Families of genus-2 curves with 5-torsion

LuCaNT, ICERM, 14 July 2023

Noam D. Elkies, Harvard University
Overview

Context: ∃ lots of parametrizations of elliptic curves with torsion (by modular curves like $X_1(p)$, $X(p)$); very few for abelian varieties of dimension $> 1$.

We find “$X_1(5)$” for genus-2 Jacobians, i.e. parametrize $(C, T)$ with $g(C) = 2$ and 5-torsion $T \in \text{Jac}(C)$. Also the involution [sic] $(C, T) \leftrightarrow (C, 2T)$ [because $(C, T) \cong (C, -T)$].

Then refine to $(C, T)$ with some level-2 structure (e.g. “$X_1(10)$”), or satisfying some extra geometric or RM condition. E.g. a 2-dim. family with $\text{Jac}(C)$ generically simple (even $\text{End}_C = \mathbb{Z}$) and torsion $(\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/5\mathbb{Z})$.

Connections elsewhere in the LMFDB (modular forms; hypergeometric periods).
Classical elliptic modular curves
vs. genus-2 modular threefolds

We’ve long known and used formulas for $X_0(p)$ for plenty of $p$ (100+) and a good number of $X_1(p)$ and even $X(p)$, some with beautiful geometry.

Not so for curves of genus 2: for $p > 3$, only partial parametrizations: notably, split Jacobians ($\text{Jac}(C)$ isogenous to $E \times E'$, Howe–Leprévost–Poonen 2000 and Howe 2015); also, $p = 5$ with $T = P - P_0$ for some Weierstrass point $P_0$ (Boxall–Grant–Leprévost 2001: $y^2 + (x^3 + a_1x^2 + a_2x)y = a_5x$ with $P_0, P$ at $x = 0, \infty$).

For $p = 2$ and $p = 3$, similar formulas are known for $g = 2 \ldots$
$p = 2$:

One 2-torsion point on $g = 1$ curve $y^2 = P_3(x) \leftrightarrow$ partial factorization, $P_3 = P_1P_2$;

One 2-torsion point on $g = 2$ curve $y^2 = P_6(x) \leftrightarrow$ partial factorization, $P_6 = P_2P_4$.

In each case, full level-2 structure (i.e. all 2-torsion rational) $\leftrightarrow$ full factorization of $P_3$ or $P_6$.

$p = 3$:

One 3-torsion point on $y^2 = P_3(x)$: choice of $a_1,a_2,a_3$ s.t. $P_3 = (x + a_2)^3 + (a_1x + a_3)^2$. Full level-3 structure: $X(3)$, tetrahedral symmetry over $\mathbb{Q}(\mu_3)$.

One 3-torsion point on $y^2 = P_6(x)$: quadratic $\beta$ s.t. $P_6 = c\gamma^3 + \beta^2$. Full level-3 structure: Burkhardt quartic, $Sp_4(\mathbb{Z}/3\mathbb{Z})$ symmetry (again needs $\mathbb{Q}(\mu_3)$).
\( p = 2: \)

One 2-torsion point on \( g = 1 \) curve \( y^2 = P_3(x) \) \( \longleftrightarrow \) partial factorization, \( P_3 = P_1P_2; \)

One 2-torsion point on \( g = 2 \) curve \( y^2 = P_6(x) \) \( \longleftrightarrow \) partial factorization, \( P_6 = P_2P_4. \)

In each case, full level-2 structure (i.e. all 2-torsion rational) \( \longleftrightarrow \) full factorization of \( P_3 \) or \( P_6. \)

\( p = 3: \)

One 3-torsion point on \( y^2 = P_3(x) \): choice of \( a_1, a_2, a_3 \) s.t. \( P_3 = (x + a_2)^3 + (a_1x + a_3)^2. \) Full level-3 structure: \( X(3), \) tetrahedral symmetry over \( \mathbb{Q}(\mu_3). \)

