A database of paramodular forms from quinary orthogonal modular forms

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Summary

We report on our computation of a moderately large data-base of paramodular forms for GSp(4) for weight ≥ 3. We compute with algebraic modular forms on orthogonal groups of positive definite quadratic forms in five variables.

Why paramodular forms?

- A natural generalisation of the modularity of elliptic curves proved by Wiles et al is understanding the relation between analytic objects and higher dimensional abelian varieties.
- Yoshida suggested that an abelian surface should be related to a Siegel modular form of degree 2 (automorphic forms for the group GSp(4)).
- Brumer and Kramer conjectured that abelian varieties should correspond to Siegel modular forms transforming under the paramodular group of degree N.
- The spaces of paramodular forms present a nice theory of newforms proven by Roberts and Schmidt.

Orthogonal spaces

Let V be a finite-dimensional $\mathbb{Q}$-vector space equipped with a quadratic form $Q\colon V \to \mathbb{Q}$. The orthogonal group of Q is the group of $\mathbb{Q}$-linear automorphisms of V which preserve Q, that is

$$\mathrm{O}(V) := \{g \in \mathrm{GL}(V) : Q(gx) = Q(x) \text{ for all } x \in V\}.$$

Also let

$$\mathrm{SO}(V) := \mathrm{O}(V) \cap \mathrm{SL}(V).$$

Orthogonal spaces

Two integral lattices $\Lambda, \Lambda' \subset V$ are isometric if there exist $g \in \mathrm{O}(V)$ such that $g\Lambda = \Lambda'$ and we denote it by $\Lambda \simeq \Lambda'$. Same with isometric for $\Lambda_p = \Lambda \otimes \mathbb{Z}_p$ over $V_p = V \otimes \mathbb{Q}_p$.

The genus of Λ is the set of lattices which are everywhere locally isometric to Λ, namely

$$\mathrm{Gen}(\Lambda) := \{\Lambda' \subset V : \Lambda_p \simeq \Lambda'_p \text{ for all primes } p\}.$$

The class set $\mathrm{Cl}(\Lambda)$ is the set of isometry classes in $\mathrm{Gen}(\Lambda)$, which is finite by the geometry of numbers.

We also let $\mathrm{SO}(\Lambda) := \{g \in \mathrm{SO}(V) : g\Lambda = \Lambda\}$.

Ortogonal modular forms

Let $\Lambda_1, \ldots, \Lambda_h$ represent $\mathrm{Cl}(\Lambda)$, with $\Lambda_1 = \Lambda$, and $\rho\colon \mathrm{SO}(Q) \to \mathrm{GL}(W)$ a finite dimensional representation.

The space of (special) orthogonal modular forms of level Λ and weight ρ is the space

$$M(\mathrm{SO}(\Lambda), \rho) = \left\{ f : \mathrm{Cl}(\Lambda) \to W : \; f([\Lambda_i]) \in W^{\mathrm{SO}(\Lambda_i)} \right\} \simeq \bigoplus_{i=1}^{h} W^{\mathrm{SO}(\Lambda_i)}.$$

For the trivial representation this is just $\mathbb{C}^h$.

Using p-neighbors we get Hecke operators in the space of orthogonal modular forms (in our case we need two types of operators, $T_{p,1}$ and $T_{p,2}$).

Radical character

We define a representation of dimension one for every $d \parallel \mathrm{disc}(\Lambda)$,

$$\theta_d : \mathrm{O}(V) \to \mathbb{C}^\times,$$

which is called the radical character (formerly known as the spin character).

