On Permutation-Invariant Parking Sequences

Summer@ICERM 2022: Computational Combinatorics

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August 3, 2022

Parking Functions

Definition (parking function)

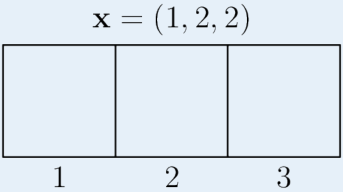
- Consider a parking lot with n spots along a one-way street.
- Consider n cars of unit length, and let $\mathbf{x} \in [n]^n$ be the sequence of the preferred spots of each car.
- We say that $\mathbf{x}$ is a *parking function* if with this preference sequence, all cars are able to park.

Parking Experiment

Example

Consider cars of unit length and the preference sequence $\mathbf{x} = (1, 2, 2)$. The parking experiment goes as follows:

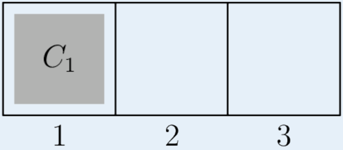
Step 1



Example

Step 2

$$\mathbf{x} = (1, 2, 2)$$

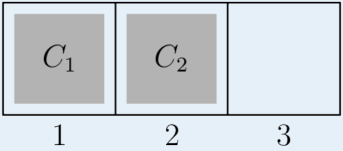


Parking Experiment

Example

Step 3

$$\mathbf{x} = (1, 2, 2)$$

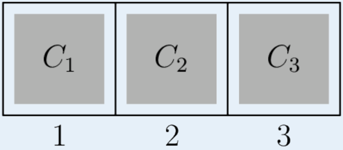


Parking Experiment

Example

Step 4

$$\mathbf{x} = (1, 2, 2)$$



What if the cars can have more than unit length?

Definition (parking sequence)

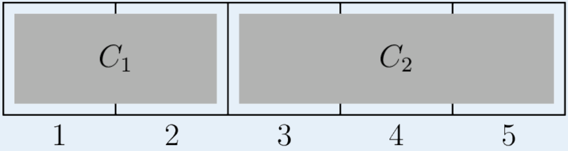
- Consider a sequence (or vector) $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{N}^n$ of arbitrary but fixed car sizes.
- Let $m = \sum_{i=1}^{n} y_i$ be the total number of parking spots along a one-way street.
- Let $\mathbf{x} \in [m]^n$ be the sequence (or vector) of the preferred spots of each car.
- We say $\mathbf{x}$ is a *parking sequence* if with this preference sequence, all cars are able to park.

What if the cars can have more than unit length?

Example

Let $\mathbf{y} = (2, 3)$. Then $\mathbf{x} = (1, 2)$ is a parking sequence.

$$\mathbf{x} = (1, 2)$$



Note that any rearrangement of the entries of a parking function also results in a parking function. Given $\mathbf{y}$, is the same true for parking sequences?

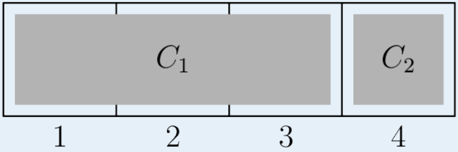
Definition (permutation-invariant parking sequence)

- Consider a length vector $\mathbf{y}$ and the total number of parking spots m as in the previous definition.
- Let $\mathbf{x} \in [m]^n$ be a parking sequence.
- We say $\mathbf{x}$ is *(permutation)-invariant* if any rearrangement of its entries also results in a parking sequence.

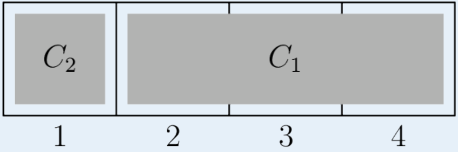
Example

Let $\mathbf{y} = (3, 1)$. Its invariant PSs are $(1, 1)$, $(1, 2)$ and $(2, 1)$.

$\mathbf{x} = (1, 1), (1, 2)$



$\mathbf{x} = (2, 1)$



Properties of Parking Sequences

Proposition

Let $\mathbf{y} \in \mathbb{N}^n$ *be an arbitrary but fixed sequence of car sizes.*

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *be a preference sequence for* $\mathbf{y}$*, where* $x_1 \leq x_2 \leq \cdots \leq x_n$.