One 3-torsion point on \( y^2 = P_6(x) \): quadratic \( \beta \) s.t. \( P_6 = c\gamma^3 + \beta^2. \) Full level-3 structure: Burkhardt quartic, \( \text{Sp}_4(\mathbb{Z}/3\mathbb{Z}) \) symmetry (again needs \( \mathbb{Q}(\mu_3). \)).
Next case is $p = 5$. For elliptic curves, $X_0(5)$ and $X_1(5)$ are well known, and even $X(5)$ is famously rational with icosahedral symmetry over $\mathbb{Q}(\mu_5)$.

In genus 2, the moduli threefold “$X(5)$” is of general type. But “$X_1(5)$” turns out to be still rational — and still parallel to the modular curve.
Example of 5-torsion on elliptic curve

5-torsion on 11.a3 (a.k.a. $X_1(11)$) : $y^2 + y = x^3 - x^2$
We reconstruct any elliptic curve $E$ with 5-torsion point $P$ from such a picture: translate $y$ so the tangent to $P$ is $y = 0$, getting $y^2 + (a_1 x + a_3)y = x^2(x - x_0)$ with $-2P$ at $(x_0, 0)$; then line through $-2P$ and $-P : (0, -a_3)$ must be tangent at $2P$, “etc.”

For 5-torsion on a genus-2 Jacobian, we typically get analogous picture from the effective divisors that represent $K + T$ and $K - 2T$. Since $2(K+T) + (K-2T) \sim \iota^*(K+T) + 2(K-2T) \sim 3K$, we get divisors of functions $y - \text{cubic}(x)$:
We reconstruct any elliptic curve $E$ with 5-torsion point $P$ from such a picture: translate $y$ so the tangent to $P$ is $y = 0$, getting $y^2 + (a_1 x + a_3)y = x^2(x - x_0)$ with $-2P$ at $(x_0, 0)$; then line through $-2P$ and $-P : (0, -a_3)$ must be tangent at $2P$, “etc.”

For 5-torsion on a genus-2 Jacobian, we typically get analogous picture from the effective divisors that represent $K + T$ and $K - 2T$. Since $2(K + T) + (K - 2T) \sim \iota^*(K + T) + 2(K - 2T) \sim 3K$, we get divisors of functions $y - \text{cubic}(x)$:
Example of 5-torsion on $g = 2$ curve

5-torsion on Jacobian of genus-2 curve 18055.b.18055.1:

$$y^2 = \frac{1}{4}(x^3 + x + 1)^2 - x^6 - x^5 + 3x^4 + 3x^3 - 6x^2 - 3x + 4$$

(blue curve: $2y = -x^3 + 3x + 1$; red: $2y = -x^3 + 2x^2 + x + 3$; $x$-coordinates satisfy $x^2 = 2$ and $x^2 + x = 1$)

Translate $y$ so blue is $y = 0$: 
Example of 5-torsion on $g = 2$ curve, cont’d

5-torsion on Jacobian of genus-2 curve 18055.b.18055.1:

\[ y^2 - (x^3 - 3x - 1)y = -x^6 - x^5 + 5x^4 + 4x^3 - 8x^2 - 4x + 4 \]

(blue curve: $y = 0$; red: $y = -x^2 - x + 1$)
[repeat] \[y^2 - (x^3 - 3x - 1)y = -x^6 - x^5 + 5x^4 + 4x^3 - 8x^2 - 4x + 4\]

