Computing Maass forms

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Classical modular forms in the LMFDB are great!

- Comprehensive over a wide range of conductor, weight, and character
- Links to elliptic curves and Artin representations
- L-functions
- Everything is rigorous to the extent that our mathematical knowledge allows; in particular, real numbers are treated with rigorous error bounds and interval arithmetic

Maass forms

$$\mathbb{H} = \{x + iy : y > 0\},\ ds^2 = \frac{dx^2 + dy^2}{y^2},\ \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\Gamma(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) : \gamma \equiv I \pmod{N}\}$

$\Gamma \supset \Gamma(N)$ congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, $X_\Gamma = \Gamma\backslash\mathbb{H}$

A Maass form is a square-integrable eigenfunction of Δ invariant under Γ, i.e. $f \in L^2(X_\Gamma)$ with $\Delta = \lambda f$.

Spectral decomposition of Δ:

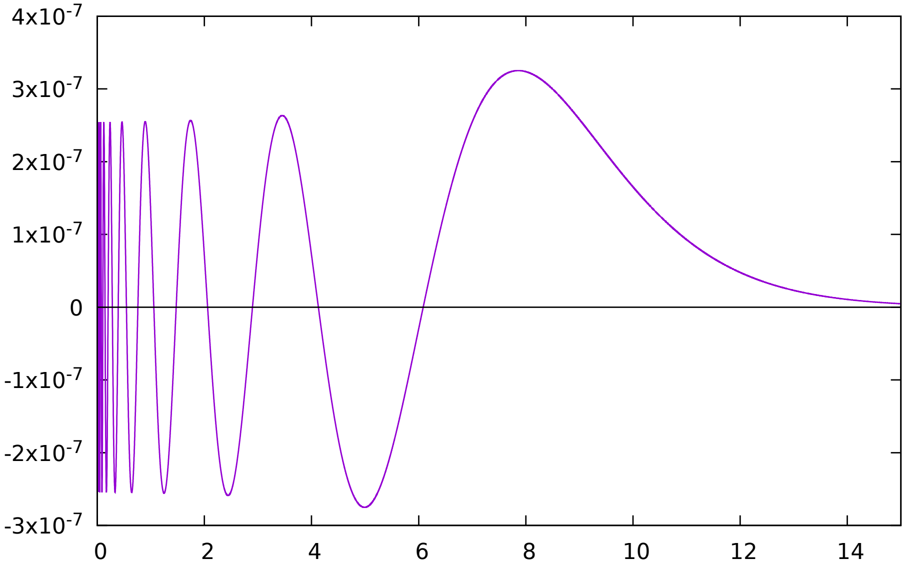
$$L^2(X_\Gamma) = (\text{continuous spectrum}) \oplus \bigoplus_{j=0}^{\infty} \mathbb{C} f_j$$

Fourier expansion: If $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$ then for $j > 0$, f_j takes the form

$$f_j(x+iy) = \sum_{n=1}^{\infty} a_j(n)\sqrt{y}K_{ir_j}(2\pi ny/h)\begin{cases}\cos(2\pi nx/h)\\ \sin(2\pi nx/h)\end{cases}$$

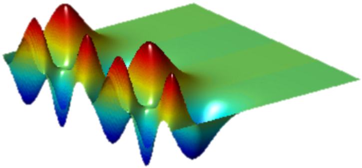
where $r_j = \sqrt{\lambda_j - \frac{1}{4}}$ and $K_{ir}(y) = \int_0^\infty e^{-y\cosh t}\cos(rt)\,dt$.

Maass forms

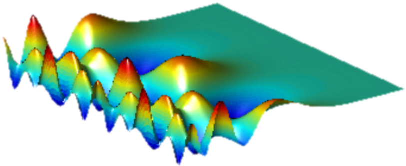


$K_{ir}(x)$ for $r \approx 9.53369526$

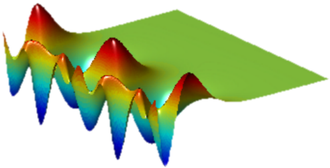
Maass forms



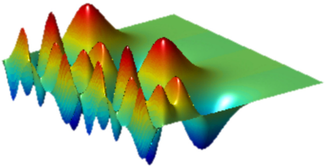
(a) Level 1, $\lambda = 91.141345\ldots$



(b) Level 1, $\lambda = 190.131547\ldots$



(c) Level 2, $\lambda = 79.867724\ldots$



(d) Level 3, $\lambda = 182.713668\ldots$

Maass forms and Artin representations

Theorem (B., 2003)

Let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ *be an even, irreducible Artin representation of conductor* N *and* L*-function* $L(s,\rho) = \sum_{n=1}^{\infty} a_\rho(n) n^{-s}$.

