The landscape of L-functions

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joint work with
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L-functions: the glue that holds the (number theory) world together

Many sources: varieties, modular forms, number fields, ...
But we want to view them as their own independent objects.

Degree 1:
Dirichlet series with an Euler product:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1-\chi(p)p^{-s})^{-1}$$

Functional equation:

$$\begin{aligned} \Lambda(s,\chi) &= N^{s/2}\Gamma_{\mathbb{R}}(s+\delta_\chi)L(s,\chi) \\ &= \varepsilon_\chi \overline{\Lambda}(1-s,\chi) \\ &= \varepsilon_\chi \Lambda(1-s,\overline{\chi}) \end{aligned}$$

χ is a (primitive) character of conductor N.

A constraint and some notation

$$\Lambda(s,\chi) = N^{s/2}\Gamma_{\mathbb{R}}(s+\delta_\chi)L(s,\chi) = \varepsilon_\chi\overline{\Lambda}(1-s,\chi)$$

$\delta_\chi = 0$ or 1, with $\chi(-1) = (-1)^{\delta_\chi}$.

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

$$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$$

Degree 2: two types

Case 1: $f \in S_k(\Gamma_0(N), \chi)$, $k \geq 2$.

Dirichlet series with an Euler product:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$
$$= \prod_p (1 - a(p)p^{-s} + \chi(p)p^{-2s})^{-1}$$

Functional equation:

$$\Lambda(s, f) = N^{s/2}\Gamma_{\mathbb{C}}(s + \kappa)L(s, f)$$
$$= \varepsilon_f \overline{\Lambda}(1 - s, f)$$
$$= \varepsilon_f \Lambda(1 - s, \overline{f})$$

$\kappa = \dfrac{k-1}{2}$. The constraint: $\chi(-1) = (-1)^{2\kappa+1}$.

Degree 2: two types

Case 2: f a Maass newform of weight 0 or 1, on $\Gamma_0(N)$ with character χ and spectral parameter $\lambda \in \mathbb{R}$.

Dirichlet series with an Euler product:

$$\begin{aligned} L(s,f) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \\ &= \prod_p (1 - a(p)p^{-s} + \chi(p)p^{-2s})^{-1} \end{aligned}$$

Functional equation:

$$\begin{aligned} \Lambda(s,f) &= N^{s/2}\Gamma_{\mathbb{R}}(s+\delta_1+i\lambda)\Gamma_{\mathbb{R}}(s+\delta_2-i\lambda)L(s,f) \\ &= \varepsilon_f \overline{\Lambda}(1-s,f) \\ &= \varepsilon_f \Lambda(1-s,\overline{f}) \end{aligned}$$

The constraint: $\chi(-1) = (-1)^{\delta_1+\delta_2}$.

Degree d

N: the **conductor**

χ: a character mod N, the **central character**

Dirichlet series with an Euler product:

$$\begin{aligned} L(s,f) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \\ &= \prod_p (1 - a(p)p^{-s} + \cdots + (-1)^d \chi(p) p^{-ds})^{-1} \end{aligned}$$

Functional equation:

$$\begin{aligned} \Lambda(s) &= N^{s/2} \Big(\Gamma - \text{factors} \Big) L(s) \\ &= \varepsilon \overline{\Lambda}(1-s) \end{aligned}$$

$\Gamma -$ factors: $\Gamma_{\mathbb{R}}(s+\delta+i\mu)$ and/or $\Gamma_{\mathbb{C}}(s+\kappa+i\lambda)$.

General degree d Γ-factor

$$\prod_{j=1}^{d_1} \Gamma_{\mathbb{R}}(s+\delta_j+i\mu_j) \prod_{k=1}^{d_2} \Gamma_{\mathbb{C}}(s+\kappa_k+i\lambda_k)$$

where $d_1 + 2d_2 = d$.

$\delta_j \in \{0, 1\}$

$\kappa_k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$

$\mu_j, \lambda_k \in \mathbb{R}$

The central character constraint:

$$\chi(-1) = (-1)^{\sum \delta_j + \sum (2\kappa_j + 1)}$$

All possible degree 3 conductor 1 functional equations

$N = 1$ so χ is the trivial character, so $\chi(-1) = 1$:

$(-1)^{\sum \delta_j + \sum(2\kappa_j+1)} = 1$

Case 1: 3 of $\Gamma_\mathbb{R}$

$$\Gamma_\mathbb{R}(s + i\mu_1)\Gamma_\mathbb{R}(s + i\mu_2)\Gamma_\mathbb{R}(s + i\mu_3)$$

$$\Gamma_\mathbb{R}(s + i\mu_1)\Gamma_\mathbb{R}(s + 1 + i\mu_2)\Gamma_\mathbb{R}(s + 1 + i\mu_3)$$