Then the RHS is a multiple of \(Q^2(x)Q'(x)\) for quadratics \(Q, Q'\) vanishing on \(x\)-coordinates of the \(K + T\) and \(K - 2T\) divisor — indeed in our example it's \(-(x^2 - 2)^2(x^2 + x - 1)\); and substituting the red cubic for \(y\) then yields a multiple of \(QQ'^2\). This gives an identity in \(Q, Q'\), and the linear coefficient of \(y\), which we soon parametrize up to scaling \(Q, Q'\): the curve is

\[y^2 + (L'Q' - LQ)y = Q^2Q'\]

where \(L, L'\) are linear polynomials with \(LL' = Q - Q'\); the divisors \(\sim K + T\) and \(K - 2T\) are \(\{Q = 0, y = 0\}\) and \(\{Q' = 0, y = LQ\}\).
The equation

\[ y^2 + (L'Q' - LQ)y = Q^2Q' \]

has seven parameters, the 2 + 2 + 3 coefficients of \( L, L', Q \); but generically \( L, L' \) are not proportional, and then using \( \text{GL}_2 \) we find an equivalent coordinate on the \( x \)-line for which

\[ (L, L', Q) = (x, 1, q_2x^2 + q_1x + q_0). \]

so \((q_0, q_1, q_2)\) are birational coordinates on the \((C, T)\) moduli space. Changing \( T \) to \( 2T \) takes \((q_0, q_1, q_2)\) to \((q_2, 1 - q_1, q_0)\); the quotient “\(X_0(5)\)” by this involution is also rational.

To recover the modular elliptic curve \(X_1(5)\), regard the equation \( y^2 + (L'Q' - LQ)y = Q^2Q' \) as an elliptic curve in the \((Q, y)\) plane, with \( L, L' \) scalars and \( Q' = Q - LL' \) as before. The 5-torsion points are again at \((Q, y) = (0, 0)\) and \((Q, y) = (LL', L^2L')\), and \( L'/L \) is a rational coordinate on \(X_1(5)\).
The equation
\[ y^2 + (L'Q' - LQ)y = Q^2Q' \]
has seven parameters, the \(2 + 2 + 3\) coefficients of \(L, L', Q\); but generically \(L, L'\) are not proportional, and then using \(\text{GL}_2\) we find an equivalent coordinate on the \(x\)-line for which
\[
(L, L', Q) = (x, 1, q_2x^2 + q_1x + q_0).
\]
so \((q_0, q_1, q_2)\) are birational coordinates on the \((C, T)\) moduli space. Changing \(T\) to \(2T\) takes \((q_0, q_1, q_2)\) to \((q_2, 1 - q_1, q_0)\); the quotient "\(X_0(5)\)" by this involution is also rational.

To recover the modular elliptic curve \(X_1(5)\), regard the equation \(y^2 + (L'Q' - LQ)y = Q^2Q'\) as an elliptic curve in the \((Q, y)\) plane, with \(L, L'\) scalars and \(Q' = Q - LL'\) as before. The 5-torsion points are again at \((Q, y) = (0, 0)\) and \((Q, y) = (LL', L^2L')\), and \(L'/L\) is a rational coordinate on \(X_1(5)\).
Now the modular curve $X_1(10)$ is still rational; it parametrizes $(E, T, T_0)$ where $E$ is an elliptic curve and $T, T_0$ are 5-torsion and 2-torsion points. That’s a 3:1 cover of $X_1(5)$, which parametrizes only $(E, T)$; explicitly

$$L' = (1 - 2t) \left( \frac{t}{t - 1} \right)^2 L,$$
$$Q = -\frac{t^4}{(t - 1)^2} L^2$$

for some rational coordinate $t$ on $X_1(10)$.

In the genus-2 setting, we can use this to get as many as 3 rational Weierstrass points. For generic $t_0, t_1, t_\infty$ choose $L, L'$ such that $L'/L = (1 - 2t_i)(t/(t - 1))^2$ at $x = i$ ($i = 0, 1, \infty$); then solve the linear equations for the coefficients of $Q$ to make $Q/L^2 = t_i^4/(t_i - 1)^2$ at each $x_i$. One or two Weiersstrass points is similar and easier; so is parametrizing $(C, T)$ with a 2-torsion point.