Then $\mathbf{x}$ *is a parking sequence if and only if for all* $i \in [n]$*, we have*

$$x_i \leq \sum_{j=1}^{i-1} y_j + 1.$$

Properties of Parking Sequences

Theorem (Adeniran and Yan, 2021)

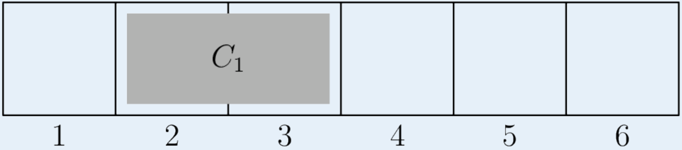
Let $\mathbf{y} = (c, c, \ldots, c) = (c^n) \in \mathbb{N}^n$. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *be a preference sequence for* $\mathbf{y}$. *Then* $\mathbf{x}$ *is an invariant parking sequence if and only if*

1. $x_i \equiv 1 \pmod{c}$ *for all* $i \in [n]$, *and*
2. $|\{i \in [n] : x_i \leq c \cdot j\}| \geq j$ *for all* $j \in [n]$.

Example

Let $\mathbf{y} = (2, 2, 2)$. Then $\mathbf{x} = (1, 2, 3) \notin \mathrm{PS}_3^{\mathrm{inv}}(\mathbf{y})$, as $2 \not\equiv 1 \bmod 2$.

$$\mathbf{x}' = (2, 1, 3)$$



Properties of Parking Sequences

Corollary

Let $\mathbf{y} = (c^n)$ for some $c \in \mathbb{N}$. Then its number of nondecreasing invariant parking sequences is C_n, where C_n is the nth Catalan number.

Properties of Parking Sequences

Theorem (Adeniran and Yan, 2021)

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ *be strictly increasing. Then its only invariant parking sequence for* $\mathbf{y}$ *is the all ones (trivial) vector.*

This theorem was a source of motivation for some of the results that follow.

Note that strictly increasing $\mathbf{y}$ are **not the only ones** that satisfy the property above!

Thus, we might ask: given $\mathbf{y}$, when do we know that the trivial vector is the only invariant PS?

Definition

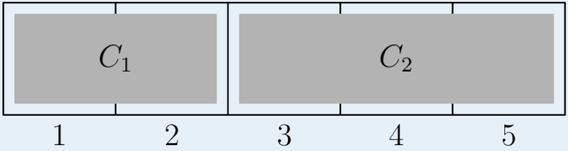
Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. We say $\mathbf{y}$ is *minimally invariant (m.i.)* if the trivial vector is its only invariant parking sequence.

Minimally Invariant Car Lengths

Example

Let $\mathbf{y} = (2, 3)$. Then $\mathbf{y}$ is m.i., as its only invariant PS is $(1, 1)$.

$\mathbf{x} = (1, 1)$



Extending m.i. Length Vectors

Lemma

Let $y_{n+1} \in \mathbb{N}$*,*

$$\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n, \quad \textit{and} \quad \mathbf{z} = (y_1, y_2, \ldots, y_n, y_{n+1}).$$

If $\mathbf{y}$ *is minimally invariant, then every nondecreasing* $\mathbf{x} \in \mathrm{PS}^{\mathrm{inv}}_{n+1}(\mathbf{z})$ *is of the form* $\mathbf{x} = (1^n, w)$ *for some* $w \in \mathbb{N}$*.*

Example

Let $\mathbf{y} = (3, 5)$ and $\mathbf{z} = (3, 5, 2)$. It is known that $\mathbf{y}$ is m.i., and the nondecreasing $\mathbf{x} \in \mathrm{PS}^{\mathrm{inv}}_3(\mathbf{z})$ are $(1, 1, 1)$, $(1, 1, 3)$, and $(1, 1, 6)$.

Characterizing m.i. Length Vectors

Theorem (main result)

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$. *Then* $\mathbf{y}$ *is minimally invariant if and only if there does not exist* $w \in \mathbb{N}_{>1}$ *such that* $(1^{n-1}, w) \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$.

Example

Let $\mathbf{y} = (5, 2, 2)$. Then its nondecreasing invariant PSs are $(1, 1, 1)$, $(1, 1, 3)$, and $(1, 3, 3)$.

If $\mathbf{y}$ is not m.i., then it must have an invariant PS with **exactly** $n - 1$ 1s!

The previous result is a concise characterization of minimal invariance in the sense that there are only n distinct permutations of $(1^{n-1}, w)$.
That is, we need only perform mn parking experiments (this is weakly polynomial).

Example

Let $\mathbf{y} = (2, 4, 1)$. To show that $\mathbf{y}$ is m.i., we only need to check that there is no $w > 1$ such that $(1, 1, w)$, $(1, w, 1)$, and $(w, 1, 1)$ are all PSs.

