Orienteering on Supersingular Isogeny Volcanoes Using One Endomorphism

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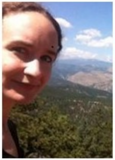
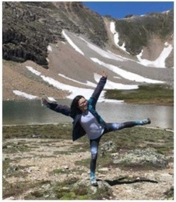
LMFDB, Computation, and Number Theory (LuCaNT)
ICERM, Providence, Rhode Island
July 14, 2023

Let the Adventure Begin . . .





Orienteering

Finding one's way to checkpoints across varied terrain using only map and compass.

- Our terrain: oriented supersingular ℓ-isogeny volcano
- Our wayfinding tool: one endomorphism
- Our task: get to a given elliptic curve
  (which we may or may not always reach)

Meheti'a,
French Polynesia

Isogeny Path Finding

Throughout, let $\mathbb{F}_q$ be a finite field ($q = p^n$ with p prime).

Isogeny Path Finding Problem

Given a set $\mathcal{L}$ of primes (small, distinct from p) and two elliptic curves E, E' over $\mathbb{F}_q$, find an $\mathcal{L}$-**isogeny path** from E to E', i.e. a sequence

$$E = E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_m} E_m = E'$$

of isogenies with $\deg(\varphi_i) \in \mathcal{L}$ for $1 \leq i \leq m$.

Questions

- How hard is this problem computationally?
- How do we solve it?

We only consider $\mathcal{L} = \{\ell\}$ (one prime).

Path Finding Applications

Cryptography:

- Hash Functions (Charles-Goren-Lauter 2006/2009)
- Cryptographic key agreement
  (Couveignes 1996/2006, Rostovtsev-Stolbunov 2006, De Feo-Jao-Plût 2011 (broken), Castryck-Lange-Martindale-Panny-Renes 2018, Colò-Kohel 2020, . . .)
- Constructing elliptic curves with a hard discrete log problem
  (Belding-Bröker-Enge-Lauter 2008)

Computing endomorphism rings (Kohel 1996, Bisson-Sutherland 2011)

Point counting (Elkies 1997, Fouquet-Morain 2002)

Computing modular polynomials (Bröker-Lauter-Sutherland 2012, Sutherland 2014)

Generating irreducible polynomials (Couveignes-Lercier 2013)

Path Finding Algorithms

E, E' ordinary (p-torsion $\mathbb{Z}/p\mathbb{Z}$):

- Classical: $\tilde{O}(q^{1/4})$ (Galbraith-Heß-Smart 2002)
- Quantum: $\exp\left(\frac{\sqrt{3}}{2}\sqrt{\log q \log\log q}\right)$ (Childs-Jao-Shoukarev 2014)

E, E' supersingular (p-torsion trivial) and defined over $\mathbb{F}_p$:

- Classical : $\tilde{O}(p^{1/4})$ (Delfts-Galbraith 2014)
- Quantum : $\exp\left(\frac{\sqrt{3}}{2}\sqrt{\log p \log\log p}\right)$ (Biasse-Jao-Sankar 2014)

E, E' supersingular, in general (i.e. defined over $\mathbb{F}_{p^2}$):

- Classical: $\tilde{O}(p^{1/2})$ (Delfts-Galbraith 2014)
- Quantum: $\tilde{O}(p^{1/4})$ (Biasse-Jao-Sankar 2014)

Path Finding With Help

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Path finding for supersingular elliptic curves is equivalent to computing endomorphism rings (Eisenträger-Hallgren-Lauter-Morrison-Petit 2018, Wesolowski 2022).

Easy **if**

- The endomorphism ring is explicitly known (Kohel-Lauter-Petit-Tignol 2014)
- One small non-integer endomorphism is known (Love-Boneh 2020)

Problem:

- Finding endomorphism rings is hard
- Small non-integer endomorphisms are rare and hard to find

Questions: Can paths be found with one (possibly large) endomorphism? If so, how?

Answers: Yes, and we have algorithms!