[This is already [NDE 2003], with similar formulas for genus-2 Jacobians with a 6-, 7-, or 8-torsion point.]
Now the modular curve $X_1(10)$ is still rational; it parametrizes $(E, T, T_0)$ where $E$ is an elliptic curve and $T, T_0$ are 5-torsion and 2-torsion points. That's a 3:1 cover of $X_1(5)$, which parametrizes only $(E, T)$; explicitly

$$L' = (1 - 2t) \left(\frac{t}{t - 1}\right)^2 L, \quad Q = -\frac{t^4}{(t - 1)^2} L^2$$

for some rational coordinate $t$ on $X_1(10)$.

In the genus-2 setting, we can use this to get as many as 3 rational Weierstrass points. For generic $t_0, t_1, t_\infty$ choose $L, L'$ such that $L'/L = (1 - 2t_i)(t/(t - 1))^2$ at $x = i$ ($i = 0, 1, \infty$); then solve the linear equations for the coefficients of $Q$ to make $Q/L^2 = t_i^4/(t_i - 1)^2$ at each $x_i$. One or two Weierstrass points is similar and easier; so is parametrizing $(C, T)$ with a 2-torsion point.

[This is already [NDE 2003], with similar formulas for genus-2 Jacobians with a 6-, 7-, or 8-torsion point.]
All that assumes that the effective divisor representing $K + T$ is disjoint from both the $K + 2T$ and the $K - 2T$ divisor. That’s the typical case, but there are atypical $(C, T)$; that happens iff either $T$ or $2T$ (but not both) is $\cong P - P_0$ with $P_0$ Weierstrass. (It then follows that the other is $\cong \pm(2P - K)$, but this is not an “iff”: atypical $(C, T)$ recover the 2-dim. Boxall–Grant–Leprévost family $y^2 + (x^3 + a_1x^2 + a_2x)y = a_5x$.

Here even full 2-level structure is possible, with moduli space open in the Clebsch–Klein cubic surface $\sum_{i=1}^5 r_i = \sum_{i=1}^5 r_i^3 = 0$; this surface is rational so there are plenty of examples (albeit all beyond current LMFDB range). The simplest, from the point $(1 : 5 : -7 : -8 : 9)$ on the C–K cubic, is

$$y^2 = x(x + 1)(x - 1)(3x - 7)(8x - 13)(24x + 25)$$

with $P_0$ at $x = -25/24$. This seems to be the record for torsion on a simple genus-2 Jacobian over $\mathbb{Q}$. 
All that assumes that the effective divisor representing $K + T$ is disjoint from both the $K + 2T$ and the $K - 2T$ divisor. That’s the typical case, but there are atypical $(C, T)$; that happens iff either $T$ or $2T$ (but not both) is $\cong P - P_0$ with $P_0$ Weierstrass. (It then follows that the other is $\cong \pm(2P - K)$, but this is not an “iff”: atypical $(C, T)$ recover the 2-dim. Boxall–Grant–Leprévost family $y^2 + (x^3 + a_1x^2 + a_2x)y = a_5x$.

Here even full 2-level structure is possible, with moduli space open in the Clebsch–Klein cubic surface $\sum_{i=1}^{5} r_i = \sum_{i=1}^{5} r_i^3 = 0$; this surface is rational so there are plenty of examples (albeit all beyond current LMFDB range). The simplest, from the point $(1 : 5 : -7 : -8 : 9)$ on the C–K cubic, is

$$y^2 = x(x + 1)(x - 1)(3x - 7)(8x - 13)(24x + 25)$$

with $P_0$ at $x = -25/24$. This seems to be the record for torsion on a simple genus-2 Jacobian over $\mathbb{Q}$. 
RM5 Jacobians with a $\sqrt{5}$-torsion point

Alex Cowan, using Mestre’s *méthode des graphes* with various old and new optimizations, computes modular forms $\varphi$ giving simple factors of $J_0(N)$ of small dimension $2+$ (prime $N \leq 2 \cdot 10^6$ and counting). Most have dim = 2, and most of those are RM5.