Case 2: 1 of $\Gamma_\mathbb{R}$ and 1 of $\Gamma_\mathbb{C}$

$$\Gamma_\mathbb{R}(s + i\mu)\Gamma_\mathbb{C}(s + \kappa + i\lambda) \qquad \text{with} \quad \kappa \in \{\tfrac{1}{2}, \tfrac{3}{2}, \ldots\}$$

$$\Gamma_\mathbb{R}(s + 1 + i\mu)\Gamma_\mathbb{C}(s + \kappa + i\lambda) \qquad \text{with} \quad \kappa \in \{1, 2, 3, \ldots\}$$

A normalization

If $L(s)$ is an L-function, then so is $L(s + iy)$ for any $y \in \mathbb{R}$. So in

$$\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s + i\lambda_2)\Gamma_{\mathbb{R}}(s + i\lambda_3)$$

we may assume $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

By rearranging and conjugating, that functional equation is specified by a pair (λ_1, λ_2) with $0 \le \lambda_1 \le \lambda_2$.

Similarly,

$$\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_2)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_3)$$

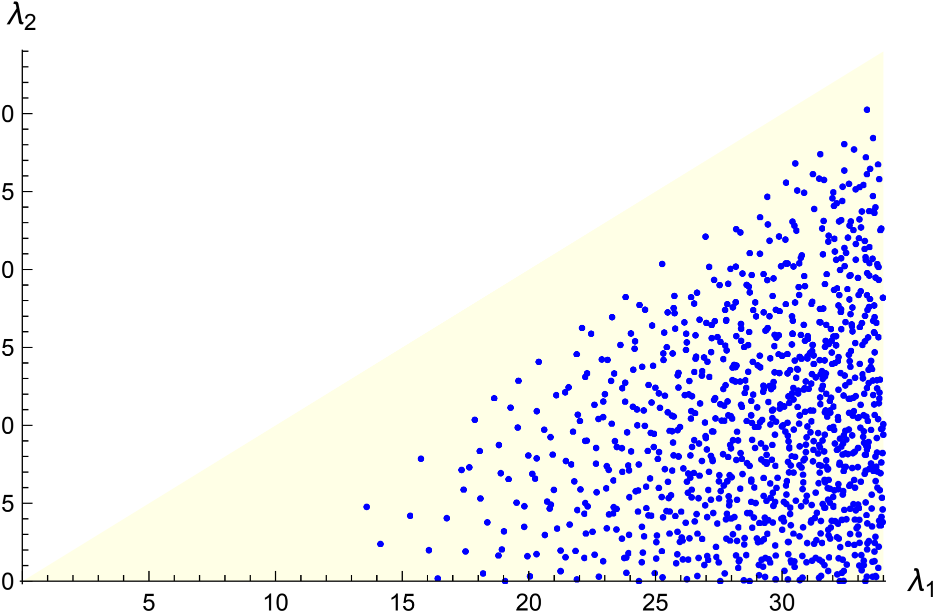
is specified by (λ_1, λ_2) with $\lambda_1 \ge 0$ and $\lambda_2 \ge -\lambda_1/2$.
Also,

$$\Gamma_{\mathbb{R}}(s + \delta + 2i\lambda)\Gamma_{\mathbb{C}}(s + \kappa - i\lambda)$$

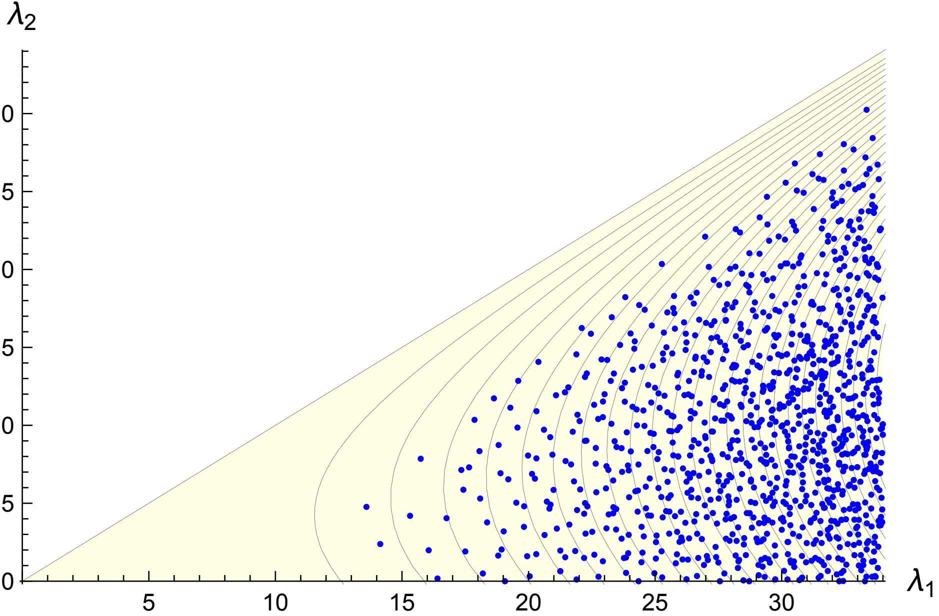
is specified by (κ, λ) with $\lambda \ge 0$.

