The landscape of L-functions

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joint work with
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L-functions: the glue that holds the (number theory) world together

Many sources: varieties, modular forms, number fields, ...
But we want to view them as their own independent objects.

Degree 1:
Dirichlet series with an Euler product:

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}
\]

Functional equation:

\[
\Lambda(s, \chi) = N^{s/2} \Gamma_{\mathbb{R}}(s + \delta_{\chi}) L(s, \chi)
= \varepsilon_{\chi} \overline{\Lambda}(1 - s, \chi)
= \varepsilon_{\chi} \Lambda(1 - s, \overline{\chi})
\]

\(\chi\) is a (primitive) character of conductor \(N\).
A constraint and some notation

\[ \Lambda(s, \chi) = N^{s/2} \Gamma_R(s + \delta \chi) L(s, \chi) = \varepsilon \chi \Lambda(1 - s, \chi) \]

\[ \delta \chi = 0 \text{ or } 1, \text{ with } \chi(-1) = (-1)^{\delta \chi}. \]

\[ \Gamma_R(s) = \pi^{-s/2} \Gamma(s/2) \]
\[ \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s) \]

\[ \Gamma_R(s) \Gamma_R(s + 1) = \Gamma_C(s) \]
Degree 2: two types

Case 1: \( f \in S_k(\Gamma_0(N), \chi) \), \( k \geq 2 \).

Dirichlet series with an Euler product:

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left( 1 - a(p)p^{-s} + \chi(p)p^{-2s} \right)^{-1}
\]

Functional equation:

\[
\Lambda(s, f) = N^{s/2}\Gamma_{\mathbb{C}}(s + \kappa)L(s, f) = \varepsilon_f \bar{\Lambda}(1 - s, f) = \varepsilon_f \bar{\Lambda}(1 - s, \bar{f})
\]

\( \kappa = \frac{k - 1}{2} \). The constraint: \( \chi(-1) = (-1)^{2\kappa+1} \).
Degree 2: two types

Case 2: \( f \) a Maass newform of weight 0 or 1, on \( \Gamma_0(N) \) with character \( \chi \) and spectral parameter \( \lambda \in \mathbb{R} \).

Dirichlet series with an Euler product:

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p}(1 - a(p)p^{-s} + \chi(p)p^{-2s})^{-1}
\]

Functional equation:

\[
\Lambda(s, f) = N^{s/2} \Gamma_{\mathbb{R}}(s + \delta_1 + i\lambda) \Gamma_{\mathbb{R}}(s + \delta_2 - i\lambda)L(s, f) = \varepsilon_f \Lambda(1 - s, f) = \varepsilon_f \Lambda(1 - s, \overline{f})
\]

The constraint: \( \chi(-1) = (-1)^{\delta_1 + \delta_2} \).
Degree \( d \)

\( N \): the **conductor**

\( \chi \): a character mod \( N \), the **central character**

Dirichlet series with an Euler product:

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + \cdots + (-1)^d \chi(p)p^{-ds})^{-1}
\]

Functional equation:

\[
\Lambda(s) = N^{s/2} \left( \Gamma - \text{factors} \right) L(s) = \varepsilon \overline{\Lambda}(1 - s)
\]

\( \Gamma - \text{factors: } \Gamma_{\mathbb{R}}(s + \delta + i\mu) \text{ and/or } \Gamma_{\mathbb{C}}(s + \kappa + i\lambda). \)
General degree $d$ $\Gamma$-factor

\[
\prod_{j=1}^{d_1} \Gamma_\mathbb{R}(s + \delta_j + i\mu_j) \prod_{k=1}^{d_2} \Gamma_\mathbb{C}(s + \kappa_k + i\lambda_k)
\]

where $d_1 + 2d_2 = d$.

$\delta_j \in \{0, 1\}$

$\kappa_k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$

$\mu_j, \lambda_k \in \mathbb{R}$

The central character constraint:

\[
\chi(-1) = (-1)^{\sum \delta_j + \sum (2\kappa_j + 1)}
\]
All possible degree 3 conductor 1 functional equations

\( N = 1 \) so \( \chi \) is the trivial character, so \( \chi(-1) = 1 \):

\((-1)^{\sum \delta_j + \sum (2\kappa_j+1)} = 1\)

Case 1: 3 of \( \Gamma_R \)

\( \Gamma_R(s + i\mu_1)\Gamma_R(s + i\mu_2)\Gamma_R(s + i\mu_3) \)

\( \Gamma_R(s + i\mu_1)\Gamma_R(s + 1 + i\mu_2)\Gamma_R(s + 1 + i\mu_3) \)

Case 2: 1 of \( \Gamma_R \) and 1 of \( \Gamma_C \)

