Isogeny classes of typical, principally polarized abelian surfaces over $\mathbb{Q}$

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LMFDB, Computation, and Number Theory (LuCaNT), ICERM, 10 July 2023

Joint work with Edgar Costa, Shiva Chidambaram, and Jean Kieffer

Isogenies

Definition

An isogeny between two abelian varieties over $\mathbb{Q}$ is a morphism $\varphi\colon A \twoheadrightarrow B$ such that $\#\ker\varphi < \infty$.

Isogenies are obtained by taking quotients by finite subgroups defined over $\mathbb{Q}$. Being isogenous is an equivalence relation.

Theorem (Faltings)

The isogeny class of A over $\mathbb{Q}$ is finite.

Two abelian varieties in the same isogeny class share many properties, including

- dimension
- L-function
- Mordell–Weil rank $\mathrm{rk}_{\mathbb{Z}} A(\mathbb{Q})$
- endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q}$

Isogeny classes

Theorem (Faltings)

The isogeny class of A over $\mathbb{Q}$ is finite.

Can construct (finite, connected) isogeny graphs:

- vertices: abelian varieties in an isogeny class,
- edges: indecomposable isogenies and labelled by degree.

Questions

- What are the possible isogeny graphs when $\dim(A)$ is fixed?
- Can we compute the isogeny graph of a given abelian variety A?

Elliptic curves over the rationals

We can explore isogeny graphs of elliptic curves over $\mathbb{Q}$ at the LMFDB.

- Ignoring degrees, we find 10 non-isomorphic graphs:

Size	1	2	3	4	6	8
Examples	37.a	26.b	11.a	27.a, 20.a, 17.a	14.a, 21.a	15.a, 30.a

- All edge labels, i.e. degrees of indecomposable isogenies, are prime.
- Not all primes ℓ appear as isogeny degrees: only

$$\ell \in \{2, \ldots, 19, 37, 43, 67, 163\}.$$

Lemma

Any isogeny $\varphi \colon E \to E'$ can be factored as $E \xrightarrow{[n]} E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} E_n = E'$, where $\deg(\varphi_i) = \ell_i$ are primes and φ_i are defined over $\mathbb{Q}$.

Elliptic curves over the rationals

Theorem (Mazur)

If $\varphi\colon E \to E'$ defined over $\mathbb{Q}$ has prime degree ℓ, then $\ell \in \{2, \ldots, 19, 37, 43, 67, 163\}$.

Theorem (Kenku)

Any isogeny class of elliptic curves over $\mathbb{Q}$ has size at most 8.

Chiloyan, Lozano-Robledo 2021

Complete classification of possible labelled isogeny graphs.

The LMFDB contains examples for all of these graphs.

Higher dimensions?

Algorithmic problem

Given an abelian surface A (i.e. $g = 2$) over $\mathbb{Q}$, compute its isogeny class.

In this work, we add two additional assumptions:

- A is principally polarized, i.e. equipped with $A \simeq A^\vee$. True for ECs and Jacobians.
- A is typical, i.e. $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Z}$.

Then A is the Jacobian of genus 2 curves over $\mathbb{Q}$:

$$y^2 = f(x), \qquad \deg(f) = 5 \text{ or } 6 \text{ and } f \text{ has distinct roots.}$$

The LMFDB contains genus 2 curves with small discriminants, grouped by isogeny class of their Jacobians, but these isogeny classes are currently not complete.

Algorithmic approach

Algorithmic problem

Given an abelian variety A over $\mathbb{Q}$, compute its isogeny class.

For an elliptic curve $E/\mathbb{Q}$:

1. Search for ℓ-isogenies $E \to E'$ for each ℓ in Mazur's list. This is a finite problem.
2. Reapply on E' as needed.

In general:

1. Classify the possible isogeny types. (E.g., "prime degree" for elliptic curves.)
2. Compute a finite number of possible degrees. We now face a finite problem.
3. Search for all isogenies of a given type and degree.
4. Reapply as needed.

Classification of isogenies

Let A be typical, principally polarized abelian surface.