Each of these pairs is an **L-point** which we can plot in the plane.

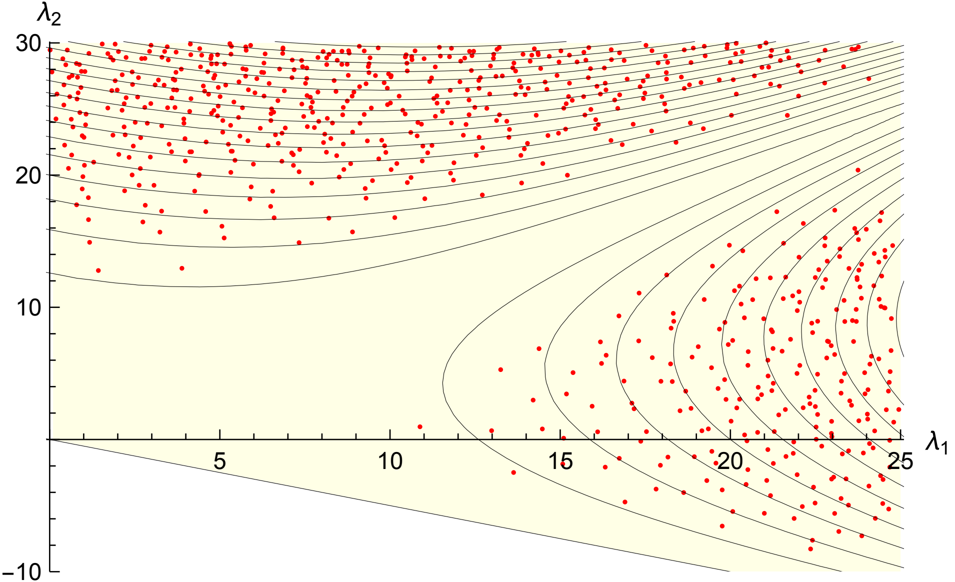
Most, but not all, of the smallest L-points of R0R0R0



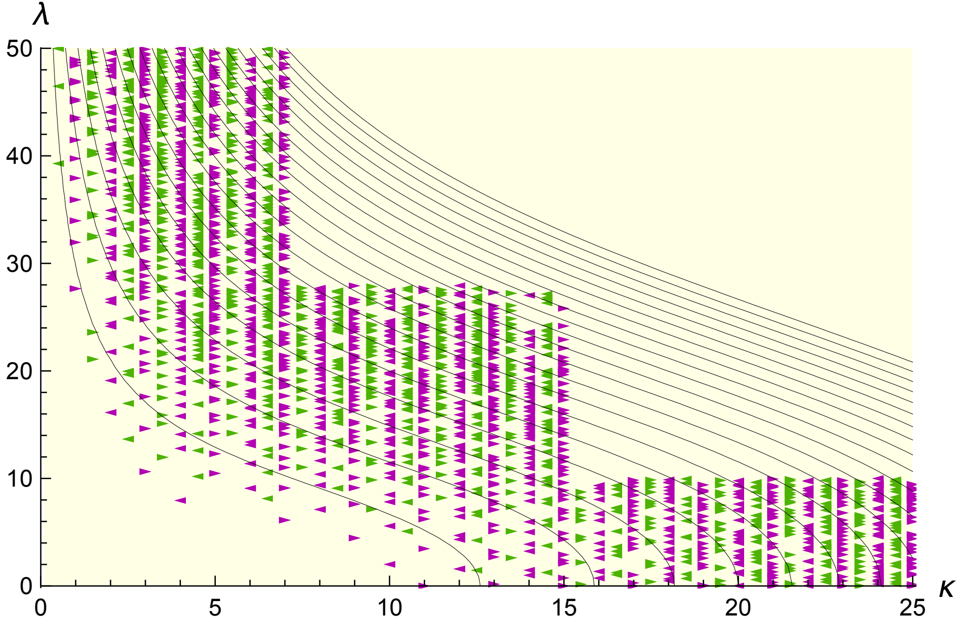
Most, but not all, of the smallest L-points of R0R0R0