(Work concurrent with Wesolowski 2022)

Isogeny Graphs

ℓ-**isogeny graph** $\mathcal{G}_\ell(\mathbb{F}_q)$ ($\ell \neq p$ prime):

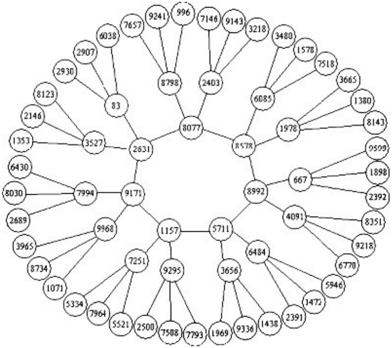
- *Vertices*: $\mathbb{F}_q$ (set of j-invariants of elliptic curves over $\mathbb{F}_q$)
- *Edges*: ℓ-isogenies, paired with their duals[1]

Properties:

- Almost $(\ell + 1)$-regular (except near 0 and 1728)
- Many ordinary components which are **volcanoes**
  - Unique cycle called the **rim** (or **crater**)
  - Vertices at level k from the rim all have CM by the same order whose conductor has ℓ-adic valuation k (Kohel 1996, Fouquet 2001, Fouquet-Morain 2002)
  - Floor has CM by Frobenius order
- One supersingular component with $\approx p/12$ vertices which is an expander graph (Ramanujan when $p \equiv 1 \pmod{12}$) (Pizer 1990)

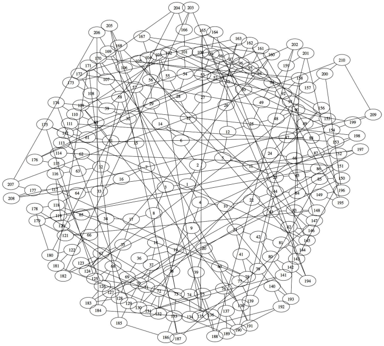
[1] Not quite right near $j = 0$ and $j = 1728$

Two Isogeny Graph Components



Ordinary component
$(\ell = 3)$

Image: Dustin Moody



Supersingular component
$(\ell = 2)$

Image: Dennis Charles

The supersingular component of $\mathcal{G}_\ell(\mathbb{F}_q)$ is an expander graph – messy!

All elliptic curves in the same ordinary component of $\mathcal{G}_\ell(\mathbb{F}_q)$ have CM by some order in a fixed imaginary quadratic field (a commutative 2D object).

Supersingular curves have CM by a maximal order in the quaternion algebra ramified at p and ∞ (a non-commutative 4D object).

- Many quadratic fields generally embed into this quaternion algebra
- We can no longer navigate this component as for ordinary curves
- Path finding is much messier!

Orientations to the rescue!

Our work: path finding with *one* endomorphism (orientation).

Oriented Elliptic Curves

Let

- $E/\mathbb{F}_q$ be an elliptic curve ($q = p^n$)
- K be an imaginary quadratic field in which p does not split
  - Then K embeds into the quaternion algebra ramified at p and ∞ (in many ways)

K-**Orientation of** E**:** $\iota : K \hookrightarrow \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$

- **Example:** ordinary $E/\mathbb{F}_q$ have $\mathbb{Q}\left(\sqrt{\mathrm{tr}(\pi)^2 - 4q}\right)$-orientations

$\mathcal{O}$-**Orientation of** E ($\mathcal{O}$ an order of K): $\iota(\mathcal{O}) \subseteq \mathrm{End}(E)$

Primitive[2] $\mathcal{O}$-**Orientation on** E: $\iota(\mathcal{O}) = \mathrm{End}(E) \cap \iota(K)$

- **Example:** for ordinary curves, $\mathrm{End}(E) \cong \mathcal{O}$ iff E is primitively $\mathcal{O}$-oriented.

[2] aka *optimal embedding* of E

Oriented Isogenies

Let

- $\varphi : E \to E'$ be an isogeny of elliptic curves
- $\iota : K \hookrightarrow \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ a K-orientation on E

K-Orientation on E' induced by φ: $\iota' = \varphi_*(\iota)$ via

$$\iota'(\alpha) = \frac{1}{[\deg(\varphi)]} \, \varphi \, \iota(\alpha) \, \hat{\varphi} \in \mathrm{End}(E')$$

for all $\alpha \in K$ (Waterhouse 1969).

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ {\scriptstyle \iota(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \iota'(\alpha)} \\ E & \xrightarrow{\varphi} & E' \end{array}$$

Write $\varphi \cdot (E, \iota) = (\varphi(E), \varphi_*(\iota)) = (E', \iota')$.

