The relative class number one problem for function fields, III

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These slides can be downloaded from https://kskedlaya.org/slides/.
Jupyter notebooks available from https://github.com/kedlaya/same-class-number.

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.
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The relative class number one problem

Let $F'/F$ be an extension of degree $d$ of function fields associated to a cover $C' \to C$ of curves$^1$ over finite fields. Let $g, g'$ be the genera of $F$ and $F'$. Let $q, q'$ be the cardinalities of the base fields$^2$ of $F, F'$.

Let $h, h'$ be the class numbers$^3$ of $F$ and $F'$. The ratio $h'/h$ equals $\#A(\mathbb{F}_q)$ for $A$ the Prym (abelian) variety of $C'/C$, and hence an integer. Following Leitzel–Madan (1976), we ask: in what cases does $h'/h = 1$?

To make this a potentially finite problem, we only specify the isomorphism classes of $F$ and $F'$, not the inclusion (this only makes a difference when $g \leq 1$). We also ignore the trivial cases where $\dim(A) = 0$:

- $g = g' = 0$;
- $q = q'$ and $1 \leq g = g'$.

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$^1$All curves are smooth, projective, and geometrically irreducible (a/k/a “nice”).

$^2$By “base field” I mean the integral closure of the prime subfield.

$^3$That is, $h = \#J(C)(\mathbb{F}_q)$ and $h' = \#J(C')(\mathbb{F}_{q'})$. 
A heuristic for finiteness

By the Weil bound, \( h'/h = \#A(\mathbb{F}_q) \geq (\sqrt{q} - 1)^{2 \dim(A)} > 1 \) if \( q \geq 5 \). So assume hereafter \( q \leq 4 \).

The condition \( h'/h = 1 \) means \( \#A(\mathbb{F}_q) \) is abnormally small. This implies (roughly) that the Frobenius trace \( T_{A,q} \) of \( A \) is abnormally large. Since

\[
T_{A,q} = T_{C',q} - T_{C,q},
T_{C',q} = q + 1 - \#C'(\mathbb{F}_q) \leq q + 1,
T_{C,q} = q + 1 - \#C(\mathbb{F}_q),
\]

this means \( T_{C,q} \) is abnormally small and so \( \#C(\mathbb{F}_q) \) is abnormally large.

Using “linear programming” bounds on \( \#C(\mathbb{F}_q) \) in terms of \( g \), one can establish an effective finiteness result. By also accounting for \( d \) (Riemann–Hurwitz, Deuring–Shafarevich, splitting behavior), one can make this bound practical.
An answer, part I

I reported some partial results at ANTS-XV (Bristol, June 2022).

- **Solved** when $F'/F$ is **constant** (i.e., $F' = F \cdot \mathbb{F}_{q'}$). We thus need only treat the case where $F'/F$ is **geometric** (i.e., $q' = q$).

- **Solved** when $q > 2$, i.e., $q \in \{3, 4\}$. Assume hereafter $q = 2$.

- **Solved** when $g \leq 1$ (we get $g' \leq 6$). Assume hereafter $g \geq 2$, so that $d := [F' : F] \leq \frac{g' - 1}{g - 1}$ by Riemann–Hurwitz.

- **Reduced to a finite computation**: the zeta functions $\zeta_F, \zeta_{F'}$ of $F, F'$ form one of 208 known pairs. In all cases, $g \leq 7, g' \leq 13$.

- **Solved** when $g \leq 5$ and $F'/F$ is a **cyclic** extension, by a table lookup for $F$ plus explicit class field theory (Magma).

For the last step, LMFDB includes a complete census of genus-$g$ curves over $\mathbb{F}_2$ for $g \leq 3$ (Sutherland), $g = 4$ (Xarles), and $g = 5$ (Dragutinović).

\[\text{4} \text{The case } g = 0 \text{ was handled by Mercuri–Stirpe and Shen–Shi; we get } g' \leq 4.\]

\[\text{5} \text{Reminder: the data of } \zeta_F \text{ and } (\#C(\mathbb{F}_{q_i}))_{i=1}^g \text{ are equivalent.}\]
I reported another partial result at AGC²T (Luminy, June 2023).

**Theorem**

Let $F'/F$ be a finite geometric extension of function fields with $q = 2$, $g > 1$, $h'/h = 1$. Then $F'/F$ is cyclic.

