An Atlas of Orthogonal Discriminants

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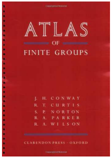
- ATLAS of finite groups
  ordinary character tables of finite simple groups
  classifying simple modules in characteristic 0



- ATLAS of Brauer tables
  modular character tables of finite simple groups
  classifying simple modules char. p dividing the group order
- Extended versions available in GAP (Thomas Breuer)
  functionalities for computing with characters:
  products, exterior powers, symmetrizations,
  permutation characters, restriction to subgroups ...

- Characters classify group homomorphisms into linear groups.
- Underlying field always contains the character field $F(\chi)$
  (a number field resp. a finite field)
- Sometimes contained in smaller classical group:
  Here: which orthogonal group?

Orthogonal representations of finite groups

ATLAS of orthogonal representations

- Current joint project with Richard Parker and Thomas Breuer:
- For all but the largest few ATLAS groups compute the discriminants of the invariant quadratic forms
- type of orthogonal group over finite fields

Recall

- $V = \bigoplus_{i=1}^{n} K v_i$ vector space
- $Q : V \to K$ quadratic form
- $B_Q : (v,w) \mapsto Q(v+w) - Q(v) - Q(w)$ polarisation
- $\mathrm{disc}(Q) := (-1)^{\binom{n}{2}} \det(B_Q(v_i,v_j)_{1\leq i,j\leq n})(K^\times)^2$ discriminant

Invariant quadratic forms

- G a finite group, K a field.
- $\rho : G \to \mathrm{GL}_n(K)$ representation, $V = K^n$ KG-module.

Space of G-invariant quadratic forms

$\mathfrak{Q}(\rho) := \{Q : V \to K \mid Q$ quadratic form $Q(v\rho(g)) = Q(v)$ for all $v \in V, g \in G\}$

ρ is called orthogonal if there is $Q \in \mathfrak{Q}(\rho)$ non-degenerate.

- ρ absolutely irreducible $\Rightarrow \mathfrak{Q}(\rho) = \{aQ \mid a \in K\}$.
- $\mathrm{disc}(aQ) = a^n \,\mathrm{disc}(Q) \in K^\times/(K^\times)^2$ well defined $\Leftrightarrow n$ even.

Orthogonal stability

An ordinary or Brauer character χ is called orthogonally stable if there is a square class $d(F(\chi)^{\times})^2$ such that for all representations $\rho : G \to \mathrm{GL}_n(L)$ with character χ and all non-degenerate $Q \in \mathfrak{Q}(\rho)$

$$\mathrm{disc}(Q) = d(L^{\times})^2.$$

$$\mathrm{disc}(\chi) := d(F(\chi)^{\times})^2$$

is called the orthogonal discriminant of χ.

Theorem

χ is orthogonally stable, if and only if all its absolutely irreducible orthogonal constituents have even degree.

Orthogonally simple characters

A character χ is called orthogonally simple if χ is orthogonal but it is not the sum of two orthogonal characters.

$$\chi = \chi_1 + \chi_2 \Rightarrow \operatorname{disc}(\chi) = \operatorname{disc}(\chi_1)\operatorname{disc}(\chi_2).$$

3 Types of orthogonally simple characters

- + $\chi \in \operatorname{Irr}(G), \operatorname{ind}(\chi) = +$.
  χ orthogonally stable $\Leftrightarrow$ $\chi(1)$ even.
- - $\chi = 2\psi$, $\psi \in \operatorname{Irr}(G), \operatorname{ind}(\psi) = -$. then $\operatorname{disc}(\chi) = 1$.
- ∘ $\chi = \psi + \overline{\psi}$, $\psi \neq \overline{\psi} \in \operatorname{Irr}(G)$, $\operatorname{ind}(\psi) = \circ$.
  $L := \mathbb{Q}(\psi)$, $K = \mathbb{Q}(\chi)$: $L = K[\sqrt{-\delta}]$ and $\operatorname{disc}(\chi) = (-\delta)^{\psi(1)}$

Task

Determine orthogonal discrimimants of absolutely irreducible orthogonal characters of even degree.