Oriented Isogeny Graph

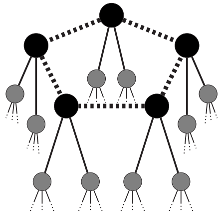
Fix an imaginary quadratic field K.

K-**oriented supersingular** ℓ-**isogeny graph** (Colò-Kohel 2020):

- *Vertices*: Ordered pairs (j, ι) with $j \in \mathbb{F}_{p^2}$ and ι a K-orientation on the supersingular isomorphism class with j-invariant j
- *Edges*: oriented ℓ-isogenies $(E, \iota) \xrightarrow{\varphi} (\varphi(E), \varphi_*(\iota))$

Structure: The components are ... infinite volcanoes! (No floor)

- Every j-invariant appears on every volcano infinitely often, each time paired with a different orientation
- $(\ell + 1)$-regular except near $j = 0, 1728$
- Vertices at level k are primitively oriented by an order $\mathcal{O}_k$ whose conductor has ℓ-adic valuation k



An oriented 3-isogeny volcano

Orientations and Endomorphisms

For a primitive orientation $\iota : \mathcal{O} = \mathbb{Z}[\omega] \xrightarrow{\sim} \mathrm{End}(E) \cap \iota(K)$, the generator image $\iota(\omega)$ defines an endomorphism of E.

Conversely, let

- $\theta \in \mathrm{End}(E) \cap \iota(K)$
- $\omega, \overline{\omega}$ be the roots of the minimal polynomial of θ

Then there are two primitive $\mathbb{Z}[\omega]$-orientations of E via

$$\iota_\theta(\omega) = \theta$$
$$\widehat{\iota_\theta}(\omega) = \hat{\theta}\,, \qquad \text{equivalently,} \quad \widehat{\iota_\theta}(\overline{\omega}) = \theta$$

Note: $(E, \iota_\theta) \neq (E, \widehat{\iota_\theta})$.

Fortunately, in terms of navigating oriented ℓ-volcanoes, the two vertices "look and behave the same locally" (same j-invariant, same level, same neighbours due to identifying dual edges etc.)

We work with endomorphisms instead of orientations because they are much more concrete and computationally amenable!

Direction Finding

Let

- $\varphi : E \to E'$ be an ℓ-isogeny
- $\theta \in \mathrm{End}(E)$ represent the orientation on E

Assume that θ satisfies a certain normal form called *ℓ-suitable* (needed for dividing by $[\ell]$, achieved via translation by a suitable integer).

The induced endomorphism on E' is $\theta'/[\ell]$ where $\theta' = \varphi\theta\hat{\varphi}$.

Proposition

If $[\ell] \nmid \theta$, then φ has the following direction:

- $\uparrow$ if $[\ell]^2 \mid \theta'$
- $\rightarrow$ or $\leftarrow$ (i.e. in the rim) if $[\ell] \mid \theta'$ and $[\ell]^2 \nmid \theta'$
- $\downarrow$ if $[\ell] \nmid \theta'$

Note: Can also use the eigenvalues of θ acting on $E[\ell]$ for direction finding (but for traversing edges, division by ℓ incurs ℓ-adic precision losses!)

Recap: Ordinary Class Group Action

Let $E/\mathbb{F}_q$ be ordinary with an isomorphism $\iota : \mathcal{O} \xrightarrow{\sim} \mathrm{End}(E)$

For any invertible $\mathcal{O}$-ideal $\mathfrak{a}$ with $p \nmid \mathrm{Norm}(\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$, the subgroup

$$E[\mathfrak{a}] = \bigcap_{\alpha \in \iota(\mathfrak{a})} \ker(\alpha) = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \iota(\mathfrak{a})\}$$

defines an isogeny $\varphi_{\mathfrak{a}} : E \to E'$ with kernel $E[\mathfrak{a}]$ and $E' \cong E/E[\mathfrak{a}]$.

This induces a faithful[3] and transitive[4] action of $\mathrm{Cl}(\mathcal{O})$ on the **CM torsor**

$$\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_q) = \{j(E) \mid E \text{ an elliptic curve over } \mathbb{F}_q \text{ with } \mathrm{End}(E) \cong \mathcal{O}\}$$

via

$$[\mathfrak{a}] \star j(E) \;\mapsto\; j(E/E[\mathfrak{a}])$$

Note: $\#\,\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_q) = \#\,\mathrm{Cl}(\mathcal{O})$, the class number of $\mathcal{O}$.