Many, but not all, of the smallest L-points of R0R1R1



Many, but not all, of the smallest L-points of CkRδ



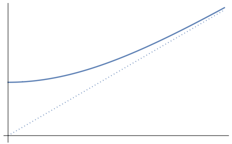
Plancherel measure, main term, in L-function terms

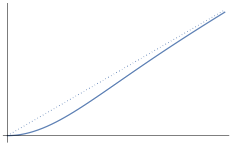
$$\mu_{R0R0R0} = \frac{P_3}{8}|\lambda_1 - \lambda_2|_+ \ |2\lambda_1 + \lambda_2|_+ \ |\lambda_1 + 2\lambda_2|_+ \ d\lambda_1 d\lambda_2$$
$$\mu_{R0R1R1} = \frac{P_3}{8}|\lambda_1 - \lambda_2|_- \ |2\lambda_1 + \lambda_2|_+ \ |\lambda_1 + 2\lambda_2|_- \ d\lambda_1 d\lambda_2$$
$$\mu_{C\kappa R\delta} = \frac{P_3}{8}\kappa(4\kappa^2 + 9\lambda^2) \ d\lambda$$

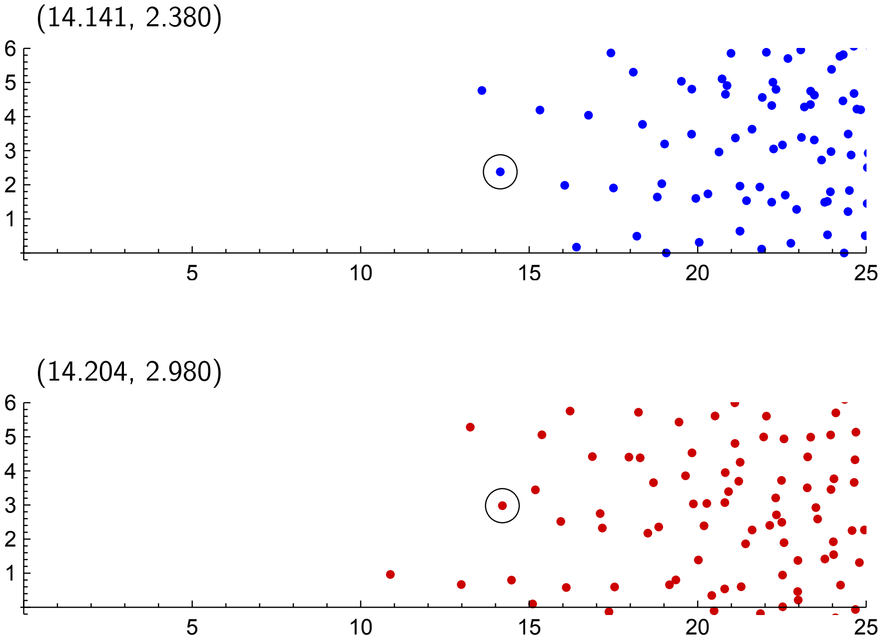
where $$P_d = \frac{d^{3/2}}{2^{(d+3)(d-1)/2}} \prod_{j=2}^{d} \frac{\zeta(j)}{\pi^j}$$

and

$$|t|_- = t\coth(\pi t/2) \qquad |t|_+ = t\tanh(\pi t/2)$$






(14.141, 2.380)
(14.204, 2.980)

One L-point in R0R0R0

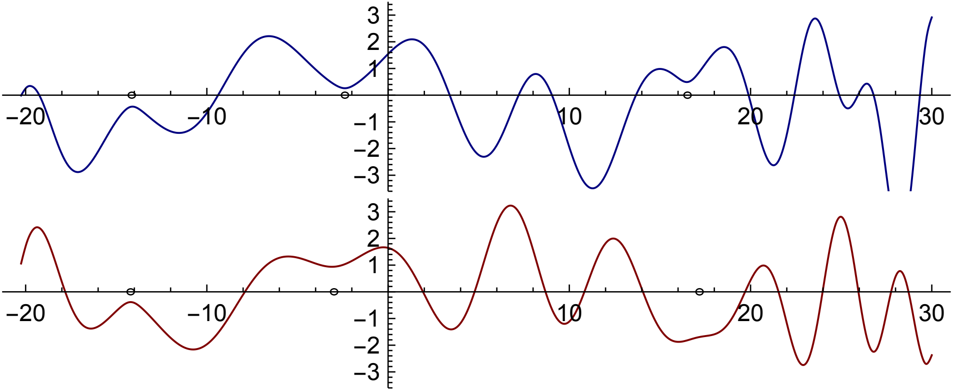
$$
\begin{aligned}
(\lambda_1, \lambda_2) &= (14.14163558812745167\ldots, 2.38038848881222505\ldots) \\
a_2 &= -0.1052409713451064\ldots \\
&\quad +i\,0.7507269443698732\ldots \\
a_3 &= 1.2359939151361498\ldots \\
&\quad -i\,0.0391121733258 76\ldots \\
a_5 &= 0.13457099978489\ldots \\
&\quad +i\,0.1546250917538\ldots \\
a_7 &= -0.9009535171391\ldots \\
&\quad -i\,0.47788143263\ldots \\
a_{11} &= 0.690303087\ldots \\
&\quad -i\,0.382679918\ldots
\end{aligned}
$$

