Computing nonsurjective primes associated to Galois representations of genus 2 curves

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The mod-ℓ Galois representation of a curve over $\mathbb{Q}$

- Let
  - $C/\mathbb{Q}$ be a smooth, projective, geometrically integral curve of genus g,
  - $A = \mathrm{Jac}(C)$.
- For each prime ℓ, there is the *mod-ℓ Galois representation*

$$\rho_{A,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(A[\ell]) \cong \mathrm{GSp}_{2g}(\mathbb{F}_\ell)$$

Question

For what ℓ is $\rho_{A,\ell}$ surjective?

Serre's Open Image Theorem addresses this question almost entirely

Theorem (Serre, 2000)

If A is typical, i.e. $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Z}$, *and if* $\dim A$ *is* 2, 6, *or odd, then* $\rho_{A,\ell}$ *is surjective for all but finitely many* ℓ.

Problem

Given a typical genus 2 *curve* $C/\mathbb{Q}$, *can we compute the finitely many nonsurjective primes* ℓ?

- The analogous problem for elliptic curves with an algorithm implemented in Sage.
- Sutherland (2015) devised an algorithm to compute $\mathrm{Im}\,\rho_{E,\ell}$.

We have an algorithm to find exactly the nonsurjective ℓ.

- "Part 1 algorithm (Dieulefait, 2002 + small improvements)":
  - Given $C/\mathbb{Q}$, generate a finite list PossiblyNonsurjectivePrimes(C) that provably contains all nonsurjective primes ℓ
- "Part 2 algorithm":
  - Given $C/\mathbb{Q}$ and $B > 0$, obtain a sublist LikelyNonsurjectivePrimes(C; B).
  - If B is big enough, then this sublist consists precisely of the nonsurjective primes.

A quick example of these ideas

Running our code on 8450.a.8450.1 from LMFDB:

$$y^2 + (x+1)y = x^5 + x^4 - 9x^3 - 5x^2 + 21x$$

Part 1 gives us

$$2, 3, 5, 7, 13$$

as the only possibly non-surjective primes.

Running Part 2 by sampling Frob_p for all $p < 1,000$

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Conclusions            Frobenius witnesses for each maximal subgroup
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Not surjective at 2    [3, 7, 7, 0, 3]
Surjective at 3        [11, 11, 29, 11, 11, 11]
Surjective at 5        [7, 3, 11, 3, 3, 3, 3, 11, 3]
Surjective at 7        [29, 29, 3, 11]
Not surjective at 13   [0, 0, 3]
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The two parts share overarching ideas

1. Use Mitchell's 1914 classification of maximal (proper) subgroups of $\mathrm{PSp}_4(\mathbb{F}_\ell)$
2. Sample characteristic polynomials $P_p(t)$ of Frobenius elements $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on $A[\ell]$.

Remark

- *We classify maximal subgroups of* $\mathrm{GSp}_4(\mathbb{F}_\ell)$ *with surjective similitude character.*
- $\rho_{A,\ell}$ *is surjective* $\Leftrightarrow$ $\mathrm{Im}\,\rho_{A,\ell}$ *is not contained in one of these maximal subgroups.*
- $P_p(t)$ *can be computed by counting* $C(\mathbb{F}_p)$ *and* $C(\mathbb{F}_{p^2})$.

$\mathrm{GSp}_4(\mathbb{F}_\ell)$ has a few types of maximal subgroups

Write (V, ω) for the 4-dimensional symplectic bilinear form space.

- Reducible maximal subgroups
  - Stabilizer of a 1-dimensional isotropic subspace for ω
  - Stabilizer of a 2-dimensional isotropic subspace for ω
- Irreducible subgroups; normalizer of stabilizer subgroup of V_1 and V_2 where $V = V_1 \oplus V_2$, $\dim V_i = 2$, and V_1 and V_2 are jointly defined over $\mathbb{F}_\ell$ and either
  - V_i are both nondegenerate for ω or
  - V_i are both isotropic for ω
- Stabilizer of a twisted cubic
- Exceptional maximal subgroups

Not all maximal subgroups generally occur as Im $\rho_{A,\ell}$

Write ω for the symplectic bilinear form.