The proof strategy: for each pair $(\zeta_F, \zeta_{F'})$ with $3 \leq d \leq 7$ listed in the ANTS-XV data, check that the noncyclic options for the Galois group lead to abelian varieties with untenable point counts.

A useful slogan here is

**the most radical [extreme] covers are radical [cyclic]:**

the class number condition puts severe pressure on point counts and splitting of places, and cyclic covers are most resistant to this pressure.

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6These are certain isogeny factors of the Jacobian of the Galois closure. Compare Paulhus’s ANTS-X paper.
# The problem at hand

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The only remaining cases of the relative class number one problem are $q = 2$, $g \in \{6, 7\}$, and $F'/F$ is unramified of degree 2. Again it will suffice to find all $F$ with a given $\zeta_F$, then use Magma to find $F'$ and $h'/h$.

If $g = 6$ then $\# C(\mathbb{F}_2), \ldots, \# C(\mathbb{F}_{2^6})$ appears in this list:

<table>
<thead>
<tr>
<th>$# C(\mathbb{F}_2)$</th>
<th>$# C(\mathbb{F}_{2^2})$</th>
<th>$# C(\mathbb{F}_{2^3})$</th>
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<td>4, 14, 16, 18, 14, 92</td>
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<td>6, 10, 9, 38, 11, 79</td>
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<td>5, 15, 5, 35, 20, 45</td>
<td>6, 14, 12, 26, 6, 66</td>
</tr>
</tbody>
</table>
Where am I now? (part 2 of 2)

If $g = 7$ then $\#C(\mathbb{F}_2), \ldots, \#C(\mathbb{F}_{2^7})$ appears in this list:

- $6, 18, 12, 18, 6, 60, 174$
- $6, 18, 12, 18, 6, 72, 132$
- $6, 18, 12, 18, 6, 84, 90$
- $7, 15, 7, 31, 12, 69, 126$
- $7, 15, 7, 31, 22, 45, 112$
- $7, 15, 7, 31, 22, 57, 70$
- $7, 15, 7, 31, 22, 57, 84$

Note that $\#C(\mathbb{F}_2)$ is “large” (in particular nonzero) but not “extremely large”: for $g \in \{6, 7\}$, the maximum number of points on a genus-$g$ curve over $\mathbb{F}_2$ is 10. Hence we do expect to find some curves $C$, so methods based on ruling out curves cannot cover the entire range.
We instead construct an iteration over a (possibly redundant) set of isomorphism representatives for genus-$g$ curves over $F_2$.

Previous calculations of this sort (e.g., in the work of Faber\textsuperscript{7}–Grantham on the gonality of curves over finite fields) use singular plane models. Here, we instead use Mukai’s descriptions of canonically embedded genus-$g$ curves in terms of linear sections of homogeneous varieties, with some extra effort paid to descending special linear systems to finite base fields.

\textsuperscript{7}This is Xander, not Carel.
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Special linear systems

Let $C$ be a curve of genus $g$ over a finite field $k$. A $g^r_d$ is a line bundle of degree $d$ whose space of global sections has dimension $r + 1$; if such a bundle is basepoint-free, then it defines a degree-$d$ map to $\mathbb{P}^r_k$. For example, the canonical bundle is a $g^r_d$ for $r = g - 1, d = 2g - 2$.

Since $k$ is finite, every Galois-invariant divisor class on $C$ contains a $k$-rational divisor. In particular, if $C_k$ admits a unique $g^r_d$ for some $r, d$, then so does $C$.

For example, the Castelnuovo–Severi inequality implies that if $g > (d - 1)^2$, then $C_k$ can have at most one $g^1_d$. We say $C$ is hyperelliptic if it admits a unique $g^1_2$ and trigonal if it is not hyperelliptic but admits a unique $g^1_3$.

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By contrast, over $\mathbb{Q}$, when $g > 2$ it is possible for a curve to be “geometrically hyperelliptic” by being a double cover of a pointless genus-0 curve.
The canonical embedding

The canonical system defines a map \( \iota : C \rightarrow \mathbb{P}_{k}^{g-1} \) which is an embedding unless \( C \) is hyperelliptic (then \( \iota \) is a 2-1 cover of a rational normal curve).

By Petri’s theorem\(^9\), \( \iota(C) \) is cut out (schematically) by quadrics unless

- \( C \) is trigonal,
- \( g = 6 \) and \( C \) is a smooth plane quintic.