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[3] Only the principal ideal class acts trivially

[4] Any two j-invariants in $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_q)$ are related by some ideal class

Let (E, ι) be supersingular and primitively oriented by $\mathcal{O}$.

For any invertible $\mathcal{O}$-ideal $\mathfrak{a}$ with $p \nmid \mathrm{Norm}(\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$, define

$$E[\mathfrak{a}] = \bigcap_{\alpha \in \iota(\mathfrak{a})} \ker(\alpha) = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \iota(\mathfrak{a})\}$$

$\mathrm{Cl}(\mathcal{O})$ acts freely[5], with one or two orbits related via Frobenius π, on

$$\mathrm{SS}_{\mathcal{O}}^{\mathrm{pr}}(p) = \{(j(E), \iota) \mid \iota \text{ is an } \mathcal{O}\text{-primitive orientation on } E\}$$

via $[\mathfrak{a}] \star j(E) \;\mapsto\; j(E/E[\mathfrak{a}])$ (Onuki 2021, ACLSST 2022).

Note: $\# \mathrm{SS}_{\mathcal{O}}^{\mathrm{pr}}(p) = \# \mathrm{Cl}(\mathcal{O})$ or $2\# Cl(\mathcal{O})$.

[5]No fixed points

Volcano Navigation

Navigation in both ordinary and oriented supersingular volcanos:

$\uparrow$ and $\downarrow$: Vélu's formulas

Rim ($\rightarrow$ or $\leftarrow$): (oriented) class group action by $\mathfrak{l} \mid \ell$

$\mathfrak{l} = \langle \ell, \omega \rangle$ ($\mathcal{O}_K$-module of rank 2)

$$E[\mathfrak{l}] = \ker([\ell]) \cap \ker(\iota(\omega)) = \ker(\iota(\omega)|_{E[\ell]})$$

More efficient than Vélu.

In the oriented setting, we also need to carry along the orientation via the Waterhouse transfer.

Supersingular Path Finding (ACLSST 2022)

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To find an ℓ-isogeny path starting at a curve E to a curve E' *with known endomorphism ring*[6], given **one** endomorphism $\theta \in \mathrm{End}(E)$:

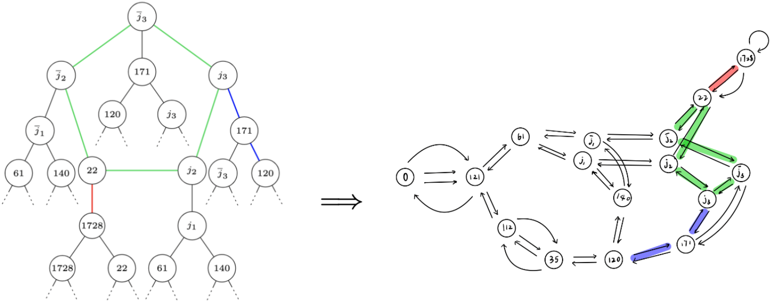
1. Pick K such that ι_θ is a K-orientation of E ($\mathrm{disc}(\theta) = f^2\,\mathrm{disc}(K)$ with $f \in \mathbb{Z}$, ideally $\mathrm{disc}(K)$ small)
2. Walk a K-oriented ℓ-isogeny path from E to the rim of its volcano
3. Generate that entire rim via class group action
4. Orient E' by K (feasible because $\mathrm{End}(E')$ is known)
5. Walk a K-oriented ℓ-isogeny path from E' to the rim of its volcano
6. If that path hits the rim of E's volcano, connect the two paths with the appropriate rim segment; else, go back to step 1 and try a different K
7. Forget all the orientations and output the unoriented path.