Plancherel measure: L-functions and lower order terms

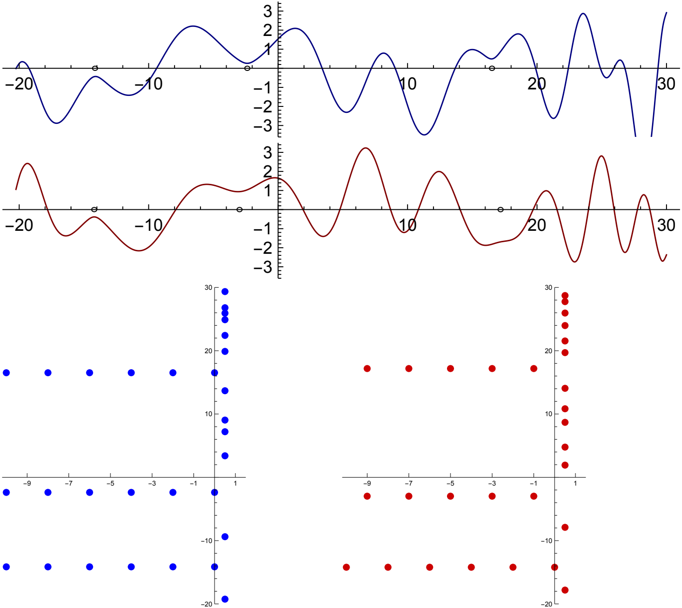
Two L-functions with similar functional equations:

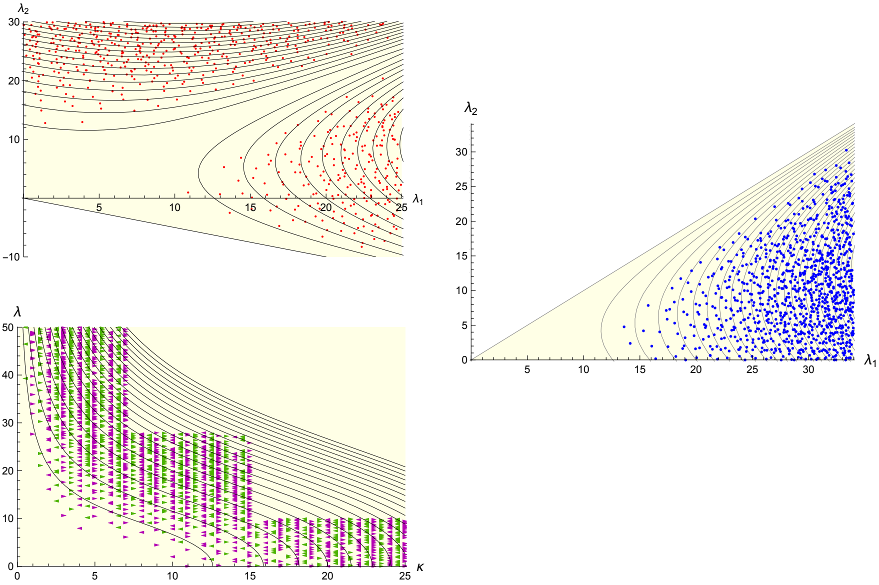
R0R0R0: $(\lambda_1, \lambda_2) \approx (14.141, 2.380)$