Application 1: Strictly Increasing $\mathbf{y}$

Theorem

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$ *be strictly increasing. Then* $\mathbf{y}$ *is minimally invariant.*

Inductive proof takes advantage of the lemma, theorem, and the recursive structure of m.i. length vectors.

Corollary

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{N}^n$*. If* $\mathbf{y}$ *is minimally invariant, then* $\mathbf{y}_{|_i} = (y_1, y_2, \ldots, y_i)$ *is minimally invariant for all* $i \in [n]$*.*

Proof.

Only need to prove for $i = n - 1$, as result can be applied inductively. Proceed by contrapositive.

Proof (continued).

Use the m.i. characterization theorem to obtain $w > 1$ such that $(1^{n-2}, w) \in \mathrm{PS}^{\mathrm{inv}}_{n-1}(\mathbf{y}_{|_{n-1}})$.

We claim that $(1^{n-1}, w) \in \mathrm{PS}^{\mathrm{inv}}_{n}(\mathbf{y})$.

Application 2: Substrings

Proof (continued).

Now consider the permutations:

- w is in the last entry. Finish using nondecreasing lemma.
- w is in the first $n-1$ entries. Finish by noting $(1^{n-2}, w) \in \mathrm{PS}^{\mathrm{inv}}_{n-1}(\mathbf{y}_{|_{n-1}})$.

□

Corollary

Let $\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2$. *Then* $\mathbf{y}$ *is minimally invariant if and only if* $y_1 < y_2$.

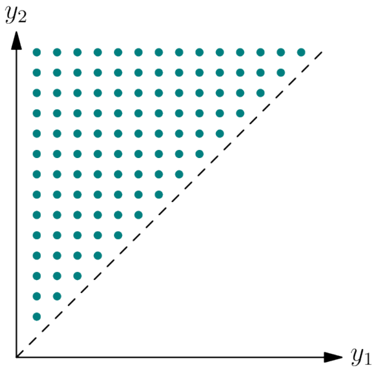


Figure: Lattice points (y_1, y_2) for m.i. $\mathbf{y}$

Application 3: Two Cars

Proof.

Idea is to only check preferences of the form $(1, w)$. □

Corollary

Let $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{N}^3$. Then $\mathbf{y}$ is minimally invariant if and only if $y_1 < y_2$, $y_1 < y_3$, and $y_1 + y_3 \neq y_2$.

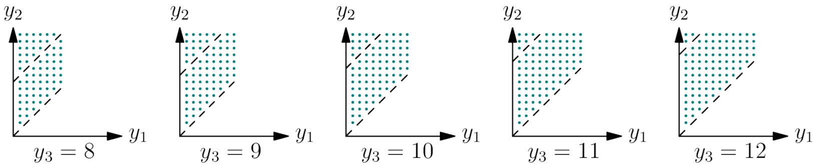


Figure: Level sets for y_3 and resulting lattice points (y_1, y_2) for m.i. $\mathbf{y}$

This Boolean condition is more direct than the one specified in the characterization.

Proof.

By the m.i. characterization theorem,

$$\mathbf{y} \text{ m.i.} \iff \nexists w \in \mathbb{N}_{>1} \text{ s.t. } (1,1,w) \in \mathrm{PS}_3^{\mathrm{inv}}(\mathbf{y}).$$

In other words, at least one of $(1,1,w)$, $(1,w,1)$, and $(w,1,1)$ is not a parking sequence.

Application 4: Three Cars

Proof (continued).

For such a w to exist, all of the following conditions must hold:

1. $(1, 1, w) \in \mathrm{PS}_3(\mathbf{y})$ if and only if $w \leq 1 + y_1 + y_2$.
2. $(1, w, 1) \in \mathrm{PS}_3(\mathbf{y})$ if and only if
    1. $w \leq 1 + y_1$, or
    2. $w = 1 + y_1 + y_3$.
3. $(1, 1, w) \in \mathrm{PS}_3(\mathbf{y})$ if and only if
    1. $w = 1 + y_2$, or
    2. $w = 1 + y_2 + y_3$, or
    3. $y_2 \geq y_3$ and $w = 1 + y_3$.

Proof (continued).

If $y_1 < y_2$, $y_1 < y_3$, and $y_1 + y_3 \neq y_2$, then all of the conditions cannot hold.