[6] e.g. $j = 0$ or $j = 1728$

$p = 179, \quad \mathbb{F}_{179^2} = \mathbb{F}_{179}(i)$ with $i^2 = -1, \quad \ell = 2.$

Find a 2-isogeny path from E to E' over $\mathbb{F}_{179^2}$ where

- $E = E_{120} : y^2 = x^3 + (7i + 86)x + (45i + 174)$
- $E' = E_{1728} : y^2 = x^3 - x$



$(j_1 = 64i + 55, \quad j_2 = 99i + 107, \quad j_3 = 5i + 109)$

(Order of algorithms steps in the example changed to 1, 2, 4, 5, 3, 6)

Step 1: Choose K

An endomorphism on E_{120} is given by $\tilde{\theta}_{120} \in \mathrm{End}(E)$ as follows:

$$\tilde{\theta}_{120}(x,y) = \left(\frac{(122i+167)x^{288} + (17i+68)x^{287} + \cdots + 174i + 157}{x^{287} + (78i+156)x^{286} + \cdots + (16i+54)}, \frac{(69i+109)x^{431} + (60i+178)x^{430} + \cdots + 98i + 124}{x^{431} + (146i+53)x^{430} + \cdots + (44i+89)} y \right).$$

Translating $\tilde{\theta}_{120}$ by $[-10]$ yields

$$\theta_{120}(x,y) = \left(\frac{159x^{188} + (29i+65)x^{187} + \cdots + 74i + 78}{x^{187} + (97i+131)x^{186} + \cdots + (161i+162)}, \frac{126ix^{281} + (163i+30)x^{280} + \cdots + 99i + 154}{x^{281} + (85i+105)x^{280} + \cdots + (36i+106)} y \right).$$

This is 2-suitable, with

$$\mathrm{disc}(\theta_{120}) = 2^2 \Delta_0 \text{ with } \Delta_0 = -4 \cdot 47 = -188 \text{ fundamental.}$$

So we orient E by $K = \mathbb{Q}(\sqrt{-47})$.

We find that θ_{120} is divisible by $[2]$ (in fact by $[2]^2$), so up we go!

Step 2: Walk from E_{120} to the Rim

We compute the blue path from 120 to the rim using Vélu's algorithm:

$$(E_{120}, \theta_{120}) \xrightarrow{\varphi_{120}} (E_{171}, \theta_{171}) \xrightarrow{\varphi_{171}} (E_{5i+109}, \theta_{5i+109})$$

where

$$\varphi_{120}(x,y) = \left(\frac{45x^2 + (-75i-1)x + (-33i-73)}{x + (58i-4)}, \frac{67x^2 + (75i+1)x + (-48i+24)}{x^2 + (-63i-8)x + (73i+53)} y \right).$$

$$E_{171} : y^2 = x^3 + (120i+119)x + (66i+112)$$

$\theta_{171} = \frac{1}{2}\,\varphi_{120}\theta_{120}\widehat{\varphi_{120}} + [1]$ divisible by exactly [2].

$$\varphi_{171}(x,y) = \left(\frac{45x^2 + (-75i+12)x + (89i+85)}{x + (58i+48)}, \frac{67x^2 + (75i-12)x + (-25i-4)}{x^2 + (-63i-83)x + (19i+14))} y \right).$$

$$E_{5i+109} : y^2 = x^3 + (120i+69)x + (5i+43)$$

$\theta_{5i+109} = \frac{1}{2}\,\varphi_{171}\theta_{171}\widehat{\varphi_{171}}$ not divisible by [2].

So $(E_{5i+109}, \theta_{5i+109})$ is at the rim.

Step 4: Orient E_{1728} by K

$$\mathrm{End}(E_{1728}) = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\,\frac{\mathbf{i}+\mathbf{ij}}{2} + \mathbb{Z}\,\frac{(1+\mathbf{j})}{2},$$

where $\mathbf{i}(x,y) = (x, iy)$ and $\mathbf{j}(x,y) = (x^{179}, y^{179})$

(Algebraically, $\mathbf{i}^2 = [-1]$, $\mathbf{j}^2 = [-179]$)

We orient E_{1728} by $K = \mathbb{Q}(\sqrt{-47})$, finding

$$\tilde{\theta}_{1728} = \mathbf{i} + \frac{\mathbf{i}+\mathbf{ij}}{2}$$

given by

$$\tilde{\theta}_{1728}(x,y) = \left(\frac{99x^{47} + 22x^{46} + \cdots + 77}{x^{46} + 40x^{45} + \cdots + 77}, \frac{113ix^{69} + 157ix^{68} + \cdots + 63i}{x^{69} + 60x^{68} \cdots + 158} y \right).$$

$\theta_{1728} := \tilde{\theta}_{1728} + [1]$ is 2-suitable.