R0R1R1: $(\lambda_1, \lambda_2) \approx (14.204, 2.980)$

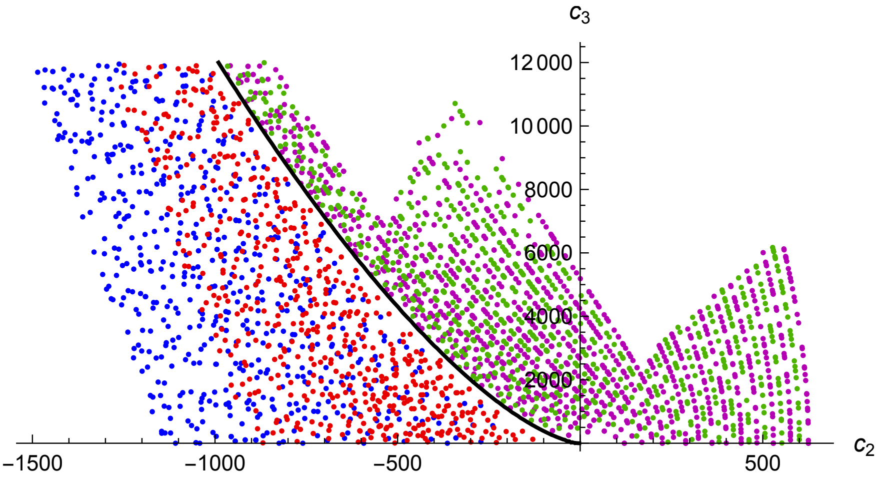


Plancherel measure: L-functions and lower order terms




λ_2
λ_1
λ
κ
λ_2
λ_1

The coefficient landscape: all of degree 3 in one place



The distribution of points is “asymptotically Euclidean.”

Coefficient landscape because the L-points are the roots of a polynomial, and the coefficients give the coordinates.

More 2-dimensional landscapes: degree 4, conductor 1, and self-dual

$$\Gamma_{\mathbb{R}}(s+i\lambda_1)\Gamma_{\mathbb{R}}(s-i\lambda_1)\Gamma_{\mathbb{R}}(s+i\lambda_2)\Gamma_{\mathbb{R}}(s-i\lambda_2)$$

$$\Gamma_{\mathbb{R}}(s+i\lambda_1)\Gamma_{\mathbb{R}}(s-i\lambda_1)\Gamma_{\mathbb{R}}(s+1+i\lambda_2)\Gamma_{\mathbb{R}}(s+1-i\lambda_2)$$

$$\Gamma_{\mathbb{R}}(s+1+i\lambda_1)\Gamma_{\mathbb{R}}(s+1-i\lambda_1)\Gamma_{\mathbb{R}}(s+1+i\lambda_2)\Gamma_{\mathbb{R}}(s+1-i\lambda_2)$$

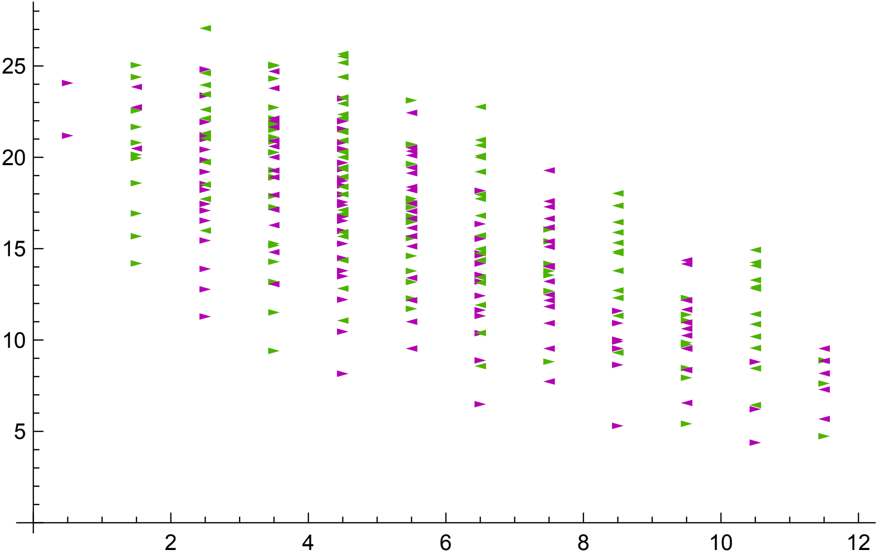
or

$$\Gamma_{\mathbb{R}}(s+\delta+i\lambda_1)\Gamma_{\mathbb{R}}(s+\delta-i\lambda_1)\Gamma_{\mathbb{C}}(s+\kappa)$$

or

$$\Gamma_{\mathbb{C}}(s+\kappa+i\lambda)\Gamma_{\mathbb{C}}(s+\kappa-i\lambda)$$

CκRδRδ: very preliminary data



CkCk: a new wrinkle

$\Delta \in S_{12}(1)$

$$\Lambda(s,\Delta) = \Gamma_{\mathbb{C}}(s+\tfrac{11}{2})L(s,\Delta) = \Lambda(1-s,\Delta)$$