The discriminant of the adjoint involution

- B non-degenerate symmetric bilinear form
- adjoint involution ι_B on $\mathrm{End}(V)$

$$B(\alpha(v), w) = B(v, \iota_B(\alpha)(w)) \text{ for all } v, w \in V.$$

$$E_-(B) := \{\alpha \in \mathrm{End}_K(V) \mid \iota_B(\alpha) = -\alpha\}$$

- basis $(v_1, \ldots, v_n)$, $\mathrm{End}(V) \cong K^{n\times n}$
- $\iota_B(A) = BA^{tr}B^{-1}$ and $E_-(B) = \{BX \mid X = -X^{tr}\}$ as
- $\iota_B(BX) = B(BX)^{tr}B^{-1} = BX^{tr}$.
- $X = -X^{tr} \Rightarrow \det(X)$ is a square

Proposition

$\dim(V)$ even $\Rightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^*)^2$ for any invertible $\alpha \in E_-(B)$.

Computing the discriminant: Example 1

Proposition

$\dim(V)$ even $\Rightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^*)^2$ for any invertible $\alpha \in E_-(B)$.

- $Q \in \mathfrak{Q}(\rho)$ non-degenerate, $\rho(G) \leq O(Q)$, $n := \dim(\rho)$ even.
- $\iota_{B_Q}(g) = g^{-1} =: \iota(g)$ for all $g \in \rho(G)$.
- Take three random elements in $\rho(G)$; g, h, k.
- Compute $X = X(g,h,k) = g - g^{-1} + h - h^{-1} + k - k^{-1}$.
- If $\det(X) \neq 0$ then $\mathrm{disc}(Q) = (-1)^{\binom{n}{2}} \det(X)(K^\times)^2$.

$U_3(3) = \{X \in \mathbb{F}_9^{3\times 3} \mid X\overline{X}^{tr} = I_3, \det(X) = 1\}, U_3(3) : \langle\alpha\rangle, \alpha(X) = \overline{X}$

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@
		6048	96	108	9	96	96	16	12	7	7	8	8	12	12			24	24	3	4	6	6
p power			A	A	A	A	A	A	AA	A	A	A	B	AB	AA			A	A	BB	C	AD	AD
p' part			A	A	A	A	A	A	AA	A	A	A	A	AA	AB			A	A	BB	A	AD	AD
	ind	1A	2A	3A	3B	4A	B**	4C	6A	7A	B**	8A	B**	12A	B**	fus	ind	2B	4D	6B	8C	12C	D**
X_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1
X_2	-	6	-2	-3	0	-2	-2	2	1	-1	-1	0	0	1	1	:	oo	0	0	0	0	i3	-i3
X_3	+	7	-1	-2	1	3	3	-1	2	0	0	-1	-1	0	0	:	++	1	-3	1	-1	0	0
X_4	o	7	3	-2	1	-2i-1	2i-1	1	0	0	0	i	-i	i-1	-i-1	.	+	0	0	0	0	0	0
																|							
X_5	o	7	3	-2	1	2i-1	-2i-1	1	0	0	0	-i	i	-i-1	i-1	.							
X_6	+	14	-2	5	-1	2	2	2	1	0	0	0	0	-1	-1	:	++	2	-2	-1	0	1	1
X_7	+	21	5	3	0	1	1	1	-1	0	0	-1	-1	1	1	:	++	3	-1	0	1	-1	-1
X_8	o	21	1	3	0	2i-3	-2i-3	-1	1	0	0	i	-i	-i	i	.	+	0	0	0	0	0	0
																|							
X_9	o	21	1	3	0	-2i-3	2i-3	-1	1	0	0	-i	i	i	-i	.							
X_10	+	27	3	0	0	3	3	-1	0	-1	-1	1	1	0	0	:	++	3	3	0	-1	0	0
X_11	o	28	-4	1	1	4i	-4i	0	-1	0	0	0	0	i	-i	.	+	0	0	0	0	0	0
																|							
X_12	o	28	-4	1	1	-4i	4i	0	-1	0	0	0	0	-i	i	.							
X_13	o	32	0	-4	-1	0	0	0	0	-b7	**	0	0	0	0	.	+	0	0	0	0	0	0
																|							
X_14	o	32	0	-4	-1	0	0	0	0	**	-b7	0	0	0	0	.							