- Reducible maximal subgroups
  - Stabilizer of a 1-dimensional isotropic subspace for ω
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  - V_i are both nondegenerate for ω or
  - V_i are both isotropic for ω
- ~~Stabilizer of a twisted cubic~~ (unless $\ell \leq 7$ or ℓ is not semistable)
- ~~Exceptional maximal subgroups~~ (unless $\ell \leq 7$ or ℓ is not semistable)

One subalgorithm for part 1 uses modularity.

Suppose that ℓ is nonsurjective good prime by virtue of:

- $\rho_{A,\ell}^{\text{semisimplification}} \cong_{\overline{\mathbb{F}}_\ell} \pi_1 \oplus \pi_2$ where
- $\dim(\pi_i) = 2$, $\det(\pi_i) = \mathrm{cyc}_\ell$.

By Khare-Wintenberger's theorem (2006, 2009), previously Serre's conjecture, there exist modular forms f_i such that $\pi_i \cong \rho_{f_i,\ell}$.

- In fact, $f_i \in S_2^{\text{new}}(\Gamma_0(N_i))$.
- Thus, for any good prime $p \neq \ell N$,

$$P_p(t) \equiv (t^2 - a_p(f_1)t + p)(t^2 - a_p(t_2)t + p) \pmod{\ell}.$$
$$\ell \mid \mathrm{Res}(P_p(t), t^2 - a_p(f_i)t + p)$$

Part 2 removes surjective primes by sampling Frob_p

- $\rho_{A,\ell}$ is not surjective $\Leftrightarrow$ $\mathrm{Im}\,\rho_{A,\ell}$ is contained in some maximal subgroup.
- There are criteria for $P_p(t)$ to satisfy for $\mathrm{Im}\,\rho_{A,\ell}$ to be contained in the various types/conjugacy classes of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_\ell)$.
- Given C and $B > 0$, and for each ℓ, we sample $P_p(t)$ for $p < B$ and remove ℓ if all of these criteria are violated for some (combination of) $P_p(t)$.

Remark

$\rho_{A,2}$ is surjective if and only if the Galois group of the splitting field of the discriminant of C is $\mathrm{Sym}(6)$.

The general criteria is purely group theoretic

Proposition

Let $\ell > 7$ be a prime and let $G \subseteq \mathrm{GSp}_4(\mathbb{F}_\ell)$. Then, $G = \mathrm{GSp}_4(\mathbb{F}_\ell)$ if and only if there exist $X, Y \in G$ such that

- $\mathrm{charpoly}(X)$ *is irreducible*
- $\mathrm{trace}(Y) \neq 0$ *and* $\mathrm{charpoly}(Y)$ *has a linear factor with multiplicity one.*

Part 2 gives exactly the nonsurjective ℓ if enough Frobenii are considered

Theorem

Given a typical genus 2 curve $C/\mathbb{Q}$, if B is sufficiently large, then $\mathrm{LikelyNonsurjectivePrimes}(C, B)$ *consists exactly of the nonsurjective primes of A.*

Question

What is "sufficiently large" here?

Chebotarev bounds give sufficiently large B

Theorem

Let q be the largest non-surjective prime for C. Assuming GRH, any B such that

$$B \geq \left(4\left[(2q^{11}-1)\log \mathrm{rad}(2qN_A) + 22q^{11}\log(2q)\right] + 5q^{11} + 5\right)^2$$