This implies that the usual classification of curves of genus up to 5 remains valid when \( k \) is finite:\(^{10}\)

- If \( g = 2 \), then \( C \) is hyperelliptic.
- If \( g = 3 \), then \( C \) is hyperelliptic or a CI\(^{11}\) of type (4) in \( \mathbb{P}_{k}^{2} \).
- If \( g = 4 \), then \( C \) is hyperelliptic or a CI of type \((2) \cap (3)\) in \( \mathbb{P}_{k}^{3} \).
- If \( g = 5 \), then \( C \) is hyperelliptic, trigonal, or a CI of type \((2) \cap (2) \cap (2)\) in \( \mathbb{P}_{k}^{4} \).

\(^9\)More precisely, by Saint-Donat’s version valid in any characteristic.

\(^{10}\)For \( k \) perfect, we must insert “geometrically” before “hyperelliptic/trigonal”.

\(^{11}\)complete intersection
The Maroni invariant of a trigonal curve

For $C$ trigonal, the quadrics vanishing on $\iota(C)$ cut out a Hirzebruch surface

$$F_n = \text{Proj}_{P^1_k} \left( \mathcal{O}_{P^1_k} \oplus \mathcal{O}(n)_{P^1_k} \right)$$

embedded in $P^{g-1}$ by $|b + (n + 1 + i)f|$ for some $i \geq 0$ where $f$ is a fiber of $F_n \to P^1_k$ and $b$ is the unique irreducible curve with $b^2 = -n$.

We call $n$ the **Maroni invariant** of $C$. We have $b \cdot C = \frac{g-3n+2}{2}$, so so $n \in \{0, \ldots, \frac{g+2}{3}\}$ and $n \equiv g \pmod{2}$.

For $n = 0$, $F_{0,k} \cong P^1_k \times P^1_k$ and $C_k$ is a $(3, \frac{g+2}{2})$-hypersurface. Since $\frac{g+2}{2} \neq 3$ for $g \geq 5$, this description descends to $k$.

For $n > 0$, $F_n$ is an $(n,1)$-hypersurface in $P^1_k \times P^2_k$. Blowing down along $b$ yields the weighted projective space $P(1 : 1 : n)_k$. 
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The Brill-Noether stratification for $g = 6$

From a corresponding result of Mukai over $\overline{k}$, we deduce that for $g = 6$, $C$ has one of the following forms.

- Hyperelliptic.
- Trigonal of Maroni invariant 2: Cl of type $(2, 1) \cap (1, 3)$ in $\mathbb{P}_k^1 \times \mathbb{P}^2_k$.
- Trigonal of Maroni invariant 0: Cl of type $(3, 4)$ in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$.
- Bielliptic: \(^{12}\) double cover of a genus 1 curve.
- Smooth quintic: Cl of type $(5)$ in $\mathbb{P}_k^2$.
- A Cl of type $(1)^4 \cap (2)$ in the Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}_k^9$ in its Plücker embedding.

\(^{12}\)Again by Castelnuovo–Severi, this cover is unique for $g > 5$, and so descends to $k$. 
The Brill-Noether stratification for $g = 7$

By Mukai again, for $g = 7$, $C$ has one of the following forms.

- **Hyperelliptic.**
- **Trigonal of Maroni invariant 3:** Cl of type $(9)$ in $\mathbb{P}(1 : 1 : 3)_k$.
- **Trigonal of Maroni invariant 1:** Cl of type $(1, 1) \cap (3, 3)$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$.
- **Bielliptic.**
- **Not bielliptic but admits a self-adjoint $g^2_6$:** Cl of type $(3) \cap (4)$ in $\mathbb{P}(1 : 1 : 1 : 2)_k$.
- **Admits two distinct $g^2_6$’s over $k$:** Cl of type $(1, 1) \cap (1, 1) \cap (2, 2)$ in $\mathbb{P}^2_k \times \mathbb{P}^2_k$.
- **Admits two distinct $g^2_6$’s only over $\overline{k}$:** Cl of type $(1, 1) \cap (1, 1) \cap (2, 2)$ in the quadratic twist of $\mathbb{P}^2_k \times \mathbb{P}^2_k$.
- **Tetragonal** (admits a $g^1_4$ but not a $g^1_3$ or $g^2_6$): Cl of type $(1, 1) \cap (1, 2) \cap (1, 2)$ in $\mathbb{P}^1_k \times \mathbb{P}^3_k$.
- **None of the above, see below.**
Generic canonical curves of genus 7

Let $V$ be the vector space $k^{10}$ equipped with the quadratic form\textsuperscript{13} $\sum_{i=1}^{5} x_i x_{5+i}$. Let $\text{SO}(V)$ be the index-2 subgroup of the orthogonal group of $V$ on which the Dickson invariant is trivial.