Computing the discriminant: Example 2

- $G = U_3(3)$, $\chi(1) = 14$, $Q \in \mathfrak{Q}(\rho_\chi)$.

- $g \in \rho(G)$, $g^7 = 1$: Eigenvalues: all 7th root of 1, multiplicity 2.
- $V = V_1 \perp V_z$, $\dim(V_z) = 12$, $g_z := g_{|V_z}$

- $\det(V_z) = \det(g_z - g_z^{-1}) = \prod_{i=1}^{6}(\zeta_7^i - \zeta_7^{-i})^2$ is a square.

- $h \in N_G(\langle g \rangle)$, $h^3 = 1$ acts on V_1 with minimal polynomial $X^2 + X + 1$
- so $\det(V_1) = \det(h_1 - h_1^{-1}) = (\zeta_3 - \zeta_3^{-1})(\zeta_3^{-1} - \zeta_3) = 3$

- $\mathrm{disc}(\chi) = (-1)\det(V_1)\det(V_z) = -3(\mathbb{Q}^\times)^2$.

Computing the discriminant: Example 3

Theorem (Eva Bayer 2015)

Assume that there is $g \in O(Q)$ such that $P(1)P(-1) \neq 0$, where P is the characteristic polynomial of g. Then $\det(g) = 1$, $n = \dim(V)$ is even, and $\det(Q) = P(1)P(-1)(K^\times)^2$.

- $P = \prod_{i=1}^{n}(X - \xi_i) \in \overline{K}[X]$
- $g \sim g^{-1} = B_Q g^{tr} B_Q^{-1} \Rightarrow$ mult(ξ_i^{-1})=mult(ξ_i).
- $\xi_i \neq \xi_i^{-1} \Rightarrow \det(g) = \prod_{j=1}^{n} \xi_j = 1$ and n is even.

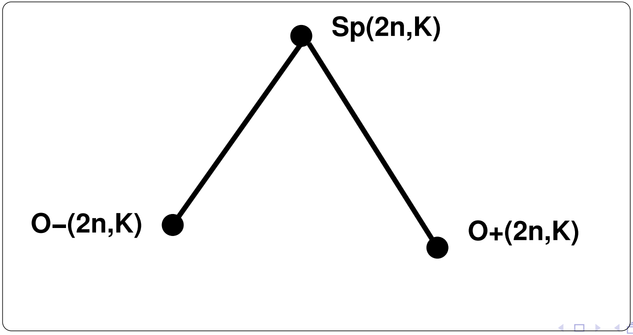
$$P(1)P(-1) = \prod_{j=1}^{n}(\xi_j^2 - 1) = (\prod_{j=1}^{n} \xi_j) \prod_{j=1}^{n}(\xi_j - \xi_j^{-1}) = \det(g)\det(g - g^{-1})$$

so $P(1)P(-1) = \det(g - g^{-1}) = \det(Q)$ up to squares.

Fields of characteristic 2: Non-Example

- K a field of characteristic 2
- $Q : V \to K$ non-degenerate quadratic form
- $Q \in K^{n \times n}$, $Q(x) = xQx^{tr}$.
- $B = Q + Q^{tr}$ Gram matrix of polarisation.
- $\ell := QB^{-1}$

Then $\dim(V) = n = 2m$ even and $\text{Arf}(Q) = s(\ell) + m(m-1)/2$ where $P_\ell = X^n + tX^{n-1} + s(\ell)X^{n-2} + \ldots$ characteristic polynomial.



