Computing invariants of Hilbert modular surfaces

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Endomorphism algebras of elliptic curves

Proposition

Let k be an algebraically closed field and E be an elliptic curve over k. Then $\mathrm{End}^0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is one of the following types.

- *I(1): $\mathbb{Q}$,*
- *III(1): a definite quaternion algebra over $\mathbb{Q}$; or*
- *IV(1,1): an imaginary quadratic field.*

Endomorphism algebras of abelian surfaces

Proposition (Albert, Oort)

Let k be an algebraically closed field and A be a simple abelian surface over k. Then $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is one of the following types.

- *I(1): $\mathbb{Q}$,*
- *I(2): a real quadratic field,*
- *II(1): an indefinite quaternion algebra over $\mathbb{Q}$; or*
- *IV(2,1): a degree 4 CM field.*

Moduli of abelian varieties

$g = 1$	$g = 2$	Dimension
	Siegel moduli spaces	3
	Hilbert modular surfaces	2
Modular curves	Shimura curves	1
CM points	CM points	0

Basic invariants

What's the first thing you want to know about a curve C?

Its genus!

$$2 - 2g(C) = \deg(\mathcal{T}_C) = c_1(\mathcal{T}_C).$$

What's the first thing you want to know about a surface S?

Its invariants!

$c_1(\mathcal{T}_S)^2$, $c_2(\mathcal{T}_S)$, Hodge diamond, Kodaira dimension.

Geometric invariants of surfaces

Let S be a smooth algebraic surface.

Hodge decomposition: $H^r(S,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(S)$.

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{l} b_0 := \dim H^0(S,\mathbb{C}) \\ b_1 := \dim H^1(S,\mathbb{C}) \\ b_2 := \dim H^2(S,\mathbb{C}) \\ b_3 := \dim H^3(S,\mathbb{C}) \\ b_4 := \dim H^4(S,\mathbb{C}) \end{array}$$

$$\frac{c_1^2 + c_2}{12} = \chi(\mathcal{O}_S) := h^{0,0} - h^{1,0} + h^{2,0}$$ (**holomorphic Euler char.**)

$$c_2 = e(S) := \sum_{i=0}^{4} (-1)^i b_i = 2 - 2b_1 + b_2$$ (**Euler number**)

$$h^{1,1} = e - 2\chi(\mathcal{O}_S) = c_2 - 2\chi(\mathcal{O}_S)$$

Geometric invariants of surfaces

Let S be a smooth (connected) Hilbert modular surface.

Hodge decomposition: $H^r(S,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(S)$.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{l} b_0 = 1 \\ b_1 = 0 \\ b_2 := \dim H^2(S,\mathbb{C}) \\ b_3 = 0 \\ b_4 = 1 \end{array}$$

$$\frac{c_1^2 + c_2}{12} = \chi(\mathcal{O}_S) := 1 - 0 + h^{2,0} \qquad \textbf{(holomorphic Euler char.)}$$

$$c_2 = e(S) := \sum_{i=0}^{4} (-1)^i b_i = 2 + b_2 \quad (\textbf{Euler number})$$

$$h^{1,1} = e - 2\chi(\mathcal{O}_S) = c_2 - 2\chi(\mathcal{O}_S)$$

Thus c_1^2 and c_2 determine the Hodge diamond.

Complex uniformization

- $\mathfrak{H}$ = complex upper half-plane
- F = real quadratic number field
- $\mathbb{Z}_F$ its ring of integers

If $A = \mathbb{C}^2/\Lambda$ has RM by $\mathbb{Z}_F$, then $\Lambda \simeq \mathbb{Z}_F \cdot (z_1, z_2) \oplus \mathfrak{b} \subseteq \mathbb{C}^2$ for some $z = (z_1, z_2) \in \mathfrak{H}^2$ and some fractional ideal $\mathfrak{b} \subseteq F$. This gives an identification of the relevant moduli space with

$$\mathrm{SL}(\mathbb{Z}_F \oplus \mathfrak{b})\backslash\mathfrak{H}^2$$

where $\mathrm{SL}(\mathbb{Z}_F \oplus \mathfrak{b}) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}^2$ coordinatewise by linear fractional transformations.

From now on, assume $\mathfrak{b} = \mathbb{Z}_F$.

(*Technical detail:* the $\mathbb{Z}_F$-linear polarizations on A equip Λ with an orientation, determining $[\mathfrak{b}] \in \mathrm{Cl}^+(F)$.)

Hilbert modular surfaces

We focus on the congruence subgroups

$$\Gamma = \Gamma_0^1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_F) : c \in \mathfrak{N} \right\}.$$

(Similar definition for Γ_0 replacing SL_2 by GL_2^+; also for $\mathfrak{b} \neq \mathbb{Z}_F$.)

Quotient: $\Gamma \backslash \mathfrak{H}^2 =: Y(\Gamma)$ is a **Hilbert modular surface**.

