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$$L' = (1-2t)\left(\frac{t}{t-1}\right)^2 L, \quad Q = -\frac{t^4}{(t-1)^2}L^2$$

for some rational coordinate t on $X_1(10)$.

In the genus-2 setting, we can use this to get as many as 3 rational Weierstrass points. For generic t_0, t_1, t_∞ choose L, L' such that $L'/L = (1-2t_i)(t/(t-1))^2$ at $x = i$ $(i = 0, 1, \infty)$; then solve the linear equations for the coefficients of Q to make $Q/L^2 = t_i^4/(t_i - 1)^2$ at each x_i. One or two Weiersstrass points is similar and easier; so is parametrizing (C, T) with a 2-torsion point.

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All that assumes that the effective divisor representing $K+T$ is disjoint from both the $K+2T$ and the $K-2T$ divisor. That's the typical case, but there are *atypical* (C,T); that happens iff either T or $2T$ (but not both) is $\cong P-P_0$ with P_0 Weierstrass. (It then follows that the other is $\cong \pm(2P-K)$, but this is *not* an "iff": atypical (C,T) recover the 2-dim. Boxall–Grant–Leprévost family $y^2+(x^3+a_1x^2+a_2x)y=a_5x$.

Here even full 2-level structure is possible, with moduli space open in the Clebsch–Klein cubic surface $\sum_{i=1}^5 r_i = \sum_{i=1}^5 r_i^3 = 0$; this surface is rational so there are plenty of examples (albeit all beyond current LMFDB range). The simplest, from the point $(1:5:-7:-8:9)$ on the C–K cubic, is

$$y^2 = x(x+1)(x-1)(3x-7)(8x-13)(24x+25)$$

with P_0 at $x=-25/24$. This seems to be the record for torsion on a simple genus-2 Jacobian over $\mathbf{Q}$.

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RM5 Jacobians with a $\sqrt{5}$-torsion point

Alex Cowan, using Mestre's *méthode des graphes* with various old and new optimizations, computes modular forms φ giving simple factors of $J_0(N)$ of small dimension 2+ (prime $N \le 2 \cdot 10^6$ and counting). Most have dim $= 2$, and most of <u>those</u> are RM5.

If nontrivial p-torsion then $\varphi \equiv E_2 \bmod p$. Very rare beyond the known $(N, p) = (29, 7)$ and $(23, 11)$ — and most have RM5 and $p = 5$.

So, find Igusa invariants of (C, T) family and intersect with RM5 locus. Get 8434-term polynomial in q_0, q_1, q_2. Happily Maxima factors it in seconds, into

$$\Delta := (4(q_0 + q_2) + 3)q_0q_2 - q_1^2q_0 - (1 - q_1)^2q_2$$

times a 7434-term complementary factor. The surface $\Delta = 0$ is rational and parametrizes RM5 C with $\sqrt{5}$-torsion T.

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Next, for each of A.Cowan's forms, reconstruct a curve (using Cremona's "indirect method" via $L(1, \varphi \otimes \chi_d)$), find its coordinates on $\Delta = 0$. Most are on the line $(q_0, (-3q_0+1)/2, -q_0)$ with $1/q_0 \in \mathbf{Z}$ odd; e.g. $q_0 = 1$ gives curve 44521.a.44521.1 $\longleftrightarrow$ form 211.2.a.a:

$$y^2 + (x^3 + x^2 + 1)y = -x^6 + 5x^4 - 5x^3 - 8x^2 + 4x + 4.$$

In general $q_0 = 1/(2a+5)$ yields C_a with disc $= (2a+5)^{12}N^2$ where

$$N = a^4 + 10a^3 + 100a^2 + 375a + 625,$$

which should be prime infinitely often.

In the RM-5 setting, Jac$(C)/\langle T\rangle$ is itself principally polarized, so Jac(C') for some curve C'. I don't yet(?) have the formula in general, but for C_a it looks like C'_a is

$$y^2 = 5x^6 + (2a+20)x^5 + (a^2+10a+50)x^4 + (2a^2+10a+50)x^3 + (a^2+25)x^2 - 10(a+1)x - (4a+15)$$

with discriminant N^4.

This is all reminiscent of the Neumann–Setzer family of elliptic curves. . .

"Recall": For elliptic curves E of prime conductor p, always disc $= \pm p$ and $E_{\text{tors}} = \{0\}$, *except* a handful of small-conductor curves ($p = 11, 17, 19, 37$) and the Neumann–Setzer curves with $p = u^2 + 64$, namely

$$E_0 : \ y^2 + xy = x^3 - \frac{u+1}{4}x^2 + 4x - u \quad (\text{disc} = -p^2),$$

$$E_1 : \ y^2 + xy = x^3 - \frac{u+1}{4}x^2 - x \qquad\qquad (\text{disc} = p)$$

with 2-torsion $(u/4, -u/8)$ and $(0, 0)$ respectively.

Remarkably C'_a itself has a 5-torsion point — so an isogeny class of (at least) three curves, so also reminiscent of $X_0(11)$ and $X_1(11)$, where there's also a third curve (which Mazur called "$X_2(11)$"). For example, for conductor 211^2 there's also

$$y^2+(x^3+x+1)y=x^6+133x^5+434x^4-312811x^3-7229489x^2+18177671x-816824059$$

with discriminant $11^{24}41^{12}211^2$.

Ari Shnidman (e-mail of 29 Aug. 2021): this had to happen for prime N "as explained in Mazur's paper"; once it's an infinite family, must work even for N composite.

Finally, what if (C, T) both RM5 and atypical? Intersect both surfaces to get one-dim. family

$$C_\jmath : y^2 = (x^3 + 5x^2 + 5x)^2 + 4\jmath x.$$

It's hypergeometric, e.g. for $|1 - \jmath| < 1$,

$$\int_0^\infty \frac{dx}{\sqrt{(x^3 + 5x^2 + 5x)^2 + 4\jmath x}} = \frac{\pi}{5 \sin(2\pi/5)} \, {}_2F_1(2/5, 3/5; 1; 1 - \jmath),$$

$$\int_0^\infty \frac{(x + 1)\, dx}{\sqrt{(x^3 + 5x^2 + 5x)^2 + 4\jmath x}} = \frac{\pi}{5 \sin(\pi/5)} \, {}_2F_1(1/5, 4/5; 1; 1 - \jmath).$$

(Proof: change of variables

$$x = z - 2 + z^{-1}, \quad x^3 + 5x^2 + 5x = (z - 1)(z^5 - 1)/z^3,$$

"etc." — see paper.)

Further directions

We got this far with hardly any theoretical machinery. How does all this connect with . . .

• Siegel and Hilbert modular forms for the relevant congruence subgroups of $\mathrm{Sp}_4(\mathbf{Z})$ and $\mathrm{SL}_2(\mathbf{Z}[\frac{1+\sqrt{5}}{2}])$?

• Your favorite compactification of the associated modular threefold? (For starters, how does the surface of "atypical" (C, P) fit on the boundary of the "typical" threefold?)

• Horrocks–Mumford surfaces ($= (1, 5)$-polarized ab. surfaces in $\mathbf{P}^4$, via $\mathrm{Jac}(C)/\langle T \rangle$)? What's isogenous C' in RM-5 case?

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