An application

Theorem

χ is orthogonally stable, if and only if all its absolutely irreducible orthogonal constituents have even degree.

Task

Determine orthogonal discrimimants of absolutely irreducible orthogonal characters of even degree.

Theorem

Absolutely irreducible orthogonal characters of even degree are orthogonally stable.

- Finite fields:
- $K = F(\chi)$, $\rho : G \to \mathrm{GL}_n(K)$, $\chi_\rho = \chi$.
- $\mathfrak{Q}(\rho) = \{aQ \mid a \in K\}$.
- $\mathrm{disc}(\chi) = \mathrm{disc}(Q)$

An application: Example 4

Theorem

Absolutely irreducible orthogonal characters of even degree are orthogonally stable.

- Number fields:
- $K = \mathbb{Q}(\chi)$ characterfield
- $A := \langle \rho(g) \mid g \in G \rangle_{\mathbb{Q}}$ is a central simple K-algebra.
- ι_B inverts the group elements
- $E_-(\rho) := E_-(B) \cap A = \langle \rho(g) - \rho(g^{-1}) \mid g \in G \rangle_{\mathbb{Q}}$
  $E_-(\rho)$ contains invertible elements X.
- $d := N_{red}(X) \in K$
- $\mathrm{disc}(\chi) = d(K^\times)^2$.