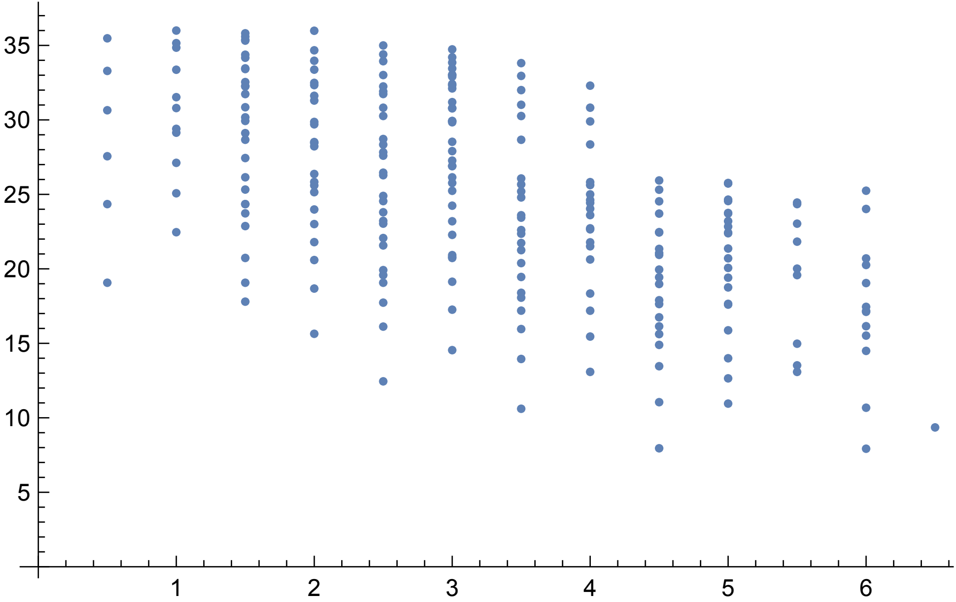
Therefore, for every $\lambda \in \mathbb{R}$

$$L(s+i\lambda,\Delta)L(s-i\lambda,\Delta)$$

satisfies a functional equation of type $C\frac{11}{2}C\frac{11}{2}$.

If you are searching directly for L-functions,
either you find a way to avoid non-primitive L-functions,
or you have to cull them later.

Some, and some false alarms, of the L-points of CkCk



The smallest $C\frac{1}{2}C\frac{1}{2}$ example we found

$$\lambda = 19.0673905\ldots$$
$$a_2 = -2.2662776959\ldots$$

Maass form on $\Gamma_0(1)$ with $R = 9.53369526135$

The Maass form on $SL(2,\mathbb{Z})$ with the smallest eigenvalue.

Maass form invariants

Level:	1
Weight:	0
Character:	1.1
Symmetry:	odd
Fricke sign:	$+1$
Spectral parameter:	9.53369526135

Maass form coefficients

$a_1 = +1.000000000$	$a_2 = -1.068333551$	$a_3 = -0.456197355$	$a_4 = +0.141336577$	$a_5 = -0.290672555$