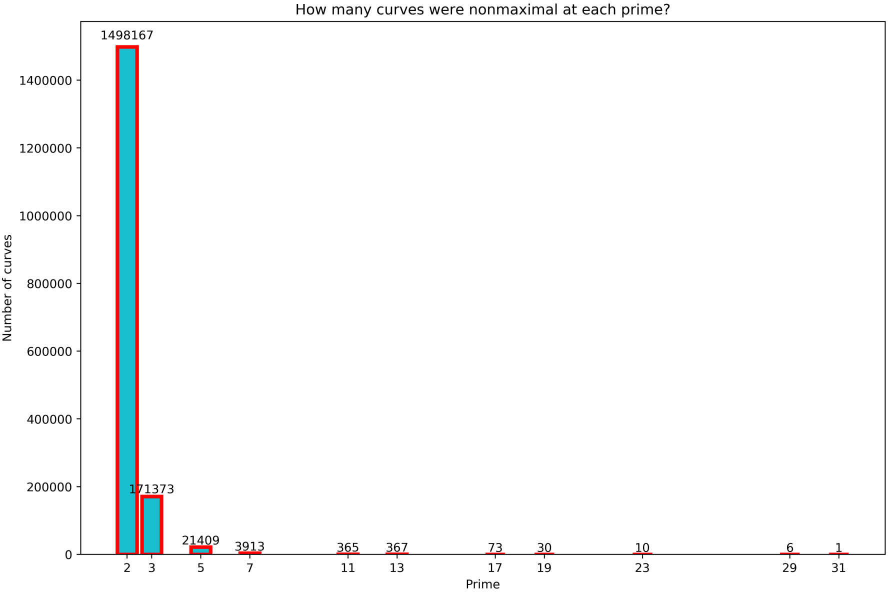
is "sufficiently large".

Theorem

Assuming that $\rho_{A,\ell}$ is surjective, there is an effective bound B_0 such that, for any $B > B_0$, if we sample $P_p(t)$ for n primes $p \in [B, 2B]$ uniformly and independently at random, then the Part 2 algorithm fails to remove ℓ with a probability $< 3 \cdot \left(\frac{9}{10}\right)^n$.

We ran our algorithm on $1,743,737$ typical genus 2 curves

- All of conductor bounded by 2^{20}.
- The curves are being prepared for addition into the LMFDB.
- Let $B = 1,000$ in the part 2 algorithm.
- Took about 35 hours on MIT's Lovelace computer.