If $L(s,\rho)$ *is entire then there is a Maass form* f_j *for* $\Gamma = \Gamma_1(N) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \Gamma(N) \rangle$ *with Laplace eigenvalue* $\lambda_j = \frac{1}{4}$ *and Fourier coefficients* $a_j(n) = a_\rho(n)$.

Some applications of Maass form computations

- Computing real quadratic class groups

Bian, B., Docherty, Jacobson, Seymour-Howell

Determined the class group structure and regulator of all real quadratic fields of discriminant $\leq 10^{11}$ (soon to be 10^{12})

Some applications of Maass form computations

- Computing real quadratic class groups
- Classifying Artin representations

B., Lee, Strömbergsson (2020)

Determined the complete list of even, nondihedral, 2-dimensional Artin representations of conductor ≤ 2862

TABLE 1. *Even, nondihedral Artin representations of conductor* $\leqslant 2862$ *up to twist. For each twist equivalence class we indicate the minimal Artin conductor and link to the* LMFDB *page of a representation in the class. It is twist minimal in all cases except those marked with* $*$.

Tetrahedral															
163	277	349	397	547	549	607	679	703	709	711	763	853	937	949	995
1009	1073	1143	1147	1197	1267	1267	1333	1343	1368	1399	1413	1699	1773	1777	1789
1879	1899	1899	1935	1951	1953	1957	1984	2051	2077	2097	2131	2135	2169	2169	2223
2311	2353	2439	2456	2587	2639	2689	2709	2743	2763	2797	2803	2817			
Octahedral															
785	1345	1940	2159*	2279	2313	2364	2424	2440	2713	2777	2777	2777	2857		
Icosahedral															
1951*	1951*	2141*	2141*	2804*	2804*										

Some applications of Maass form computations

- Computing real quadratic class groups
- Classifying Artin representations
- Verifying the Selberg eigenvalue conjecture

Theorem (B., Lee, Strömbergsson, 2020)

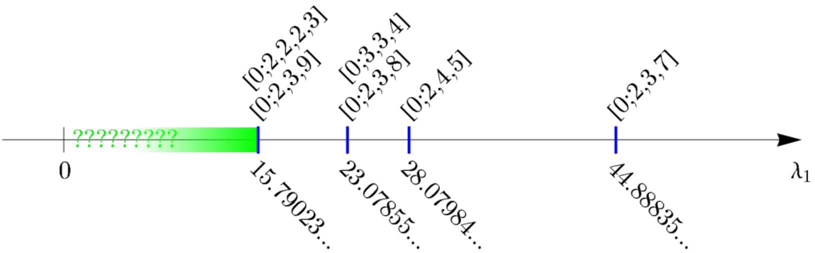
The Selberg eigenvalue conjecture is true for $\Gamma_1(N)$ *for* $N \leq 880$*, and for* $\Gamma(N)$ *for* $N \leq 226$*.*

Some applications of Maass form computations

- Computing real quadratic class groups
- Classifying Artin representations
- Verifying the Selberg eigenvalue conjecture
- Studying the distribution of low-lying eigenvalues

Kravchuk, Mazac, Pal (2021)

Determined the bass note dual of finite-volume hyperbolic surfaces



An incomplete history of Maass form computations

Pierre Cartier (1975-78), odd spectrum
Hartmut Haas (1977), odd and even (with an interesting mistake)
Dennis Hejhal (1979)
Hejhal, Berg (1982)
Golovchanskii, Smotrov (1982)
Stark (1984)
Winkler (1988) – Hecke groups Γ_q for $q = 3, 4, 5, 6, 7, 8$ ($\Gamma_3 = \mathrm{SL}(2, \mathbb{Z})$)

All of the above: $r \lesssim 25$

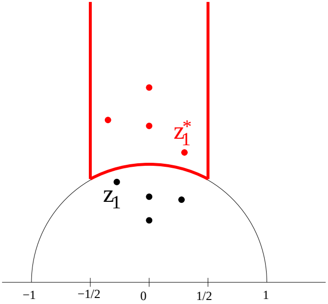
Hejhal (1991) – all $r < 50$; examples up to $r \lesssim 500$
Csordás, Graham, Szépfalusy (1991): All odd $0 < r \lesssim 200$
Steil (1994): all $r < 350$ and $500 < r < 510$; examples up to $r \approx 4000$
Hejhal (1999) – examples up to $r \approx 11000$
Then (2012) – all $r < 1400$; examples up to $r \approx 40000$