Proposition

The isogeny class of A can be enumerated using isogenies φ of the following types:

1. 1-step: $K := \ker(\varphi)$ is a maximal isotropic subgroup of $A[\ell]$, so $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$,
2. 2-step: K is a maximal isotropic subgroup of $A[\ell^2]$ and $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2 \times \mathbb{Z}/\ell^2\mathbb{Z}$.

These isogenies are of degree ℓ^2 and ℓ^4 respectively. Here "isotropic" means: isotropic w.r.t. the Weil pairing on $A[\ell]$ or $A[\ell^2]$, so that the quotient abelian surface A/K is still principally polarized.

We need to know which primes ℓ can arise. However no analogue of Mazur's isogeny theorem is known for $g > 1$.

Dieulefait's algorithm

Serre's open image theorem

If A is a typical abelian surface, then $A[\ell]$ has a nontrivial subgroup defined over $\mathbb{Q}$ only for finitely many primes ℓ.

This is good: if φ is a 1-step isogeny, then $A[\ell]$ contains a 2-dimensional subspace defined over $\mathbb{Q}$. If φ is 2-step, then $A[\ell]$ contains a 1-dimensional subspace over $\mathbb{Q}$.

Algorithm (Dieulefait, 2002)

Input: Genus 2 curve C such that $A = \mathrm{Jac}(C)$

Output: Finite set of primes ℓ containing those for which $A[\ell]$ has nontrivial subgroups defined over $\mathbb{Q}$.

Example where the only possibilities are isogenies of degree 31^2:

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45.$$

The only reasonable algorithm to actually find isogenies is to use analytic methods, i.e. $\mathbb{Q} \hookrightarrow \mathbb{C}$.

We have $A(\mathbb{C}) = \mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$ for some period matrix $\tau \in \mathbb{H}_2$: this means τ is a 2×2 complex, symmetric matrix such that $\mathrm{Im}(\tau)$ is positive definite. $\mathbb{H}_2$ carries an action of $\mathrm{GSp}_4(\mathbb{R})^+$, analogous to the "usual" action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}_1$.

Lemma

There are explicit sets $S_1(\ell)$ and $S_2(\ell) \subset \mathrm{GSp}_4(\mathbb{Q})^+$ such that for $i = 1, 2$,

$$\{\text{ab. surfaces } i\text{-step } \ell\text{-isogenous to } \mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)\} = \{\mathbb{C}^2/\left(\mathbb{Z}^2 + \gamma\tau\mathbb{Z}^2\right)\}_{\gamma \in S_i(\ell)}.$$

We need to decide when $\gamma\tau \in \mathbb{H}_2$ is attached to an abelian surface defined over $\mathbb{Q}$, and if so, reconstruct the associated genus 2 curve.

Finding isogenous curves

Task

Decide which $\gamma\tau$, for $\gamma \in S_1(\ell)$ or $S_2(\ell)$, are period matrices of $\mathbf{Jac}(C)$ for some genus 2 curve $C/\mathbb{Q}$.

Problem

Modular polynomials are of size $\mathcal{O}(\ell^{15+\varepsilon})$, which is too big! ($\gg$ 29 GB for $\ell = 7$)

1. Evaluate Siegel modular forms at $\gamma\tau$. This yields $\mathbb{C}$-valued invariants of the curve C. (Think: the j-invariant of elliptic curves is also an analytic function.) Call these invariants $N(j, \gamma)$ for $j \in \{4, 6, 10, 12\}$.
2. If C is defined over $\mathbb{Q}$, then $N(j, \gamma)$ is a rational number, and even an integer if properly constructed. We can certify this with interval arithmetic.
3. Given these invariants in $\mathbb{Z}$, reconstruct an equation for C by "standard methods" (Mestre's algorithm, computing the correct twist.)