Relation with paramodular forms

- The relation between modular forms on SO(5) and automorphic forms on GSp(4) with trivial central character is predicted by Langlands functoriality.
- An explicit correspondence was conjectured by Ibukiyama (1980) involving two steps: a correspondence between (para)modular forms of GSp(4) and its compact twist $\mathrm{GU}(2, B)$, where B is a definite quaternion algebra; and a correspondence between modular forms of $\mathrm{GU}(2, B)$ and those of $\mathrm{SO}(Q)$ for a suitable chosen quinary quadratic form Q.
- The first correspondence was proved by van Hoften (2021) and Rösner–Weissauer (2021), and extended by Dummigan–Pacetti–R–Tornaría (2021), where the second correspondence was also proved.

Dummigan-Pacetti-R-Tornaría

More precisely we can compute the space $S_{k,j}^{\mathrm{new}}(K(N))$ of paramodular newforms of level N and weight (k, j) under the following assumptions:

1. There is a prime p_0 such that $p_0 \parallel N$, and
2. $k \geq 3$ and $j \in 2\mathbb{Z}_{\geq 0}$.

p-neighbors

- Algorithms to compute with orthogonal modular forms using lattice methods were exhibited by Greenberg–Voight.
- These algorithms take as input an integral, positive definite quadratic form on a lattice Λ and compute the action of Hecke operators on spaces of functions on the class set of Λ, with values in a weight representation.
- The Hecke operators are computed as p-neighbors, after Kneser, using an algorithm to test lattice isomorphism due to Plesken–Souvignier.
- For evaluating the Hecke operator $T_{p,1}$, the running time complexity is dominated by $O(hp^3)$ isometry tests, where h is the class number of the lattice; for $T_{p,2}$, it is dominated by $O(hp^4)$ isometry tests.

Algorithmic improvements

- Two p-isotropic vectors will produce the same target when we apply the p-neighbor relation if they are in the same orbit of the isometry group $\mathrm{Aut}(\Lambda)$. So, given a p-isotropic vector v we compute its orbit under $\mathrm{Aut}(\Lambda)$ when v is minimal with respect to the lexicographic order.
- We can precompute the automorphism group of all lattices in the genus, and their conjugations into a single quadratic space, saving the cost of conjugation when computing the spinor norm.

For one of the implementations, for $N = 61$ we get a speedup factor of 11, with $\# \mathrm{Aut}(\Lambda) = 48$.

Implementation

For reliability, we carried out and compared two separate implementations to compute the data, one in C and one in PARI/GP. Eventually, these gave the same output.

We used Magma for some of the linear algebra, and SageMath for auxiliary operations on the data we produced.

Introduction

Orthogonal modular forms

Algorithms

Computations

An Example

Consider the space $S_{3,0}(K(312))$. Since $312 = 2^3 \cdot 3 \cdot 13$, we choose the quadratic form given by DPRT. Explicitly we take $\Lambda = \mathbb{Z}^5$ equipped with the quadratic form having the following Gram matrix:

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 \\ 1 & -1 & 4 & -1 & 0 \\ 0 & -1 & -1 & 6 & 0 \\ 1 & 1 & 0 & 0 & 12 \end{pmatrix}$$

The class set of Λ has cardinality 15.

An Example

Computing the spaces $M(\mathrm{SO}(\Lambda), \theta_d)$ for squarefree $d \mid 312$ we find that their dimensions are given as in the following table.

d	1	2	3	6	13	26	39	78	Total
(G)-new	1	0	0	$5 = 3 + 1 + 1$	0	1	$4 = 3 + 1$	0	11
(G)-old	1	1	1	1	1	2	0	1	8
(P)-new	2	0	0	3	0	2	2	0	9
(P)-old	8	0	0	2	0	7	2	0	19
(Y)-new	1	0	0	1	0	1	3	0	6
(Y)-old	1	1	0	0	1	1	0	0	4
Total	14	2	1	12	2	14	11	1	57