Basic algorithm

For simplicity, consider the case of an even Maass form f for $SL(2, \mathbb{Z})$. Fix an error tolerance ε and height cutoff Y. Fix $M = M(\varepsilon, Y)$ so that

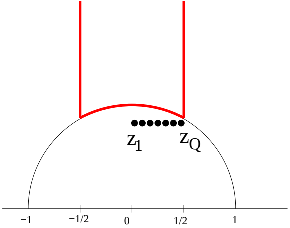
$$f(x+iy) = \sum_{1 \le m \le M} a_m \sqrt{y} K_{ir}(2\pi|m|y)\cos(2\pi mx) + \Big[|\text{error}| < \varepsilon\Big], \quad \forall y \ge Y.$$

Now use $f(z_j) = f(z_j^*)$ for appropriate $z_j = x_j + iy_j \notin \mathcal{F}$ with $y_j \ge Y$.



Hejhal's algorithm (1999)

Take $z_j = x_j + iY = \frac{j-\frac{1}{2}}{2Q} + iY$ $(j = 1, 2, \ldots, Q)$, with e.g. $Y = 0.86 < \frac{1}{2}\sqrt{3}$ and $Q > M(\varepsilon, Y)$.



Now:

$$f(z_j) = \sum_{1 \le m \le M} a_m \sqrt{Y} K_{ir}(2\pi|m|Y)\cos(2\pi m x_j) + \Big[|\text{error}| < \varepsilon\Big]$$

$$\Longrightarrow a_n \sqrt{Y} K_{ir}(2\pi|n|Y) = \frac{2}{Q}\sum_{j=1}^{Q} f(z_j)\cos(2\pi n x_j) + \Big[|\text{error}| \le 2\varepsilon\Big]$$

for $n = 0, 1, \ldots, M$.

Hejhal's algorithm (1999)

$$a_n\sqrt{Y}K_{ir}(2\pi|n|Y) = \frac{2}{Q}\sum_{j=1}^{Q} f(z_j)\cos(2\pi n x_j) + \Big[\big|\text{error}\big| < 2\varepsilon\Big]$$
$$= \frac{2}{Q}\sum_{j=1}^{Q} f(z_j^*)\cos(2\pi n x_j) + \Big[\big|\text{error}\big| < 2\varepsilon\Big].$$

Thus:

$$a_n\sqrt{Y}K_{ir}(2\pi|n|Y) = \sum_{1\le m\le M} a_m V_{nm} + \Big[\big|\text{error}\big| < 4\varepsilon\Big],$$

$$\text{where}\quad V_{nm} = \frac{2}{Q}\sum_{j=1}^{Q}\sqrt{y_j^*}\,K_{ir}(2\pi m y_j^*)\cos(2\pi m x_j^*)\cos(2\pi n x_j).$$

Theorem (B., Strömbergsson, Venkatesh, 2005)

Suppose that $\tilde{\lambda}$ and $\tilde{a}_n$ for $n \le M$ are numbers approximating to B bits the Laplacian and Hecke eigenvalues λ and a_n of a Maass form for $\mathrm{SL}(2,\mathbb{Z})$, i.e.

$$|\tilde{\lambda} - \lambda| < 2^{-B} \quad \text{and} \quad |\tilde{a}_n - a_n| < 2^{-B} \quad \forall n \le M.$$

For any $\varepsilon > 0$ and M, B sufficiently large (depending in a precise way on λ and ε), there is an algorithm that verifies in polynomial time in λ, M and B that

$$|\tilde{\lambda} - \lambda| < 2^{-(1-\varepsilon)B} \quad \text{and} \quad |\tilde{a}_n - a_n| < 2^{-(1-\varepsilon)B} \quad \forall n \le (1-\varepsilon)M.$$

Certification

Theorem (B., Strömbergsson, Venkatesh, 2005)