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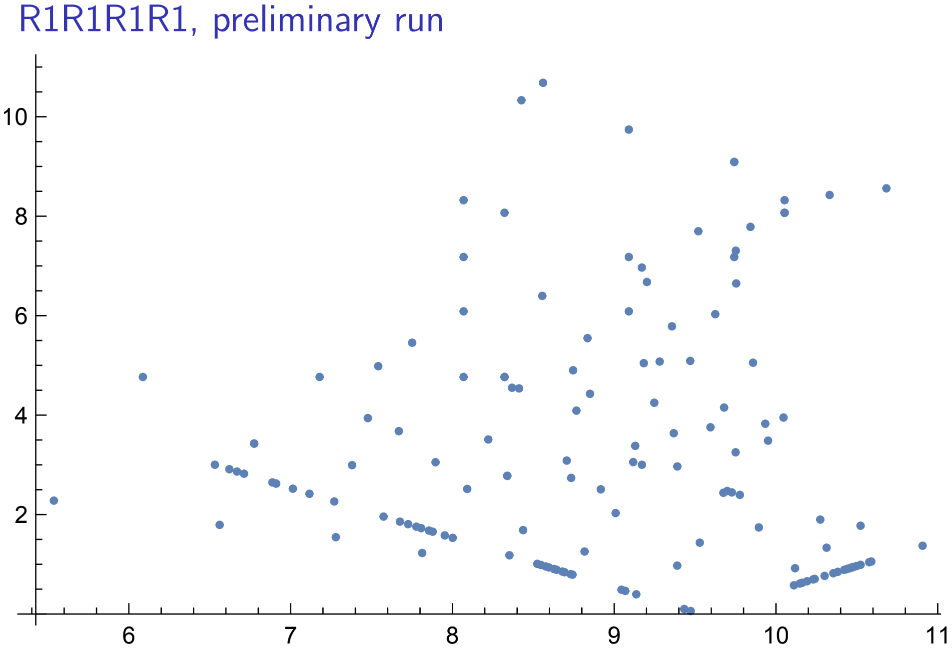
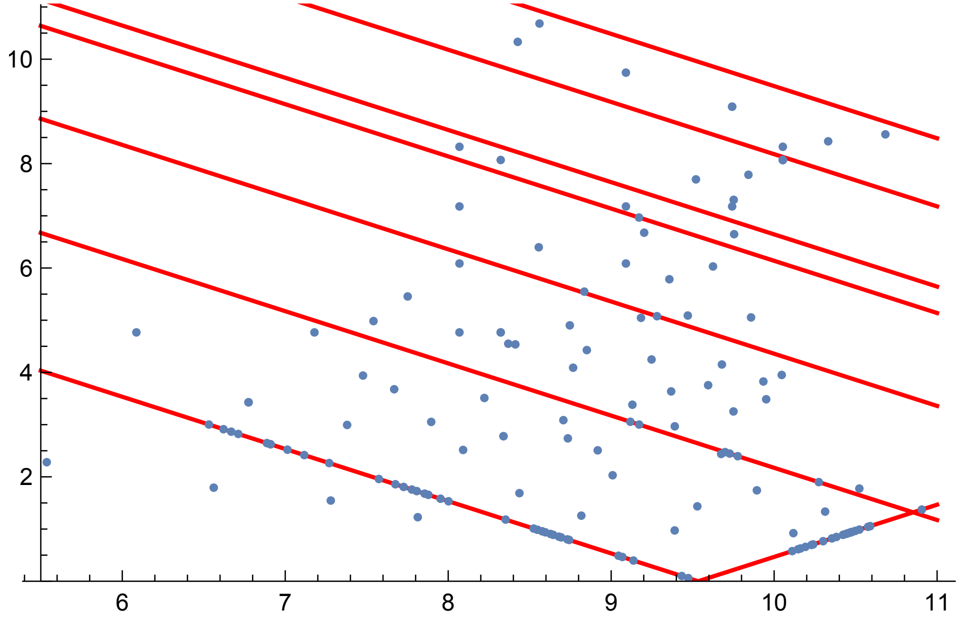
$\lambda = 19.0673905\ldots = 2 \times 9.5336952\ldots$

$a_2 = -2.2662776\ldots = (\sqrt{2} + 1/\sqrt{2}) \times (-1.06833\ldots)$

$L(s - \frac{1}{2}, f)L(s + \frac{1}{2}, f)$


R1R1R1R1, preliminary run
10
8
6
4
2
6
7
8
9
10
11

R1R1R1R1, preliminary run



How to find the L-points

More ingredients:

$$L(s) = \prod_p (1 - a_p p^{-s} + \cdots + (-1)^d \chi(p) p^{-ds})^{-1}$$
$$= f_p(p^{-s})^{-1}$$

where for $p \nmid N$, f_p is **self-reciprocal**:

$$f_p(x) = (-1)^d \overline{\chi}(p) x^d f_p(x^{-1}).$$

If $d = 2,\ 3$, then $\{a_p\}$ determine $L(s)$,
if $d = 4,\ 5$, then $\{a_p\}$ and $\{a_{p^2}\}$ determine $L(s)$, etc

Ramanujan conjecture: if $p \nmid N$ then the roots of f_p lie on $|z| = 1$.

The Selberg conjecture restricts the possible Γ-factors.

Ramanujan + Selberg = Tempered.

How to find the L-points

Main idea: secant method to find approximations to L-points.

Step 1: use an approximate L-point functional equation and make equations for the coefficients

assume $L(s)$ satisfies that functional equation

$\rightsquigarrow$ a linear equation in the Dirichlet coefficients

$\rightsquigarrow$ a nonlinear equation in the a_p (and a_{p^2} if necessary)

Step 2: find a solution to the system (or more than one)

Step 3: make auxiliary equations

Step 4. look at the residuals of the auxiliary equations

Step 5. combine the residuals from multiple inputs to make a better guess.

Some challenges

Need to work to high precision: routinely lose 40 digits.

Ramanujan bound: don't know how to make use of it.

Non-primitive L-functions: don't know how to avoid them.

Completeness: don't know when we have missed something.

Can't get started: don't find a solution to the nonlinear system

Parameter choice: how many coefficients, where to evaluate, how to evaluate,...