An Example

We can calculate the space of Yoshida lifts:

$$
\begin{aligned}
M^{\mathrm{new}}(\mathrm{SO}(\Lambda))_{\mathbf{(Y)}} \simeq (S_2^{\mathrm{new}}(\Gamma_0(26)) \otimes S_4^{\mathrm{new}}(\Gamma_0(12))) \\
\oplus (S_2^{\mathrm{new}}(\Gamma_0(39)) \otimes S_4^{\mathrm{new}}(\Gamma_0(8))) \\
\oplus (S_2^{\mathrm{new}}(\Gamma_0(52)) \otimes S_4^{\mathrm{new}}(\Gamma_0(6))),
\end{aligned}
$$

leading to the corresponding dimension counts in the last table. Furthermore, the only Yoshida lifts that occur as oldforms are the images of the forms in

$$
M^{\mathrm{new}}(\mathrm{SO}(\Lambda_{156}))_{\mathbf{(Y)}} \simeq S_2^{\mathrm{new}}(\Gamma_0(26)) \otimes S_4^{\mathrm{new}}(\Gamma_0(6)).
$$

An Example

Similarly, we find that

$$M^{\mathrm{new}}(\mathrm{SO}(\Lambda))_{\mathbf{(P)}} \simeq S_4^{\mathrm{new},-}(\Gamma_0(312)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(24)),$$

(plus and minus signs for the Atkin–Lehner involution), and

$$M^{\mathrm{old}}(\mathrm{SO}(\Lambda))_{\mathbf{(P)}} \simeq \bigoplus_{d|24, d\neq 24} S_4^{\mathrm{new},-}(\Gamma_0(13d)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(d))$$

$$\oplus \bigoplus_{d|6} S_4^{\mathrm{new},-}(\Gamma_0(13d)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(d)).$$

An Example

Finally, since we are able to compute the Hecke eigenvalues at good primes for each of the newforms, we can decompose $S_{3,0}^{\mathrm{new}}(K(312))_{\mathbf{(G)}}$ to newform subspaces.
The following table lists the Galois orbits of the Hecke eigenforms in this space, giving rise to such a decomposition.

dimension	field	Traces				A-L signs		
		a_5	a_7	a_{11}	a_{17}	2	3	13
1	$\mathbb{Q}$	-1	-13	-6	63	$+$	$+$	$-$
1	$\mathbb{Q}$	-11	3	-16	3	$+$	$-$	$+$
1	$\mathbb{Q}$	1	-15	14	135	$-$	$+$	$+$
1	$\mathbb{Q}$	2	-6	-52	44	$-$	$-$	$-$
1	$\mathbb{Q}$	-13	-3	-4	-37	$-$	$-$	$-$
3	3.3.961.1	-1	-25	-24	-71	$+$	$-$	$+$
3	3.3.961.1	-12	28	2	-90	$-$	$-$	$-$

Computations

We computed the spaces of paramodular forms of level N and weight (k, j), the Hecke eigenforms and the eigenvalues of the Hecke operators in the following ranges:

- $(k, j) = (3, 0)$, $D = N \leq 1000$, good $T_{p,i}$ with $p^i < 200$
- $(k, j) = (4, 0)$, $D = N \leq 1000$, good $T_{p,1}$ with $p < 100$, good $T_{p,2}$ with $p < 30$
- $(k, j) = (3, 2)$, $D = N \leq 500$, good $T_{p,1}$ with $p < 100$, good $T_{p,2}$ with $p < 30$

Computations

Newspace and newform data computed:

(k, j)	Newspaces			Newforms		
	sqfree N	nonsqfree N	total	sqfree N	nonsqfree N	total
$(3, 0)$	2 764	4 820	7 584	52 181	23 853	76 034
$(3, 2)$	1 363	3 072	4 435	72 551	29 226	101 777
$(4, 0)$	2 856	7 783	10 639	287 974	132 380	420 354

Future directions

- Compute the Hecke eigenvalues in the bad primes.
- Compute for more weights.
- Implement level-raising maps between spaces of orthogonal modular forms.

`https://github.com/assaferan/omf5_data`
`https://gitlab.fing.edu.uy/grama/quinary`

Thanks!