How many curves were nonmaximal at each prime?
Number of curves
Prime
1498167
171373
21409
3913
365
367
73
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10
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200000
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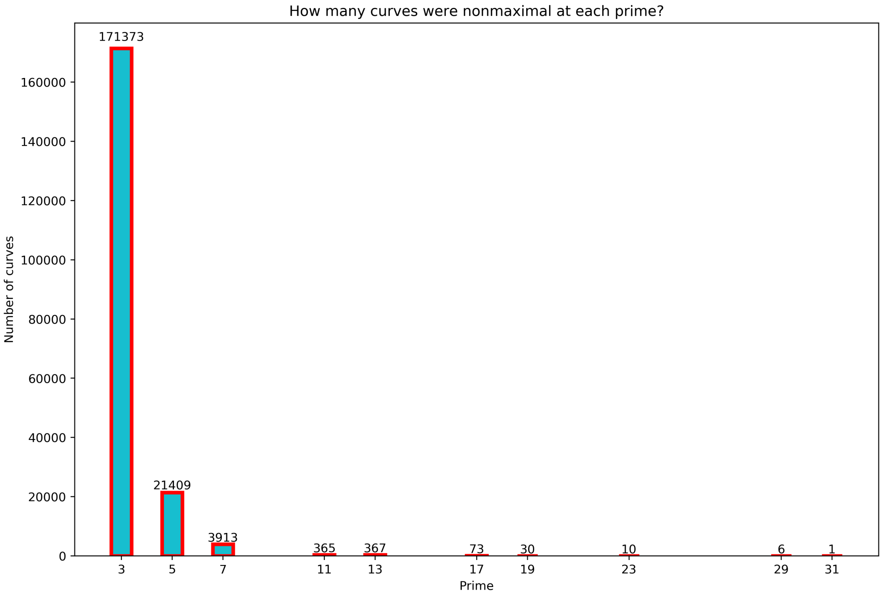
How many curves were nonmaximal at each prime?
Number of curves
Prime
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20000
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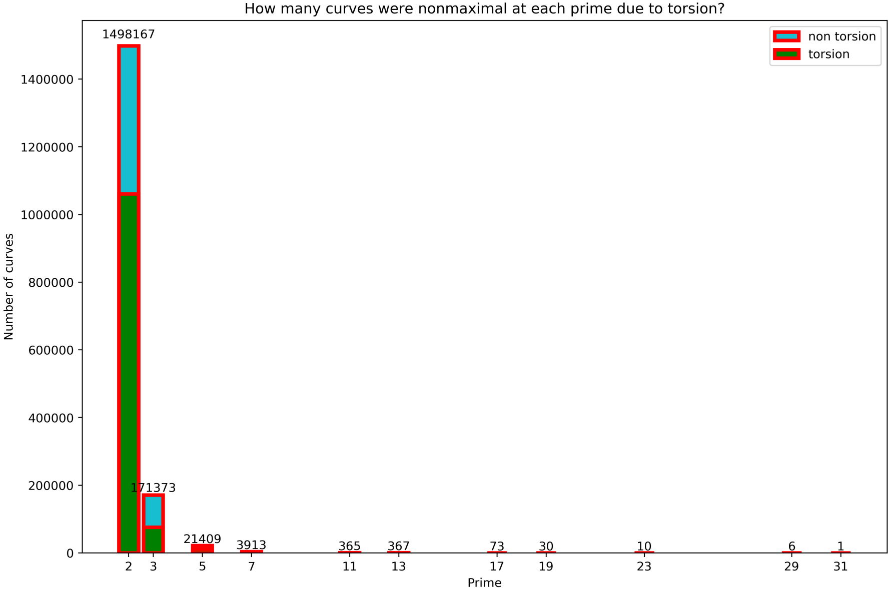
How many curves were nonmaximal at each prime due to torsion?
non torsion
torsion
Number of curves
Prime
1498167
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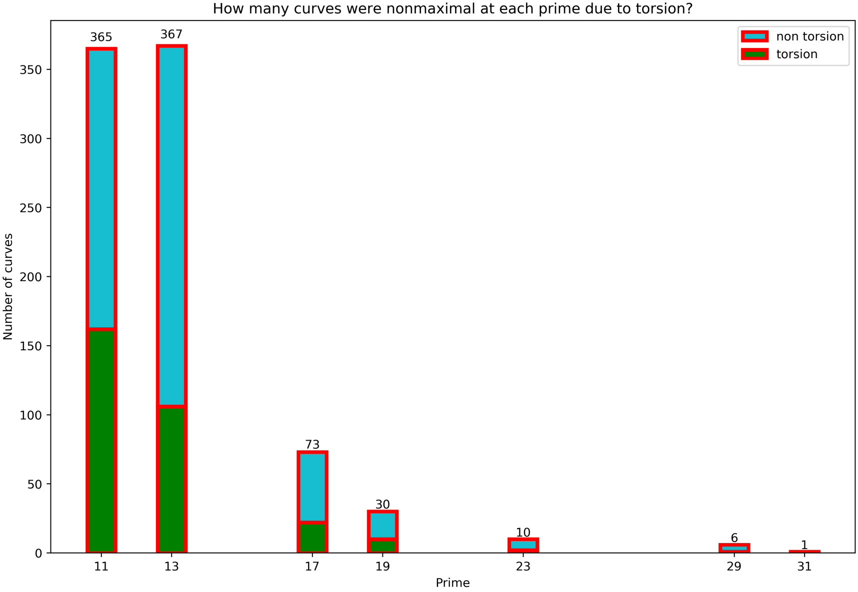
How many curves were nonmaximal at each prime due to torsion?
non torsion
torsion
Number of curves