The character table of J_2

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@	@	@	@	@	@	
			1		1																															
		604800	920	240	080	36	96	300	300	50	50	24	12	7	8	20	20	10	10	12	15	15			336	48	12	6	48	16	6	6	7	12	12	
p power			A	A	A	A	A	A	A	A	A	AA	BB	A	A	BB	AB	DA	CA	AA	BA	AA			A	A	B	BC	A	A	AB	BC	AC	AB	AB	
p' part			A	A	A	A	A	A	A	A	A	AA	BB	A	A	AB	BB	CA	DA	AA	AA	BA			A	A	A	BC	A	A	AB	BC	AC	AB	AB	
ind		1A	2A	2B	3A	3B	4A	5A	B*	5C	D*	6A	6B	7A	8A	10A	B*	10C	D*	12A	15A	B*	fus	ind	2C	4B	4C	6C	8B	8C	12B	12C	14A	24A	B*	
χ_{1}	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1	1	1	1	1	1	χ_{1}
χ_{2}	+	14	-2	2	5	-1	2	-3b5	*	b5+2	*	1	-1	0	0	b5	*	-b5	*	-1	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{2}
χ_{3}	+	14	-2	2	5	-1	2	*	-3b5	*	b5+2	1	-1	0	0	*	b5	*	-b5	-1	0	0	|													χ_{3}
χ_{4}	+	21	5	-3	3	0	1	b5+4	*	-2b5	*	-1	0	0	-1	b5	*	0	0	1	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{4}
χ_{5}	+	21	5	-3	3	0	1	*	b5+4	*	-2b5	-1	0	0	-1	*	b5	0	0	1	-b5	*	|													χ_{5}
χ_{6}	+	36	4	0	9	0	4	-4	-4	1	1	1	0	1	0	0	0	-1	-1	1	-1	-1	:	++	6	-2	0	0	2	2	1	0	-1	-1	-1	χ_{6}
χ_{7}	+	63	15	-1	0	3	3	3	3	-2	-2	0	-1	0	1	-1	-1	0	0	0	0	0	:	++	7	3	-1	1	-3	1	0	-1	0	0	0	χ_{7}
χ_{8}	+	70	-10	-2	7	1	2	-5b5	*	0	0	-1	1	0	0	-b5	*	0	0	-1	b5	*	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{8}
χ_{9}	+	70	-10	-2	7	1	2	*	-5b5	0	0	-1	1	0	0	*	-b5	0	0	-1	*	b5	|													χ_{9}
χ_{10}	+	90	10	6	9	0	-2	5	5	0	0	1	0	-1	0	1	1	0	0	1	-1	-1	:	++	6	2	0	0	4	0	-1	0	-1	1	1	χ_{10}
χ_{11}	+	126	14	6	-9	0	2	1	1	1	1	-1	0	0	0	1	1	-1	-1	-1	1	1	:	++	0	4	0	0	2	-2	1	0	0	-1	-1	χ_{11}
χ_{12}	+	160	0	4	16	1	0	-5	-5	0	0	0	1	-1	0	-1	-1	0	0	0	1	1	:	++	8	0	2	-1	0	0	0	-1	1	0	0	χ_{12}
χ_{13}	+	175	15	-5	-5	1	-1	0	0	0	0	3	1	0	-1	0	0	0	0	-1	0	0	:	++	7	-1	1	1	-1	-1	-1	1	0	-1	-1	χ_{13}
χ_{14}	+	189	-3	-3	0	0	-3	-3b5	*	b5+2	*	0	0	0	1	b5	*	b5	*	0	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{14}
χ_{15}	+	189	-3	-3	0	0	-3	*	-3b5	*	b5+2	0	0	0	1	*	b5	*	b5	0	0	0	|													χ_{15}
χ_{16}	+	224	0	-4	8	-1	0	2r5-1	-2r5-1	2b5	*	0	-1	0	0	1	1	0	0	0	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{16}
χ_{17}	+	224	0	-4	8	-1	0	-2r5-1	2r5-1	*	2b5	0	-1	0	0	1	1	0	0	0	-b5	*	|													χ_{17}
χ_{18}	+	225	-15	5	0	3	-3	0	0	0	0	0	-1	1	-1	0	0	0	0	0	0	0	:	++	1	-3	-1	1	3	-1	0	-1	1	0	0	χ_{18}
χ_{19}	+	288	0	4	0	-3	0	3	3	-2	-2	0	1	1	0	-1	-1	0	0	0	0	0	:	++	8	0	-2	-1	0	0	0	1	1	0	0	χ_{19}
χ_{20}	+	300	-20	0	-15	0	4	0	0	0	0	1	0	-1	0	0	0	0	0	1	0	0	:	++	6	-2	0	0	-2	-2	1	0	-1	1	1	χ_{20}
χ_{21} $Q_{2,3}$	+	336	16	0	-6	0	0	-4	-4	1	1	-2	0	0	0	0	0	1	1	0	-1	-1	:	++	0	0	0	0	0	0	0	0	0	r6	-r6	χ_{21}
												6	12	7	8	20	20	10	10	12	15	15	fus	ind	4	4	8	12	8	8	12	24	28	24	24	
ind		1	2	4	3	3	4	5	5	5	5	6		14	8			10	10	12	30	30					8		8		12	24	28	24	24	
		2	2		6	6	4	10	10	10	10																									
		6	2	0	-3	0	2	-2b5	*	*	b5-1	1	0	-1	0	0	0	-b5	*	-1	b5	*	|	-	0	0	0	0	0	0	0	0	0	0	0	χ_{22}
																																				χ_{23}

The character table of J_2

- Rational Schur index of 336 is 2.
- So need 672 dimensional rational representation, dimension of space of invariant quadratic forms is 3.
- But over $\mathbb{F}_p$, $p \geq 7$ this is a well defined character of an orthogonal representation of degree 336.

- Compute $\mathrm{disc}(336)$:
- Construct 672-dimensional rational representation ρ

- Choose $g, h, k \in \rho(G)$, compute $X = X(g, h, k) = g - g^{-1} + h - h^{-1} + k - k^{-1} \in \rho(E_{-}(\mathbb{Q}G))$.
- $\mathrm{disc}(336) = N_{red}(X)(\mathbb{Q}^{\times})^2$.