\( \Gamma_R(s + i\mu)\Gamma_C(s + \kappa + i\lambda) \quad \text{with} \quad \kappa \in \{\frac{1}{2}, \frac{3}{2}, \ldots\} \)

\( \Gamma_R(s + 1 + i\mu)\Gamma_C(s + \kappa + i\lambda) \quad \text{with} \quad \kappa \in \{1, 2, 3, \ldots\} \)
A normalization

If $L(s)$ is an $L$-function, then so is $L(s + iy)$ for any $y \in \mathbb{R}$. So in

$$\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s + i\lambda_2)\Gamma_{\mathbb{R}}(s + i\lambda_3)$$

we may assume $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

By rearranging and conjugating, that functional equation is specified by a pair $(\lambda_1, \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$.

Similarly,

$$\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_2)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_3)$$

is specified by $(\lambda_1, \lambda_2)$ with $\lambda_1 \geq 0$ and $\lambda_2 \geq -\lambda_1/2$.

Also,

$$\Gamma_{\mathbb{R}}(s + \delta + 2i\lambda)\Gamma_{\mathbb{C}}(s + \kappa - i\lambda)$$

is specified by $(\kappa, \lambda)$ with $\lambda \geq 0$.

Each of these pairs is an L-point which we can plot in the plane.
Most, but not all, of the smallest L-points of $R_0 R_0 R_0$
Most, but not all, of the smallest L-points of $R_0 R_0 R_0$
Many, but not all, of the smallest L-points of R0R1R1
Many, but not all, of the smallest L-points of $C_kR\delta$
Plancherel measure, main term, in L-function terms

\[ \mu_{R0R0R0} = \frac{P_3}{8} |\lambda_1 - \lambda_2|_+ |2\lambda_1 + \lambda_2|_+ |\lambda_1 + 2\lambda_2|_+ \, d\lambda_1 d\lambda_2 \]

\[ \mu_{R0R1R1} = \frac{P_3}{8} |\lambda_1 - \lambda_2|_- |2\lambda_1 + \lambda_2|_+ |\lambda_1 + 2\lambda_2|_- \, d\lambda_1 d\lambda_2 \]

\[ \mu_{C_{\kappa R\delta}} = \frac{P_3}{8} \kappa(4\kappa^2 + 9\lambda^2) \, d\lambda \]

where

\[ P_d = \frac{d^{3/2}}{2(d+3)(d-1)/2} \prod_{j=2}^{d} \frac{\zeta(j)}{\pi j} \]

and

\[ |t|_- = t \coth(\pi t/2) \quad |t|_+ = t \tanh(\pi t/2) \]
One L-point in $R_0 R_0 R_0$

$$(\lambda_1, \lambda_2) = (14.14163558812745167 \ldots, 2.38038848881222505 \ldots)$$

$$a_2 = -0.1052409713451064 \ldots$$
$$+ i 0.7507269443698732 \ldots$$

$$a_3 = 1.2359939151361498 \ldots$$
$$- i 0.039112173325876 \ldots$$
$$- i 0.039112173325876 \ldots$$

$$a_5 = 0.13457099978489 \ldots$$
$$+ i 0.1546250917538 \ldots$$

$$a_7 = -0.9009535171391 \ldots$$
$$- i 0.47788143263 \ldots$$

$$a_{11} = 0.690303087 \ldots$$
$$- i 0.382679918 \ldots$$
Plancherel measure: L-functions and lower order terms

Two L-functions with similar functional equations:

\[ \text{R0R0R0: } (\lambda_1, \lambda_2) \approx (14.141, 2.380) \]

\[ \text{R0R1R1: } (\lambda_1, \lambda_2) \approx (14.204, 2.980) \]
Plancherel measure: L-functions and lower order terms
The coefficient landscape: all of degree 3 in one place

The distribution of points is “asymptotically Euclidean.”

**Coefficient landscape** because the L-points are the roots of a polynomial, and the coefficients give the coordinates.
More 2-dimensional landscapes: degree 4, conductor 1, and self-dual

\[
\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s - i\lambda_1)\Gamma_{\mathbb{R}}(s + i\lambda_2)\Gamma_{\mathbb{R}}(s - i\lambda_2) \\
\Gamma_{\mathbb{R}}(s + i\lambda_1)\Gamma_{\mathbb{R}}(s - i\lambda_1)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_2)\Gamma_{\mathbb{R}}(s + 1 - i\lambda_2) \\
\Gamma_{\mathbb{R}}(s + 1 + i\lambda_1)\Gamma_{\mathbb{R}}(s + 1 - i\lambda_1)\Gamma_{\mathbb{R}}(s + 1 + i\lambda_2)\Gamma_{\mathbb{R}}(s + 1 - i\lambda_2)
\]