The 10-dimensional orthogonal Grassmannian $\text{OG}$ parametrizes Lagrangian (maximal isotropic) subspaces of $V$. It admits a canonical spinor embedding $\text{OG} \hookrightarrow \mathbb{P}_k^{15}$ on which $\text{SO}(V)$ acts transitively.

There are two connected components of $\text{OG}$, stabilized by $\text{SO}(V)$. Given $L_0 \in \text{OG}(k)$, we may characterize the component $\text{OG}^+$ containing $L_0$ as parametrizing $L$ with $\dim_k(L \cap L_0) \equiv 1 \pmod{2}$.

**Theorem (after Mukai)**

*Every canonical genus-7 curve over $k$ arises as a CI of type $(1)^9$ in $\text{OG}^+$.***

\textsuperscript{13}For $k$ finite, there is a second form with no Lagrangian subspaces defined over $k$; but the fact that curves always have points over large odd-degree extensions means we don’t need to worry about the second form.
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### Review of point count conditions

For $g = 6$, we are looking for $C$ for which $\#C(\mathbb{F}_2), \ldots, \#C(\mathbb{F}_{2^6})$ appears in:

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For $g = 7$, we are looking for $C$ for which $\#C(\mathbb{F}_2), \ldots, \#C(\mathbb{F}_{2^7})$ appears in:

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</tr>
</tbody>
</table>
Initial cases

- If $g = 6$, then $C$ cannot be hyperelliptic: we have $\#C(\mathbb{F}_4) > 10 = 2\#\mathbb{P}^1(\mathbb{F}_4)$ except in three cases where $\#C(\mathbb{F}_{16}) = 38 > 34 = 2\#\mathbb{P}^1(\mathbb{F}_{16})$.

- If $g = 7$, then $C$ cannot be hyperelliptic: we have $\#C(\mathbb{F}_4) \geq 15 > 10 = 2\#\mathbb{P}^1(\mathbb{F}_4)$.

- If $g = 7$ and $\#C(\mathbb{F}_2) = 6$, then $C$ cannot be trigonal: we have $\#C(\mathbb{F}_4) = 18 > 15 = 3\#\mathbb{P}^1(\mathbb{F}_4)$.

- If $g = 7$ and $\#C(\mathbb{F}_2) = 7$, then $C$ cannot be trigonal of Maroni invariant 3: we have $\#C(\mathbb{F}_2) = 7$ which exceeds the number of smooth points of $\mathbb{P}(1 : 1 : 3)(\mathbb{F}_2)$.

Also, for $C$ bielliptic, we can identify options for the genus-1 curve, then use Magma to compute all double covers of the right genus.
A paradigm for the remaining cases

In each remaining case, we are looking for certain complete intersections $X_1 \cap \cdots \cap X_m$ inside some homogeneous variety $X$ over $\mathbb{F}_2$.

- Compute $S := X(\mathbb{F}_2)$ and $G := \text{Aut}(X)(\mathbb{F}_2)$.
- Compute orbit representatives for the $G$-action on subsets of $S$ of size at most $g$. More on this below.$^{14}$
- For each representative subset of size in $\{4, 5, 6\}$ (if $g = 6$) or $\{6, 7\}$ (if $g = 7$), use linear algebra to find all tuples of hypersurfaces $X_1, \ldots, X_{m-1}$ of the desired degrees containing these $\mathbb{F}_2$-points.
- For each choice, impose linear conditions on $X_m$ to ensure that $X_1 \cap \cdots \cap X_m$ has exactly the specified set of $\mathbb{F}_2$-rational points. This crucially exploits the fact that the base field is $\mathbb{F}_2$; a similar strategy is used by Faber–Grantham.$^{14}$

$^{14}$For $g = 7$, $X = \text{OG}^+$, we use a slightly different setup that requires only the action on 6-element subsets.
Let $G$ be a finite group acting on a finite set $S$. We need to compute orbit representatives for the action of $G$ on $k$-element subsets of $S$ without instantiating in memory the full list of $k$-element subsets.