Compactify: Let $(\mathfrak{H}^2)^* := \mathfrak{H}^2 \cup \mathbb{P}^1(F)$ and

$$\overline{Y}(\Gamma) := \Gamma \backslash (\mathfrak{H}^2)^*$$

The **cusps** of Γ are the orbits of $\mathbb{P}^1(F)$ under Γ.

Resolve: $\overline{Y}(\Gamma)$ is singular at the cusps (and elliptic points). Need to compute a minimal resolution

$$\pi \colon X(\Gamma) \to \overline{Y}(\Gamma).$$

Chern numbers: key takeaway

Theorem

Let $X(\Gamma)$ be the minimal desingularization of $\overline{\Gamma\backslash\mathfrak{H}^2}$. Then

$$c_1^2(X(\Gamma)) = 2\,\mathrm{vol}(\Gamma\backslash\mathfrak{H}^2) + \textit{cusps term} + \textit{elliptic points term},$$
$$c_2(X(\Gamma)) = \mathrm{vol}(\Gamma\backslash\mathfrak{H}^2) + \textit{cusps term} + \textit{elliptic points term.}$$

Ingredients for these formulas:

1. The volume: this is given by $2\,[\mathrm{SL}_2(\mathbb{Z}_F) : \Gamma]\,\zeta_F(-1)$.
2. Cuspidal singularities and their resolutions.
3. Elliptic singularities and their resolutions.

Cusps and stabilizers

The **cusps** of Γ are the orbits of $\mathbb{P}^1(F)$ under Γ. We compute them by adapting work of Dasgupta–Kakde.

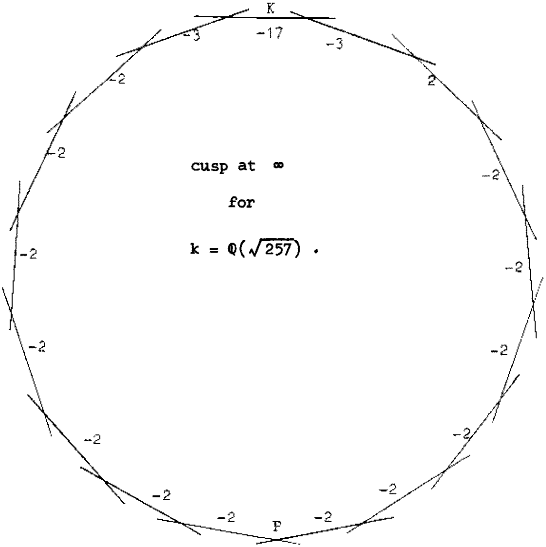
Consider $(1:0) = \infty$. Its **stabilizer** can be written as

$$\mathrm{Stab}_\Gamma(\infty)/\{\pm 1\} = G(M,V) = \left\{ \begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} : v \in V,\ m \in M \right\}$$

where $V \leq \mathbb{Z}_{F,>0}^\times$ and $M \subset F$ is a fractional ideal. We determine M and V, then apply Hirzebruch's method to compute the resolution using continued fractions and toric geometry.

Note: for other congruence subgroups, $G(M,V) \lneq \mathrm{Stab}_\Gamma(c)$ with finite index.

Resolution of a cusp



Elliptic points

A point $z \in \mathfrak{H}^2$ is an **elliptic point** of Γ if $\mathrm{Stab}_\Gamma(z) \neq \{\pm 1\}$.

In this case, $\mathrm{Stab}_\Gamma(z)/\{\pm 1\} = \langle \gamma \rangle$ is finite cyclic.

The group $G = \mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (\zeta_m^a z_1, \zeta_m^b z_2)$$

where ζ_m is a primitive m^{th} root of 1 and $a, b \in \{0, \ldots, m-1\}$.

The elliptic point z is of **type** $(m; a, b)$ if locally its image in $\Gamma\backslash\mathfrak{H}^2$ looks like the image of the origin in $G\backslash\mathbb{C}^2$ with the above action. The type determines the resolution.

Let $\mathcal{O}$ be the order generated by Γ in $\mathrm{M}_2(F)$. To determine the number of elliptic points of each type, we compute the number of embeddings of quadratic orders $\mathbb{Z}_F[\gamma] \hookrightarrow \mathcal{O}$ up to Γ-conjugacy.

Kodaira classification of minimal surfaces

A surface S is **minimal** if it contains no curves with self-intersection -1.

Kodaira dim	type	χ	K^2
$\kappa = -1$	rational	1	8 or 9
$\kappa = 0$	Enriques	1	0
	K3	2	0
$\kappa = 1$	honestly elliptic	≥ 1	0
$\kappa = 2$	general type	≥ 1	≥ 1

When S is a Hilbert modular suface, χ and K^2 completely determine the Kodaira dimension, except when:

1. $\chi = 1, \quad K^2 = 8, 9$: rational or general type,
2. $\chi = 2, \quad K^2 = 0$: K3 or honestly elliptic.

van der Geer's classification

Lemma

If S is an algebraic surface with $h^{0,1} = 0$, $\chi(S) > 1$ and $c_1^2(S) > 0$, then S is a surface of general type.