The first ten cuspidal eigenvalues $(\lambda_j = \frac{1}{4} + r_j^2)$ *on* $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ *are as follows, correct to 100 decimal places:*

$$
\begin{aligned}
r_1 &= 9.53369526135355755434423523592877032382125639510725198237579046413534899129834778176925550997543536\ldots\\
r_2 &= 12.17300832467967784952795117639554812398247167309994790041359894085944536082660887402607610119914083\ldots\\
r_3 &= 13.77975135189073894424367328151771259715513256879348706925238822161445033353997009415783160955742757\ldots\\
r_4 &= 14.35850951825981277986694256903716549561438589919676624781520226663201120679288581901319549358192409\ldots\\
r_5 &= 16.13807317152103058019829428598600394563144288541378695827382712175947030542755279355556642723837034\ldots\\
r_6 &= 16.64425920189981994352627455936865570143168145997231928907651455001829017618970424409102246827670179\ldots\\
r_7 &= 17.73856338105737789321732636154654617200548005325129188079624689810214157759980377279197640233860653\ldots\\
r_8 &= 18.18091783453070386031830826819331393824992456541781787106792508774526862910670490089247820557750868\ldots\\
r_9 &= 19.42348147082825519163378035720852444158555256007633319757794759378626261134705961296947480791604794 1\ldots\\
r_{10} &= 19.48471385474101336412852642787287621877406238534520661308580751557226039659320657991642152653178001\ldots
\end{aligned}
$$

Certification: sketch of proof

Putative eigenfunction:

$$f(x+iy) := \sum_{m=1}^{M} \tilde{a}_m \sqrt{y} K_{i\tilde{r}}(2\pi my)\cos(2\pi mx).$$

Note $\Delta f \equiv \tilde{\lambda} f$, with $\tilde{\lambda} = \frac{1}{4} + \tilde{r}^2$.

Set $\widetilde{f} = \Gamma$-invariant extension of $f_{|\mathcal{F}}$
and $\widetilde{f}_S = \widetilde{f}$ smoothed in Γ-invariant way.

The (standard) quasimode idea: there exists λ in continuous or discrete spectrum such that

$$\left|\tilde{\lambda} - \lambda\right| \le \frac{\left\|(\Delta - \lambda)\widetilde{f}_S\right\|_{L^2}}{\left\|\widetilde{f}_S\right\|_{L^2}}.$$

Note: In the odd case there is no continuous spectrum.

Certification: estimating the right-hand side

Lemma

Suppose $\widetilde{f}_S$ is obtained from $\widetilde{f}$ by convolving with a point-pair invariant k with compact support of size δ, and let $B(\delta)$ be the δ-neighborhood of $\{z \in \mathbb{H} : |z| = 1, |\Re z| \le \frac{1}{2}\}$. Then

$$\left\|(\Delta - \lambda)\widetilde{f}_S\right\|_{L^2} \le \sqrt{\mathrm{Area}(B(\delta) \cap \mathcal{F})} \int_{\mathbb{H}} \left|(\Delta - \lambda)k(z, i)\right| d\mu(z) \cdot \sup_{z \in B(\delta)} \left|\widetilde{f}(z) - f(z)\right|.$$

Certification: separating out the continuous spectrum

The proof (but not the computer implementation) is harder for even forms, because $\widetilde{f}$ has a continuous spectrum component.

We use the magic Lindenstrauss–Venkatesh operator:

$$\diamond = 2\cos\left((\log p)\sqrt{\Delta - \tfrac{1}{4}}\right) - T_p,$$

which annihilates the continuous spectrum.

Applying it to $\widetilde{f}_S$, the argument goes through as before, at the price of a factor $\frac{1}{p^{ir}+p^{-ir}-a_p}$.

Recent developments

- Strömberg (2005) extended Hejhal's algorithm to groups with many cusps, including $\Gamma_0(N)$. His algorithm underlies all of the Maass form data currently in the LMFDB.
- Then (2012) showed how to linearize the search for r, greatly improving the efficiency of Hejhal's algorithm.
- Child (2022) extended the BSV certification method to groups with multiple cusps.
- Berghaus, Monien, Radchenko (2022) made many practical improvements to Hejhal's algorithm.
- Seymour-Howell (2023) has shown that Hejhal's algorithm converges on the first few eigenfunctions of level 1.

Seymour-Howell's algorithm

Fix N and let $\{f_j\}$ be a Hecke eigenbasis of $L^2_{\text{cusp}}(\Gamma_0(N)\backslash\mathbb{H})$ with Laplace eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ and Hecke eigenvalues $a_j(n)$.

For a suitably nice test function H, the Selberg trace formula allows us to compute

$$t(n,H) := \sum_{j=1}^{\infty} a_j(n) H(\lambda_j)$$

for fixed $n \neq 0$ with $(n,N) = 1$.