If $y_1 \geq y_2$ or $y_1 \geq y_3$ or $y_1 + y_3 = y_2$, then all of the conditions hold. □

Characterizing Invariant PSs: Two Cars

Theorem

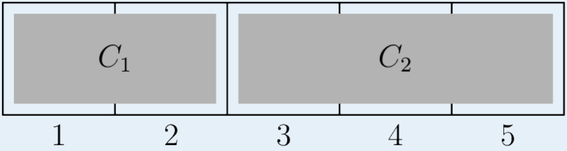
Let $\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2$*.*

- $y_1 < y_2 \implies \mathrm{PS}_2^{\mathrm{inv}}(\mathbf{y}) = \{(1,1)\}$.
- $y_1 \geq y_2 \implies \mathrm{PS}_2^{\mathrm{inv}}(\mathbf{y}) = \{(1,1), (1, y_2+1), (y_2+1, 1)\}$.

Example

Let $\mathbf{y} = (2,3)$. Then its only invariant PS is $(1,1)$.

$$\mathbf{x} = (1,1)$$



Characterizing Invariant PSs: Three Cars

Let a, b, and c be positive integers such that $a < b < c$.

y	Invariant PSs (up to permutation)
(a, a, a)	$(1,1,1), (1,1,1+a), (1,1,1+2a), (1,1+a,1+a), (1,1+a,1+2a)$
(a, a, b)	$(1,1,1), (1,1,1+a)$
$(a, b, a),\ b = 2a$	$(1,1,1), (1,1,1+a), (1,1,1+2a)$
$(a, b, a),\ b \neq 2a$	$(1,1,1), (1,1,1+a)$
$(b, a, a),\ 2a \leq b$	$(1,1,1), (1,1,1+a), (1,1,1+2a), (1,1+a,1+a), (1,1+a,1+2a)$
$(b, a, a),\ 2a > b$	$(1,1,1), (1,1,1+a), (1,1+a,1+a)$
(a, b, b)	$(1,1,1)$
(b, a, b)	$(1,1,1), (1,1,1+a)$
(b, b, a)	$(1,1,1), (1,1,1+a), (1,1,1+b), (1,1,1+a+b)$
(a, b, c)	$(1,1,1)$
$(a, c, b),\ a+b = c$	$(1,1,1), (1,1,1+a+b)$
$(a, c, b),\ a+b \neq c$	$(1,1,1)$
(b, a, c)	$(1,1,1), (1,1,1+a)$
$(b, c, a),\ a+b = c$	$(1,1,1), (1,1,1+a), (1,1,1+a+b)$
$(b, c, a),\ a+b \neq c$	$(1,1,1), (1,1,1+a)$
$(c, a, b),\ a+b \leq c$	$(1,1,1), (1,1,1+a), (1,1,1+a+b)$
$(c, a, b),\ a+b > c$	$(1,1,1), (1,1,1+a)$
$(c, b, a),\ a+b \leq c$	$(1,1,1), (1,1,1+a), (1,1,1+b), (1,1,1+a+b)$
$(c, b, a),\ a+b > c$	$(1,1,1), (1,1,1+a), (1,1,1+b)$

Characterizing Invariant PSs: Three Cars

Example

Let $\mathbf{y} = (3, 2, 4)$. This is of the form $\mathbf{y} = (b, a, c)$, where $b = 3, a = 2, c = 4$.

Thus, its invariant PSs are:

- $(1, 1, 1)$
- $(1, 1, 1 + a) = (1, 1, 3)$

This is true!

A Closure Result

Theorem

Given $\mathbf{y} \in \mathbb{N}^n$, *suppose* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$. *For all* $i \in [n]$, *define* $r_i : \mathbb{N}^n \to \mathbb{N}^n$ *by* $r_i(\mathbf{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, x_n)$. *Then for all* $i \in [n]$, $r_i(\mathbf{x}) \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$.

Put simply:

- Replacing any entry of an invariant PS with a 1 gives another invariant PS.
- $\mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$ is closed under replacement of an entry by 1.

Example

Let $\mathbf{y} = (3, 2, 4)$.
We know $\mathbf{x} = (1, 1, 3) \in \mathrm{PS}_3^{\mathrm{inv}}(\mathbf{y})$.
Thus, $r_3(\mathbf{x}) = (1, 1, 1) \in \mathrm{PS}_3^{\mathrm{inv}}(\mathbf{y})$.

Note: Inductively, replacing any number of entries with a 1 gives another invariant PS.

A Closure Result: Corollary

Adding 1s to an invariant PS gives a new invariant PS.