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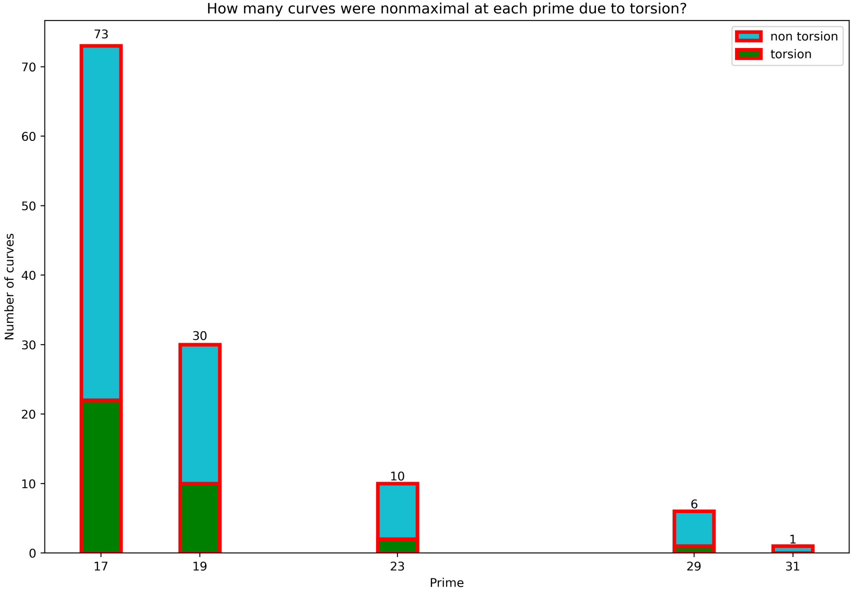
How many curves were nonmaximal at each prime due to torsion?
non torsion
torsion
Number of curves
Prime
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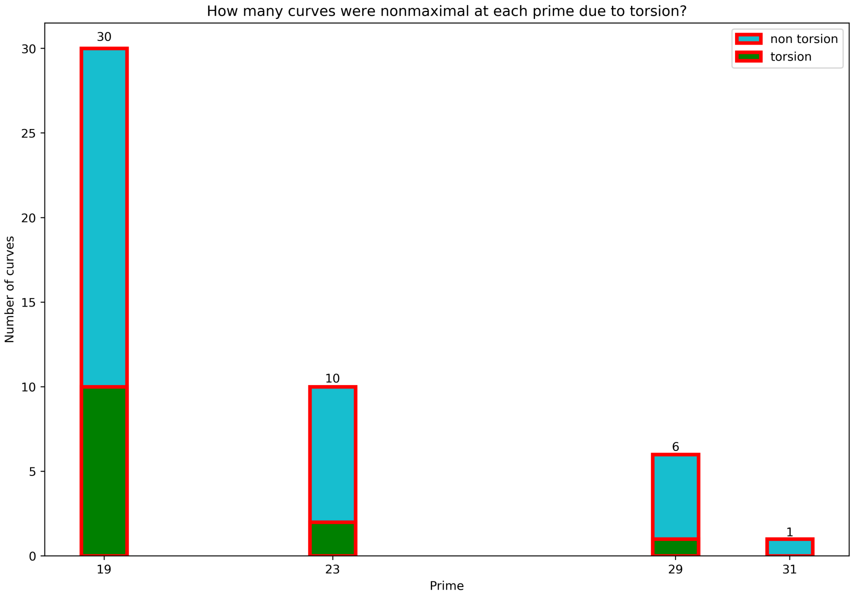
How many curves were nonmaximal at each prime due to torsion?
non torsion
torsion
Number of curves
Prime
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31

Find our computational results on the LMFDB!

- We had also run our code all 63,107 typical genus 2 curves in the LMFDB
- `https://www.lmfdb.org/`
- `https://www.lmfdb.org/Genus2Curve/Q/439587/d/439587/1`

Further questions

- Can we compute $\operatorname{Im} \rho_{A,\ell}$ when ℓ is not surjective?
- $\dim(A) > 2$?
- Other number fields?
- Bounds on the index of Galois image?

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