Example, continued

Let $\ell = 31$, $i = 1$ and

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45.$$

Working at 300 bits of precision, there is only one $\gamma_0 \in S_1(\ell)$ such that the invariants $N(j,\gamma_0)$ for $j \in \{4, 6, 10, 12\}$ could possibly be integers:

$$N(4,\gamma_0) = \alpha^2 \cdot 318972640 + \varepsilon \quad \text{with } |\varepsilon| \leq 7.8 \times 10^{-47},$$
$$N(6,\gamma_0) = \alpha^3 \cdot 1225361851336 + \varepsilon \quad \text{with } |\varepsilon| \leq 5.5 \times 10^{-39},$$
$$N(10,\gamma_0) = \alpha^5 \cdot 10241530643525839 + \varepsilon \quad \text{with } |\varepsilon| \leq 1.6 \times 10^{-29},$$
$$N(12,\gamma_0) = -\alpha^6 \cdot 307105165233242232724 + \varepsilon \quad \text{with } |\varepsilon| \leq 4.6 \times 10^{-22}$$

where $\alpha = 2^2 \cdot 3^2 \cdot 31$.

We certify equality by working at 4 128 800 bits of precision using certified quasi-linear time algorithms for the evaluation of modular forms (Kieffer 2022).

Example, finding the curve

Given $(m'_4, m'_6, m'_{10}, m'_{12}) = (318972640, 1225361851336, 10241530643525839, \ldots)$, find a corresponding curve C' such that $\mathbf{Jac}(C)$ and $\mathbf{Jac}(C')$ are isogenous over $\mathbb{Q}$.

Mestre's algorithm yields

$$y^2 = -1624248x^6+5412412x^5-6032781x^4+876836x^3-1229044x^2-5289572x-1087304,$$

a quadratic twist by -83761 of the desired curve

$$C' : y^2+xy = -x^5+2573x^4+92187x^3+2161654285x^2+406259311249x+93951289752862.$$

We reapply the algorithm to C', and we only find the original curve.

Remarks

- 113 minutes of CPU time for this example
- 90% of the time is spent certifying the results

LMFDB data

Originally 63 107 typical genus 2 curves in 62 600 isogeny classes.

By computing isogeny classes, we found 21 923 new curves.

Size	1	2	3	4	5	6	7	8	9	10	12	16	18
Count	51 549	2 672	6 936	420	756	164	40	45	3	2	3	9	1

Observation

A 2-step 2-isogeny (of degree 16) always implies an existence of a second one. This explains the 6913 $\triangle$ and the 756 $\bowtie$ we found.

The whole computation took 75 hours. Only 3 classes took more than 10 minutes:

- 349.a: 56 min, isogeny of degree 13^4.
- 353.a: 23 min, isogeny of degree 11^4.
- 976.a: 19 min, checking that no isogeny of degree 29^4 exists.

Upcoming to LMFDB

A new set of 1 743 737 typical genus 2 curves due to Sutherland is soon to be added to the LMFDB, split in 1 440 894 isogeny classes. We found 600 948 new curves (in 111 CPU days). Counts per size:

1	2	3	4	5	6	7	8	≥ 9
1 032 456	116 847	197 253	54 543	15 547	14 323	430	5 594	3 901

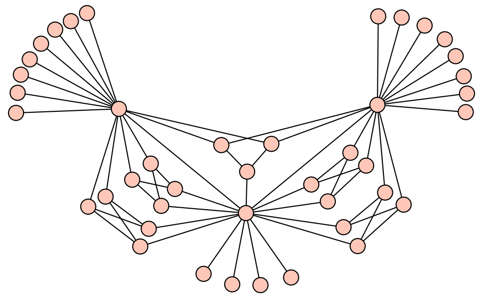
We discovered indecomposable isogenies of degree

$$2^2 \text{ (= Richelot isogenies)}, \ 2^4, 3^2, 3^4, 5^2, 5^4, 7^2, 7^4, 11^4, 13^2, 13^4, 17^2, 31^2.$$

- Size 2: 75% have degree 2^2, 22% have degree 3^4, and then 3^2, 5^4, 5^2, 7^4, 7^2, ...
- Size 3: 99% are $\triangle$ of degree 2^4 isogenies.
- Size 4: 98% are $\succ\!\!-$ of Richelot isogenies.
- Size 5: 99.8% are $\bowtie$ of degree 2^4 isogenies.
- Size 6: 75% + 15% are two graphs consisting of Richelot isogenies.

Life, the universe, and everything

Isogeny graph consisting of 42 Richelot isogenous curves (outside our database):



Preprint: https://arxiv.org/abs/2301.10118

Code and data: https://github.com/edgarcosta/genus2isogenies