or

\[
\Gamma_{\mathbb{R}}(s + \delta + i\lambda_1)\Gamma_{\mathbb{R}}(s + \delta - i\lambda_1)\Gamma_{\mathbb{C}}(s + \kappa)
\]

or

\[
\Gamma_{\mathbb{C}}(s + \kappa + i\lambda)\Gamma_{\mathbb{C}}(s + \kappa - i\lambda)
\]
$C\kappa R\delta R\delta$: very preliminary data
CkCk: a new wrinkle

\[ \Delta \in S_{12}(1) \]

\[ \Lambda(s, \Delta) = \Gamma_C(s + \frac{11}{2})L(s, \Delta) = \Lambda(1 - s, \Delta) \]

Therefore, for every \( \lambda \in \mathbb{R} \)

\[ L(s + i\lambda, \Delta)L(s - i\lambda, \Delta) \]

satisfies a functional equation of type \( C \frac{11}{2} C \frac{11}{2} \).

If you are searching directly for L-functions, either you find a way to avoid non-primitive L-functions, or you have to cull them later.
Some, and some false alarms, of the L-points of C\textsuperscript{k}C\textsuperscript{k}
The smallest $C\frac{1}{2}C\frac{1}{2}$ example we found

\[ \lambda = 19.0673905 \ldots \]
\[ a_2 = -2.2662776959 \ldots \]

**Maass form on $\Gamma_0(1)$ with $R = 9.53369526135$**

The Maass form on $SL(2, \mathbb{Z})$ with the smallest eigenvalue.

**Maass form invariants**

- **Level:** 1
- **Weight:** 0
- **Character:** 1.1
- **Symmetry:** odd
- **Fricke sign:** +1
- **Spectral parameter:** 9.53369526135

**Maass form coefficients**

\[ a_1 = +1.00000000 \quad a_2 = -1.068333551 \quad a_3 = -0.456197355 \quad a_4 = +0.141336577 \quad a_5 = -0.290672555 \]
The smallest $C^{1/2}C^{1/2}$ example we found

$$\lambda = 19.0673905 \ldots$$
$$a_2 = -2.2662776959 \ldots$$

Maass form on $\Gamma_0(1)$ with $R = 9.53369526135$

The Maass form on $SL(2, \mathbb{Z})$ with the smallest eigenvalue.

Maass form invariants

- Level: 1
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- Spectral parameter: 9.53369526135

Maass form coefficients

\[
\begin{align*}
\lambda &= 19.0673905 \ldots = 2 \times 9.5336952 \ldots \\
a_2 &= -2.2662776 \ldots = (\sqrt{2} + 1/\sqrt{2}) \times (-1.06833 \ldots) \\
L(s - \frac{1}{2}, f)L(s + \frac{1}{2}, f)
\end{align*}
\]
R1R1R1R1, preliminary run
R1R1R1R1, preliminary run
How to find the $L$-points

More ingredients:

$$L(s) = \prod_p (1 - a_p p^{-s} + \cdots + (-1)^d \chi(p) p^{-ds})^{-1}$$

$$= f_p(p^{-s})^{-1}$$

where for $p \nmid N$, $f_p$ is self-reciprocal:

$$f_p(x) = (-1)^d \chi(p) x^d f_p(x^{-1}).$$

If $d = 2, 3$, then $\{a_p\}$ determine $L(s)$,
if $d = 4, 5$, then $\{a_p\}$ and $\{a_p^2\}$ determine $L(s)$, etc.

Ramanujan conjecture: if $p \nmid N$ then the roots of $f_p$ lie on $|z| = 1$.

The Selberg conjecture restricts the possible $\Gamma$-factors.
Ramanujan + Selberg = Tempered.
How to find the L-points

Main idea: secant method to find approximations to L-points.

Step 1: use an approximate L-point functional equation and make equations for the coefficients

  assume $L(s)$ satisfies that functional equation
  $\implies$ a linear equation in the Dirichlet coefficients
  $\implies$ a nonlinear equation in the $a_p$ (and $a_{p^2}$ if necessary)

Step 2: find a solution to the system (or more than one)

Step 3: make auxiliary equations

Step 4. look at the residuals of the auxiliary equations

Step 5. combine the residuals from multiple inputs to make a better guess.
Some challenges

Need to work to high precision: routinely lose 40 digits.

Ramanujan bound: don’t know how to make use of it.

Non-primitive L-functions: don’t know how to avoid them.

Completeness: don’t know when we have missed something.

Can’t get started: don’t find a solution to the nonlinear system

Parameter choice: how many coefficients, where to evaluate, how to evaluate,...