Step 4: Orient E_{1728} by K (cont'd)

An alternative approach to walking up is to give our endomorphisms in power-smooth factored form; in this case, as a product of $\{2, 3\}$-power degree isogenies, and refactor in each step:

$\theta_{1728} = \psi_{171}\psi_{1728}$, of degree $3 \cdot 2^4$,

with $\psi_{171} : E_{171} \to E_{1728}$ of degree 3 given by

$$\psi_{171}(x, y) = \left(\frac{x^3 + (102i + 30)x^2 + (31i + 74)x + 10i + 158}{x^2 + (102i + 30)x + (98i + 130)}, \frac{x^3 + (153i + 45)x^2 + (3i + 88)x + 102i + 108}{x^3 + (153i + 45)x^2 + (115i + 32)x + (45i + 174)} y \right).$$

and $\psi_{1728} : E_{1728} \to E_{171}$ of degree 16 given by

$$\psi_{1728}(x, y) = \left(\frac{x^{16} + (156i + 63)x^{15} + \cdots + 56i + 36}{x^{15} + (156i + 63)x^{14} + \cdots + (10i + 71)}, \frac{x^{23} + (55i + 95)x^{22} + \cdots + 105i + 82}{x^{23} + (55i + 95)x^{22} + \cdots + (26i + 87)} y \right)$$

We find that ψ_{1728} is divisible by [2], and hence so is θ_{1728}. So up we go!

Step 5: Walk from E_{1728} to the Rim

We compute the red path from 1728 to the rim:

$$(E_{1728}, \theta_{1728}) \xrightarrow{\varphi_{1728}} (E_{22}, \theta_{22})$$

where

$E_{22} : y^2 = x^3 + 168x + 14$

and, again in already $\{2,3\}$-power-smooth factored and 2-suitable form,

$\theta_{22} = \psi_{174i+109}\psi_{22}$ of degree 12, with isogenies

$\psi_{174i+109} : E_{174i+109} \to E_{22}$ of degree 3,

$\psi_{22} = [4]^{-1}\sigma_{171}\psi_{1728}\widehat{\varphi_{1728}}$ of degree 4,

where $\sigma_{171} : E_{171} \to E_{174i+109}$ has degree 2.

θ_{22} is not divisible by [2], so (E_{22}, θ_{22}) is at the rim.

Step 3: Generate the Rim

The rim order is the maximal order $\mathcal{O}_K$.

Using the $\mathrm{Cl}(\mathcal{O}_K)$-action of $\mathfrak{l} = \langle 2, (1+\sqrt{-47})/2 \rangle$ generates the rim

$$E_{22} \xrightarrow{\varphi_{22}} E_{99i+107} \xrightarrow{\varphi_{99i+107}} E_{5i+109} \xrightarrow{\varphi_{5i+109}} E_{174i+109} \xrightarrow{\varphi_{174i+109}} E_{80i+107} \xrightarrow{\varphi_{80i+107}} E'_{22} \cong E_{22}$$

of length 5, where each curve E_j has an attached endomorphism θ_j (not written here).

Note: $K = \mathbb{Q}(\sqrt{-47})$ has class number 5, and the ideal class of $\mathfrak{l}$ generates $\mathrm{Cl}(K)$.

Happily, $(E_{5i+109}, \theta_{5i+109})$ and (E_{22}, θ_{22}) lie on the same rim!

A path from E_{120} to E_{1728} in $\mathcal{G}_2(179^2)$ is thus given by

$$E_{120} \xrightarrow{\varphi_{120}} E_{171} \xrightarrow{\varphi_{171}} E_{5i+109} \xrightarrow{\widehat{\varphi_{99i+107}}} E_{99i+107} \xrightarrow{\widehat{\varphi_{22}}} E_{22} \xrightarrow{\widehat{\varphi_{1728}}} E_{1728}$$