For this we use an inductive combinatorial construction called an **orbit lookup tree**. It answers the question: given a sequence $x_1, \ldots, x_k$, find a permutation $\pi$ of $\{1, \ldots, k\}$ and an element $g \in G$ such that for each $i$, \( \{g(x_{\pi(1)}), \ldots, g(x_{\pi(i)})\} \) is an orbit representative for $i$-element subsets.

In some cases, a strategy introduced by Auel–Kulkarni–Petok–Weinbaum based on decomposing $k[G]$-modules may be superior.
### Summary of the computation

<table>
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<tr>
<th>Type of $C$</th>
<th>Dim</th>
<th>$#C$</th>
<th>$#C'$</th>
<th>Time$^{15}$</th>
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<td>11</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
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<tr>
<td>$g = 7$, irrational $g_6^2$</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>45m</td>
</tr>
<tr>
<td>$g = 7$, tetragonal, no $g_6^2$</td>
<td>17</td>
<td>1</td>
<td>0</td>
<td>2h</td>
</tr>
<tr>
<td>$g = 7$, generic</td>
<td>18</td>
<td>1</td>
<td>0</td>
<td>1h</td>
</tr>
</tbody>
</table>

$^{15}$These are wall times on a laptop. Don’t take them too seriously; there are many confounding factors at work.
The final results

Theorem

(a) There are two isomorphism classes of curves $C$ of genus 6 over $\mathbb{F}_2$ admitting an étale double covering $C' \to C$ such that $\#J(C')(\mathbb{F}_2) = \#J(C)(\mathbb{F}_2)$. The curves $C$ are Brill–Noether general with automorphism groups $C_3$ and $C_5$.

(b) There is a unique isomorphism class of curves $C$ of genus 7 over $\mathbb{F}_2$ admitting an étale double covering $C' \to C$ such that $\#J(C')(\mathbb{F}_2) = \#J(C)(\mathbb{F}_2)$. The curve $C$ is bielliptic with automorphism group $D_6$.

In the latter case, $C$ admits the affine model

$$\text{Spec } \frac{\mathbb{F}_2[x, y, z]}{(y^2 + (x^3 + x^2 + 1)y + x^2(x^2 + x + 1), z^2 + z + x^2(x + 1)y)}.$$
A full census of genus-6 and genus-7 curves

It would be desirable to have a full census of genus-$g$ curves over $\mathbb{F}_2$ for $g = 6, 7$. This would provide a valuable consistency check, and also serve as a rich resource for future investigation (ideally as part of LMFDB).

A further consistency check\(^{16}\) would be provided by computing\(^{17}\) $\#M_g(\mathbb{F}_2)$ using explicit generators/relations for the Chow ring. For $g = 6$, this has been achieved using very recent work of Canning–H. Larson.\(^{18}\)

It should be possible to upgrade our existing code to remove the filtering on zeta functions to achieve a full census. For $g = 6$, this is work in progress with Jun Bo Lau, but extra help would be welcome.

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\(^{16}\)Such a count can even be used to **certify** the validity of a census: it is easy to compute automorphism groups and check pairwise nonisomorphism for an explicit list of curves, this providing a concrete lower bound on stacky $\#M_g(\mathbb{F}_2)$.

\(^{17}\)This point count is **stacky**: the isomorphism class of a curve $C$ has weight $\frac{1}{\# \text{Aut}(C)}$.

\(^{18}\)Odd coincidence: Hannah is also lecturing in Providence at this hour!
Into the wild: beyond genus 7

Since $M_g$ has dimension $3g - 3$, we expect $\#M_g(\mathbb{F}_2)$ to be roughly $2^{3g-3}$. So it might be feasible to compile a census\textsuperscript{19} of genus-$g$ curves over $\mathbb{F}_2$ for $g = 8, 9, 10$.

Conveniently, Mukai also has similar descriptions of canonical curves in these genera. For example, a general canonical genus-8 curve is a linear section of $\text{Gr}(2, 6) \subset \mathbb{P}_k^{14}$.

However, it will take significant implementation skill to keep the complexity down to a manageable level.

\textsuperscript{19}Faber-Grantham encountered a single zeta function that they had to show did not occur in genus 9. Fortunately they were able to do this by “pure thought”.