Seymour-Howell's algorithm

By the Hecke relations, for any sequence of real numbers $\{c(m)\}_{m=1}^{M}$ such that $(m, N) > 1 \implies c(m) = 0$, we have

$$\left(\sum_{m=1}^{M} c(m)a_j(m)\right)^2 = \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} c(m_1)c(m_2) \sum_{d|(m_1,m_2)} a_j\left(\frac{m_1 m_2}{d^2}\right).$$

We define

$$\begin{aligned} Q(c,H) &:= \sum_{j=1}^{\infty}\left(\sum_{m=1}^{M} c(m)a_j(m)\right)^2 H(\lambda_j) \\ &= \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} c(m_1)c(m_2) \sum_{d|(m_1,m_2)} a_j\left(\frac{m_1 m_2}{d^2}\right) H(\lambda_j) \\ &= \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} c(m_1)c(m_2) \sum_{d|(m_1,m_2)} t\left(\frac{m_1 m_2}{d^2}, H\right). \end{aligned}$$

Seymour-Howell's algorithm

We choose the test function H to be non-negative, and define $\widetilde{H}(\lambda) = H(\lambda)(\lambda - \tilde{\lambda})^2$, where $\tilde{\lambda}$ is a putative approximate Laplace eigenvalue.

Defining $\varepsilon = \varepsilon(c) := \sqrt{Q(c, \widetilde{H})/Q(c, H)}$, we see that for any choice of the coefficients c, ε^2 is a weighted average of $(\lambda_j - \tilde{\lambda})^2$. Hence, there must exist a λ_j in the interval $[\tilde{\lambda} - \varepsilon, \tilde{\lambda} + \varepsilon]$.

For a given $\tilde{\lambda}$ we can find a choice of c that minimizes the Rayleigh quotient ε^2.

A related idea helps to choose $\tilde{\lambda}$: define $\widehat{H}(\lambda) = \lambda H(\lambda)$, and let Q and $\widehat{Q}$ denote the respective matrices of the quadratic forms $Q(c, H)$ and $Q(c, \widehat{H})$. Then we choose $\tilde{\lambda}$ to be the solutions to the generalized symmetric eigenvalue problem $\widehat{Q}x = \lambda Qx$.

Bootstrapping

Idea: given an approximate Laplace eigenvalue $\tilde{\lambda}$ that has been certified to low precision (10 decimal places, say), we can carry out Hejhal's algorithm in interval arithmetic to provably refine the precision of $\tilde{\lambda}$ and compute the associated Hecke eigenvalues.

The main theoretical input that's needed is bounds for $K_{ir}(x)$ and $\frac{\partial}{\partial r}K_{ir}(x)$.

Lowry-Duda has carried this out in wide generality and used it to refine output from Seymour-Howell's algorithm.

Other methods

- Kuznetsov trace formula (Golovchanskii and Smotrov, 1982)
- Power series (Voight and Willis, 2014)
- Finite element method (Levitin and Strohmaier, 2021)

L-functions

We have good algorithms for rigorously computing motivic L-functions in the LMFDB, including L-functions of classical modular forms, thanks to work of Platt and Costa.

Maass forms yield the first large class of L-functions to which those algorithms don't apply (and could not be easily extended without serious deficiencies in performance).

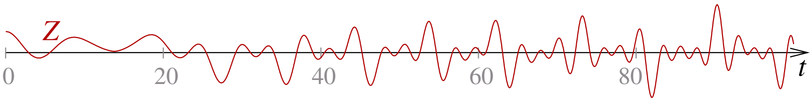
Fortunately, we have a different suite of algorithms for this specific family. One of the main ideas is to multiply the complete L-function by

$$ {}_2F_1\left(\frac{s+\epsilon+ir}{2}, \frac{s+\epsilon-ir}{2}; \frac{1}{2}+\epsilon; -\tan^2\theta\right) $$

for a suitable $\theta \in [0, \pi/2)$, rather than the exponential factor $e^{-i\theta s}$ that we use for motivic L-functions.

This has the right shape to compensate for the amplitude variations of the Γ-factors that occur for Maass forms.

L-functions



Theorem (B., Then, 2018)

For f a Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ with spectral parameter $r \in [0, 178]$, all non-trivial zeros of $L(s,f)$ with imaginary part bounded by 30000 are simple and lie on the critical line.

Summary of current status

- Rigorous computations for level 1 going back to 2005
- Work in progress of Lowry-Duda and Seymour-Howell gives the first major progress beyond level 1 ($\Gamma_0(N)$ for all squarefree $N \leq 105$)
- So far L-functions have been computed for level 1 only, but the algorithms have been worked out more generally

- Theoretical: explicit trace formulas
- Algorithmic: extend Seymour-Howell's algorithm to non-squarefree level and character
- Computational: large scale runs to get to higher level
- Applications: 1951 and all that
- Do everything again for weight 1