Algorithmic Ingredients

1. Standard elliptic curve stuff: point arithmetic, Vélu, endomorphism translates $\theta + [n]$, torsion subgroups, isogeny kernels, dual isogenies, evaluating isogenies on ℓ-torsion points, composing isogenies
2. Dividing an ℓ-suitable endomorphism by $[\ell]$ (to go up)
   (McMurdy 2014 for $\ell = 2$, ACLSST 2022 for $\ell > 2$)
3. Waterhouse transfer (i.e. computing induced orientations)
4. Oriented class group action (for traversing rims)
5. Computing an $\mathcal{O}$-orientation/endomorphism on a curve with known endomorphism ring (uses Cornacchia's algorithm)
6. Computing a primitive orientation from an orientation
   (not considered in Wesolowski 2022)
7. Factoring power-smooth isogenies
8. Finding power-smooth suitable translates via sieving

SageMath code at `https://github.com/SarahArpin/WIN5`

Classical Path Finding

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Theorem 1 (ACLSST 2022, La Matematica)

Let $\theta \in \mathrm{End}(E)$ have degree $d = \deg(\theta)$ and discriminant $\Delta = \mathrm{disc}(\theta)$. Suppose d is sufficiently large and θ can be evaluated efficiently on points on E. Let Δ' be the ℓ-fundamental factor of Δ, and assume that $|\Delta'| \leq p^{2+\varepsilon}$. Then there is a heuristic classical algorithm that finds an ℓ-isogeny path of length $O(\log p + h_{\Delta'})$ from E to a curve of known endomorphism ring.

Run time: $h_{\Delta'} \exp\left(C\sqrt{\log d \log\log d}\right) \mathrm{poly}(\log p)$.

- $\Delta = \ell^{2r}\Delta'$ where $v_\ell(\Delta') = 0$ or $v_\ell(\Delta') \in \{3, 2\}$ if $\ell = 2 \mid \Delta$
- $h_{\Delta'}$ is the class number of the quadratic order of discriminant Δ'; $h_{\Delta'} < \sqrt{|\Delta'|}\log|\Delta'|/3$

Runtime improves to $h_{\Delta'}\mathrm{poly}(B)\log p$ if θ is given as a B-powersmooth product.

Quantum Smooth Isogeny Finding

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Theorem 2 (ACLSST 2022, La Matematica)

Let $\theta \in \mathrm{End}(E)$ have degree $d = \deg(\theta)$ and discriminant $\Delta = \mathrm{disc}(\theta)$. Suppose $d \ll |\Delta| \leq p^{2+\varepsilon}$ and θ can be evaluated efficiently on points on E. Then there is a heuristic quantum algorithm that finds a smooth isogeny of norm $O(\sqrt{|\Delta|})$ from E to a curve of known endomorphism ring.

Smoothness bound: $\exp(C\sqrt{\log|\Delta| \log\log|\Delta|})$.

Run time: $\exp(C'\sqrt{\log|\Delta| \log\log|\Delta|})\,\mathrm{poly}(\log p)$.

Uses *oriented vectorization* to solve the following new problem:

Primitive Orientation Problem

Given a supersingular elliptic curve E and an endomorphism θ on E, find the imaginary quadratic order $\mathcal{O}$ so that the orientation ι_θ is $\mathcal{O}$-primitive.

Classically, exponential in the size of the largest prime power factor of Δ.

Rims and Cycles

Theorem 3 (ACLSST 2022, WIN5 Proceedings)

For any $r \geq 3$, there is a bijection between the following two sets:

- Primitive non-backtracking closed walks of length r in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$;
- Directed rims of length r, identified with conjugates, in $\bigcup_K \mathcal{G}_{\ell,K}(\mathbb{F}_{p^2})$.

Corollary 1

1. The cardinality c_r of the sets of Theorem 3 is a weighted average of class numbers of certain imaginary quadratic orders.
2. If $p \equiv 1 \pmod{12}$, then $c_r \sim \ell^r/2r$ as $r \to \infty$ (expected count for Ramanujan graphs).
3. $c_r \leq \dfrac{2\pi e^\gamma \log(4\ell)}{3} \left(\log\log(2\sqrt{\ell}) + \dfrac{7}{3} + \log r \right) \ell^r + O(\ell^{3r/4} \log r)$, as $r \to \infty$, where the O-constant is explicit.

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  To appear in *Research Directions in Number Theory — Proceedings of Women in Numbers 5*

Thank You — Questions (or Answers)?