Further applications

Split extensions $G:2$ (Example 5)

Groups with non-trivial center (ind. o, Spinor norm)

Reduction to simple groups

Enough to compute $\mathrm{disc}(\chi)$ for absolutely irreducible even degree orthogonal characters of simple groups.

p-groups ($G \neq 1 \Rightarrow Z(G) \neq 1$)

- Let G be a p-group, χ orthogonally stable
- explicit formula for $\mathrm{disc}(\chi)$
- $\mathrm{disc}(\chi) = (-p)^{\chi(1)/2}$ if $p \equiv 3 \pmod 4$
- $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$ for $p = 2$.
- Similar for $p \equiv 1 \pmod 4$
- $\mathbb{Q}(\chi) = \mathbb{Q}$, $p \equiv 1 \pmod 4 \Rightarrow \mathrm{disc}(\chi) = p^{\chi(1)/(p-1)}$.

Orthogonal Condensation

- Large KG-module V with composition factors $S_1, \ldots, S_t$
- compute idempotent $e \in KG$
- condensed module Ve is module for $eKGe$
- composition factors $\{S_i e \mid 1 \leq i \leq t\} - \{0\}$.
- Problem:
- $\{g_1, \ldots, g_s\}$ K-algebra generating set of KG
- then set $\{eg_i e \mid 1 \leq i \leq s\}$ does not generate $eKGe$
- However we know the involution on $eKGe$, $ege \mapsto eg^{-1}e$.

Orthogonal Condensation: Example 6

- V a permutation module, p odd prime
- $H \in \mathrm{Syl}_p(G)$, $e := \frac{1}{|H|} \sum_{h \in H} h$.
- $V(1-e)_{|H}$ orthogonally stable of known orthogonal discriminant
- Ve spanned by the H-orbit sums
- yields eg_ie, $i = 1, \ldots, 10$
- $\iota(eg_ie) = eg_i^{-1}e$
- $A = \langle eg_ie, eg_i^{-1}e \mid 1 \leq i \leq 10 \rangle_{\mathbb{Q}-algebra}$
- Reduce modulo primes $\neq p$
- Compute determinant of skew element of A on composition factors
- Enough primes $\Rightarrow$ discriminant of this composition factor.

Harada Norton group

- $G = HN$, $\dim(V) = 108, 345, 600$, $p = 5$.
- $\dim(Ve) = 7008$.
- $\mathrm{disc}(\chi) = 4\sqrt{5} + 17$ for $\chi(1) = 5, 103, 000$.

The discriminant field

Definition

χ ordinary orthogonally stable character, $K = \mathbb{Q}(\chi)$ character field, $d(K^\times)^2 = \mathrm{disc}(\chi)$. Then $\Delta(\chi) := K[\sqrt{d}]$ discriminant field of χ.

Theorem

$\wp \trianglelefteq \mathbb{Z}_K$ prime ideal such that $\chi \pmod{\wp}$ is orthogonally stable. Then

- $\wp$ is unramified in $\Delta(\chi)/K$.
- $\wp$ is split in $\Delta(\chi)/K$ if and only if $\mathrm{disc}(\chi \pmod{\wp}) = 1$.

In particular only prime divisors of the group order ramify in $\Delta(\chi)/K$.

$\mathbb{Q}(\chi) = \mathbb{Q}$, $\chi \pmod{2}$ orthogonally stable $\Rightarrow \mathrm{disc}(\chi) \equiv 1 \pmod{4}$.

The discriminant field

Definition

χ ordinary orthogonally stable character, $K = \mathbb{Q}(\chi)$ character field, $d(K^\times)^2 = \mathrm{disc}(\chi)$. Then $\Delta(\chi) := K[\sqrt{d}]$ discriminant field of χ.

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- $\wp$ is split in $\Delta(\chi)/K$ if and only if $\mathrm{disc}(\chi \pmod{\wp}) = 1$.

In particular only prime divisors of the group order ramify in $\Delta(\chi)/K$.

If χ lies in a $\wp$-block of defect 1 then $\chi \pmod{\wp}$ is orthogonally stable if and only if $\wp$ is unramified in $\Delta(\chi)/K$.

Ordinary character table of J_1.