If nontrivial $p$-torsion then $\varphi \equiv E_2 \mod p$. Very rare beyond the known $(N, p) = (29, 7)$ and $(23, 11)$ — and most have RM5 and $p = 5$.

So, find Igusa invariants of $(C, T)$ family and intersect with RM5 locus. Get 8434-term polynomial in $q_0, q_1, q_2$. Happily Maxima factors it in seconds, into

$$\Delta := (4(q_0 + q_2) + 3)q_0q_2 - q_1^2q_0 - (1 - q_1)^2q_2$$

times a 7434-term complementary factor. The surface $\Delta = 0$ is rational and parametrizes RM5 $C$ with $\sqrt{5}$-torsion $T$. 

13
RM5 Jacobians with a $\sqrt{5}$-torsion point

Alex Cowan, using Mestre’s *méthode des graphes* with various old and new optimizations, computes modular forms $\varphi$ giving simple factors of $J_0(N)$ of small dimension $2+$ (prime $N \leq 2 \cdot 10^6$ and counting). Most have $\dim = 2$, and most of those are RM5.

If nontrivial $p$-torsion then $\varphi \equiv E_2 \mod p$. Very rare beyond the known $(N,p) = (29,7)$ and $(23,11)$ — and most have RM5 and $p = 5$.

So, find Igusa invariants of $(C,T)$ family and intersect with RM5 locus. Get 8434-term polynomial in $q_0,q_1,q_2$. Happily Maxima factors it in seconds, into

$$\Delta := (4(q_0 + q_2) + 3)q_0q_2 - q_1^2q_0 - (1 - q_1)^2q_2$$

times a 7434-term complementary factor. The surface $\Delta = 0$ is rational and parametrizes RM5 $C$ with $\sqrt{5}$-torsion $T$.  

13
RM5 Jacobians with a $\sqrt{5}$-torsion point

Next, for each of A. Cowan’s forms, reconstruct a curve (using Cremona’s “indirect method” via $L(1, \varphi \otimes \chi_d)$), find its coordinates on $\Delta = 0$. Most are on the line $(q_0, (-3q_0 + 1)/2, -q_0)$ with $1/q_0 \in \mathbb{Z}$ odd; e.g. $q_0 = 1$ gives curve 44521.a.44521.1 $\leftrightarrow$ form 211.2.a.a:

$$y^2 + (x^3 + x^2 + 1)y = -x^6 + 5x^4 - 5x^3 - 8x^2 + 4x + 4.$$  

In general $q_0 = 1/(2a + 5)$ yields $C_a$ with disc $= (2a + 5)^{12}N^2$ where

$$N = a^4 + 10a^3 + 100a^2 + 375a + 625,$$

which should be prime infinitely often.
In the RM-5 setting, $\text{Jac}(C)/\langle T \rangle$ is itself principally polarized, so $\text{Jac}(C')$ for some curve $C'$. I don’t yet(?) have the formula in general, but for $C_a$ it looks like $C'_a$ is

$$y^2 = 5x^6 + (2a + 20)x^5 + (a^2 + 10a + 50)x^4$$

$$+ (2a^2 + 10a + 50)x^3 + (a^2 + 25)x^2 - 10(a + 1)x - (4a + 15)$$

with discriminant $N^4$.