Proposition (van der Geer)

Let $\Gamma = \mathrm{PSL}(\mathbb{Z}_F \oplus \mathfrak{b})$ where $\mathfrak{b}$ is a nonzero fractional ideal. If F has discriminant $D > 500$, then $\chi(Y(\Gamma)) > 1$ and $c_1^2(Y(\Gamma)) > 0$, and thus $Y(\Gamma)$ is of general type.

Conjecture

If F has discriminant D and $D^{3/2}\,\mathrm{Nm}(\mathfrak{N}) > 500^{3/2}$, then $\chi(X_0^1(\mathfrak{N})) > 1$ and $c_1^2(X_0^1(\mathfrak{N})) > 0$, and thus $X_0^1(\mathfrak{N})$ is of general type.

Results

Type	Kodaira dimension	Γ_0	Γ_0^1
rational	$\kappa = -1$	18	15
honestly elliptic	$\kappa = 1$	7	16
general type	$\kappa = 2$	4308	4311
unknown	$\kappa \in \{-1, 2\}$	61	44
	$\kappa \in \{0, 1\}$	5	12
	$\kappa \in \{0, 1, 2\}$	51	43
	$\kappa \in \{1, 2\}$	67	76
	total unknown	184	175
total		4517	4517

Summary:

- New ideas to enumerate all cusps.
- New ideas to enumerate all elliptic points.
- Computing invariants for many Hilbert modular surfaces.
- Identify the Kodaira dimension in many cases.

Future directions:

- Computing the Kodaira dimension. (Better, Hilbert series for canonical sheaf.)
- Equations!

`https://teal.lmfdb.xyz/HilbertModularSurface/Q/`
`https://github.com/edgarcosta/hilbertmodularforms`

Thank you!

Rational (?) surfaces

d_F	Genus of $\mathfrak{b}$	$\mathfrak{N}$
5	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_5$, $\mathfrak{p}_3$, $\mathfrak{p}_{11}$, $\mathfrak{p}_2^2$, $\mathfrak{p}_{19}$, $\mathfrak{p}_2\mathfrak{p}_5$, $\mathfrak{p}_5^2$, $\mathfrak{p}_{29}$, $\mathfrak{p}_2\mathfrak{p}_{11}$, $\mathfrak{p}_{59}$
8	$+$	$\mathfrak{p}_2^2$, $\mathfrak{p}_7$, $\mathfrak{p}_2^3$, $\mathfrak{p}_2\mathfrak{p}_7$, $\mathfrak{p}_2^4$, $\mathfrak{p}_{23}$
12	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2^2$, $\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_3^2$, $\mathfrak{p}_{11}$
12	$--$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_{11}$
13	$+$	$\mathfrak{p}_3$, $\mathfrak{p}_3^2$
17	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_2^2$
24	$++$	$\mathfrak{p}_3$
28	$++$	$\mathfrak{p}_2$

Table: Hilbert surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ for $\mathfrak{N} \neq (1)$ satisfying $\chi = 1$

K3 (?) surfaces

d_F	Genus of $\mathfrak{b}$	$\mathfrak{N}$
5	$+$	$\mathfrak{p}_{31}$
8	$+$	$\mathfrak{p}_3$, $\mathfrak{p}_{17}$, $\mathfrak{p}_2^5$
12	$++$	$\mathfrak{p}_2^3$, $\mathfrak{p}_2^2\mathfrak{p}_3$, $\mathfrak{p}_2^4$
12	$--$	$\mathfrak{p}_2^2$, $\mathfrak{p}_2^3$, $\mathfrak{p}_2^2\mathfrak{p}_3$
13	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_3\overline{\mathfrak{p}_3} = (3)$
17	$+$	$\mathfrak{p}_2\overline{\mathfrak{p}_2} = (2)$, $\mathfrak{p}_2^3$
21	$++$	$\mathfrak{p}_3$, $\mathfrak{p}_2$, $\mathfrak{p}_5$, $\mathfrak{p}_7$, $\mathfrak{p}_3^2$
21	$--$	$\mathfrak{p}_3$, $\mathfrak{p}_5$
24	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_2^2$
24	$--$	$\mathfrak{p}_2$
28	$++$	$\mathfrak{p}_3$, $\mathfrak{p}_2^2$, $\mathfrak{p}_3^2$
28	$--$	$\mathfrak{p}_3$
33	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2^2$
33	$--$	$\mathfrak{p}_2$

Table: Hilbert surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ for $\mathfrak{N} \neq (1)$ satisfying $\chi = 2$ and $K^2 \leq 0$