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@
		175560	120	30	30	30	6	7	10	10	11	15	15	19	19	19
	p power		A	A	A	A	AA	A	BA	AA	A	BA	AA	A	A	A
	p' part		A	A	A	A	AA	A	AA	BA	A	AA	BA	A	A	A
	ind	1A	2A	3A	5A	B*	6A	7A	10A	B*	11A	15A	B*	19A	B*5	C*4
X_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X_2	+	56	0	2	-2b5	*	0	0	0	0	1	b5	*	-1	-1	-1
X_3	+	56	0	2	*	-2b5	0	0	0	0	1	*	b5	-1	-1	-1
X_4	+	76	4	1	1	1	1	-1	-1	-1	-1	1	1	0	0	0
X_5	+	76	-4	1	1	1	-1	-1	1	1	-1	1	1	0	0	0
X_6	+	77	5	-1	2	2	-1	0	0	0	0	-1	-1	1	1	1
X_7	+	77	-3	2	b5	*	0	0	b5	*	0	b5	*	1	1	1
X_8	+	77	-3	2	*	b5	0	0	*	b5	0	*	b5	1	1	1
X_9	+	120	0	0	0	0	0	1	0	0	-1	0	0	c19	*5	*4
X_10	+	120	0	0	0	0	0	1	0	0	-1	0	0	*4	c19	*5
X_11	+	120	0	0	0	0	0	1	0	0	-1	0	0	*5	*4	c19
X_12	+	133	5	1	-2	-2	-1	0	0	0	1	1	1	0	0	0
X_13	+	133	-3	-2	-b5	*	0	0	b5	*	1	-b5	*	0	0	0
X_14	+	133	-3	-2	*	-b5	0	0	*	b5	1	*	-b5	0	0	0
X_15	+	209	1	-1	-1	-1	1	-1	1	1	0	-1	-1	0	0	0

$$\mathrm{b5} = \frac{1}{2}(-1+\sqrt{5})$$

$$\mathrm{c19} = \zeta+\zeta^7+\zeta^8+\zeta^{11}+\zeta^{12}+\zeta^{18},\ \zeta = \exp(2\pi i/19)$$

χ	$56a$	$56b$	$76a$	$76b$	$120a$	$120b$	$120c$
$\mathbb{Q}(\chi)$	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}[c_{19}]$	$\mathbb{Q}[c_{19}]$	$\mathbb{Q}[c_{19}]$
$p=2$	$56a$	$56b$	$76a$	$76b$	$120a$	$120b$	$120c$
$p=3$	$56a$	$56b$	$76a$	$76b$	$120a$	$120b$	$120c$
$p=5$	$56a$	$56a$	$76a$	$76b$	$120a$	$120b$	$120c$
$p=7$	$56a$	$56b$	$1+75$	$31+45$	$45+75$	$31+89$	120
$p=11$	56	$7+49$	$27+49$	$7+69$	$1+119$	$56+64$	$14+106$
$p=19$	$1+55$	$22+34$	$76a$	$76b$	$43+77$	$43+77$	$43+77$
$\mathrm{disc}(\chi)$	17-4$\sqrt{5}$	17+4$\sqrt{5}$	77	77	29-18c_{19}-9c'_{19}	47+9c_{19}+18c'_{19}	38+9c_{19}-9c'_{19}

$\Delta(\chi)/\mathbb{Q}$ is not Galois for $\chi \in \{56a, 56b, 120a, 120b, 120c\}$.

Theorem (Marie Roth)

If G is a solvable group then $\Delta(\chi)/\mathbb{Q}$ is Abelian.

Observation

Non-Galois extensions only occur for sporadic groups.

Parker's Conjecture

If $\mathrm{disc}(\chi) = d(K^\times)^2$ then for all dyadic valuations ν of K we have that $\nu(d)$ is even.

Theorem (N.)

Parker's conjecture holds for solvable groups.

Observation

No counterexamples to Parker's conjecture so far.