This is all reminiscent of the Neumann–Setzer family of elliptic curves...
“Recall”: For elliptic curves $E$ of prime conductor $p$, always \( \text{disc} = \pm p \) and $E_{\text{tors}} = \{0\}$, except a handful of small-conductor curves ($p = 11, 17, 19, 37$) and the Neumann–Setzer curves with $p = u^2 + 64$, namely

\[
E_0 : \quad y^2 + xy = x^3 - \frac{u + 1}{4} x^2 + 4x - u \quad (\text{disc} = -p^2),
\]

\[
E_1 : \quad y^2 + xy = x^3 - \frac{u + 1}{4} x^2 - x \quad (\text{disc} = p)
\]

with 2-torsion \((u/4, -u/8)\) and \((0,0)\) respectively.
Remarkably $C'_a$ itself has a 5-torsion point — so an isogeny class of (at least) three curves, so also reminiscent of $X_0(11)$ and $X_1(11)$, where there’s also a third curve (which Mazur called “$X_2(11)$”). For example, for conductor $211^2$ there’s also

$$y^2 + (x^3 + x + 1)y = x^6 + 133x^5 + 434x^4 - 312811x^3 - 7229489x^2 + 18177671x - 816824059$$

with discriminant $11^2 41^{12} 211^2$.

Ari Shnidman (e-mail of 29 Aug. 2021): this had to happen for prime $N$ “as explained in Mazur’s paper”; once it’s an infinite family, must work even for $N$ composite.
Finally, what if \((C, T)\) both RM5 and atypical? Intersect both surfaces to get one-dim. family

\[
C_j : y^2 = (x^3 + 5x^2 + 5x)^2 + 4jx.
\]

It’s hypergeometric, e.g. for \(|1 - j| < 1\),

\[
\int_0^\infty \frac{dx}{\sqrt{(x^3 + 5x^2 + 5x)^2 + 4jx}} = \frac{\pi}{5 \sin(2\pi/5)} {}_2F_1\left(\frac{2}{5}, \frac{3}{5}; 1; 1-j\right),
\]

\[
\int_0^\infty \frac{(x + 1) dx}{\sqrt{(x^3 + 5x^2 + 5x)^2 + 4jx}} = \frac{\pi}{5 \sin(\pi/5)} {}_2F_1\left(\frac{1}{5}, \frac{4}{5}; 1; 1-j\right).
\]

(Proof: change of variables

\[x = z - 2 + z^{-1}, \quad x^3 + 5x^2 + 5x = (z - 1)(z^5 - 1)/z^3,\]

“etc.” — see paper.)
Further directions

We got this far with hardly any theoretical machinery. How does all this connect with . . .

• Siegel and Hilbert modular forms for the relevant congruence subgroups of $\text{Sp}_4(\mathbb{Z})$ and $\text{SL}_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}])$?

• Your favorite compactification of the associated modular threefold? (For starters, how does the surface of “atypical” $(C, P)$ fit on the boundary of the “typical” threefold?)

• Horrocks–Mumford surfaces ($= (1, 5)$-polarized ab. surfaces in $\mathbb{P}^4$, via $\text{Jac}(C)/\langle T \rangle$)? What’s isogenous $C'$ in RM-5 case?

I thank: NSF, Simons; GP-PARI, Maxima, Sage; Alex Cowan; Ari Shnidman, Bjorn Poonen and Drew Sutherland; LuCaNT organizers and referees; and you.
Further directions

We got this far with hardly any theoretical machinery. How does all this connect with …

• Siegel and Hilbert modular forms for the relevant congruence subgroups of $\text{Sp}_4(\mathbb{Z})$ and $\text{SL}_2(\mathbb{Z}[^{1+\sqrt{5}}_2])$?

• Your favorite compactification of the associated modular threefold? (For starters, how does the surface of “atypical” $(C, P)$ fit on the boundary of the “typical” threefold?)

• Horrocks–Mumford surfaces ($=(1, 5)$-polarized ab. surfaces in $\mathbb{P}^4$, via $\text{Jac}(C) / \langle T \rangle$)? What’s isogenous $C'$ in RM-5 case?

I thank: NSF, Simons; GP-PARI, Maxima, Sage; Alex Cowan; Ari Shnidman, Bjorn Poonen and Drew Sutherland; LuCaNT organizers and referees; and you.