Theorem

Let $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{N}^k$*, and let* $\mathbf{y}^+ = (y_1, \ldots, y_k, \ldots, y_n) \in \mathbb{N}^n$ *be an extension of* $\mathbf{y}$*. Suppose* $\mathbf{x} = (x_1, \ldots, x_k) \in \mathrm{PS}_k^{\mathrm{inv}}(\mathbf{y})$*. Then* $\mathbf{x}^+ = (x_1, \ldots, x_k, 1^{n-k}) \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y}^+)$.

Example

Let $\mathbf{y} = (3, 2, 4)$.
We know $\mathbf{x} = (1, 1, 3) \in \mathrm{PS}_3^{\mathrm{inv}}(\mathbf{y})$.
Let $\mathbf{y}^+ = (3, 2, 4, 3)$.
Then $\mathbf{x}^+ = (1, 1, 3, 1) \in \mathrm{PS}_4^{\mathrm{inv}}(\mathbf{y}^+)$.

We have a Boolean characterization of minimal invariance for **y** of length 2 and 3.
This is our guess for **y** of length 4:

Conjecture

Let $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{N}^4$. *Then* **y** *is minimally invariant if and only if:*

1. $y_1 < y_2, y_3, y_4$, *and*
2. $y_2 \neq y_1 + y_3, y_1 + y_3 + y_4$, *and*
3. $y_1 + y_3 \geq y_2$ *or* $y_3 \neq y_1 + y_4$, *and*
4. $y_1 + y_3 \leq y_2$ *or* ($y_2 \neq y_1 + y_4$ *and* ($y_2 < y_3$ *or* $y_3 \neq y_1 + y_4$)).

It is challenging to produce a concise characterization of $\mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$. Maybe this problem is hard...

Definition (NP)

NP is the set of all "yes-no" problems for which it is easy to verify "yes" answers.

Example (Subset Sum Problem)

Given a finite set S of integers, does any subset of S have sum k?
Input: $S = \{3, 4, -2\}$, $k = 1$
Output: yes; $\{3, -2\} \subseteq S$

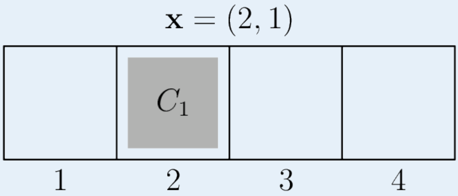
Open Questions: NP-Completeness

Definition (co-NP)

co-NP is the set of all "yes-no" problems for which it is easy to verify "no" answers.

Example (Checking for Invariance)

Given $\mathbf{y} = (1, 3)$, is $\mathbf{x} = (1, 2)$ invariant?
No: parking fails for $(2, 1)$.



Open Questions: NP-Completeness

Definition (NP-complete)

A problem is *NP-complete* if it is "at least as hard" as all problems in NP.

No efficient algorithms are known that solve NP-complete problems.

Question: Is $\mathbf{x} \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$?

1. This problem is co-NP.
   - Can easily verify a proof that $\mathbf{x}$ is not invariant.
2. Is it co-NP complete?
   - If so, there is (probably) no concise characterization of $\mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$.

Open Questions: Weaker Forms of Invariance

Note that $\mathbf{x} \in \mathrm{PS}_n^{\mathrm{inv}}(\mathbf{y})$ means that $\mathbf{x}$ is invariant under $\mathfrak{S}_n$.
But we can also consider a subgroup $T \trianglelefteq \mathfrak{S}_n$...

Example

Let $\mathbf{y} = (2, 2, 2)$. Then $\mathbf{x} = (1, 3, 2)$ is not invariant, but it is invariant under swaps of the first two entries.

Ideas: For any subgroup $T \trianglelefteq \mathfrak{S}_n$,

- Characterize invariance under T.
- How many $\mathbf{x}$ are invariant under T?

References

[1] Ayomikun Adeniran and Catherine H. Yan. On increasing and invariant parking sequences. *Australasian Journal of Combinatorics*, 79(1):167–182, 2021.

[2] Richard Ehrenborg and Alex Happ. Parking cars of different sizes. *The American Mathematical Monthly*, 123(10):1045–1048, 2016.

[3] Alan G. Konheim and Benjamin Weiss. An occupancy discipline and applications. *Society for Industrial and Applied Mathematics Journal on Applied Mathematics*, 14(6):1266–1274, 1966.

Acknowledgements

- This project was supported by the Institute for Computational and Experimental Research in Mathematics (ICERM). Special thanks to our advisor Pamela Harris and our TA Juan Carlos Martínez Mori for their mentoring throughout this research.
- This project was funded by the NSF.

Thank You!

Questions?