The G-Shi Arrangement:
Games on Paths, Trees, and More

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August 3rd, 2022

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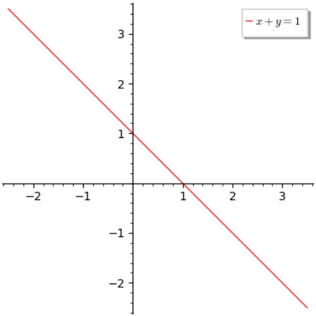
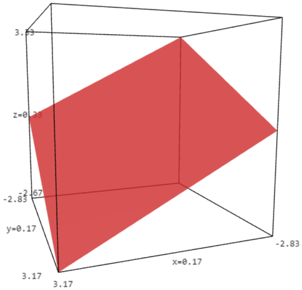
Background

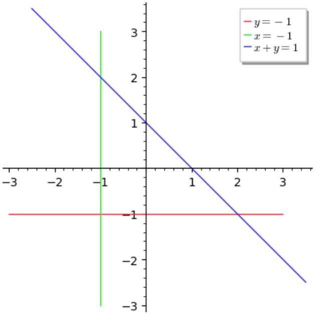
Hyperplanes

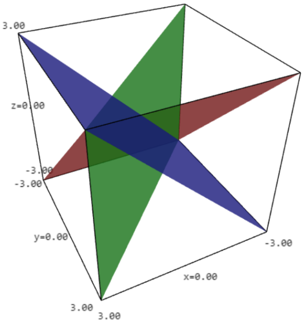




A *hyperplane* in $\mathbb{R}^n$ is an affine subspace of dimension $n-1$.

Hyperplane Arrangements

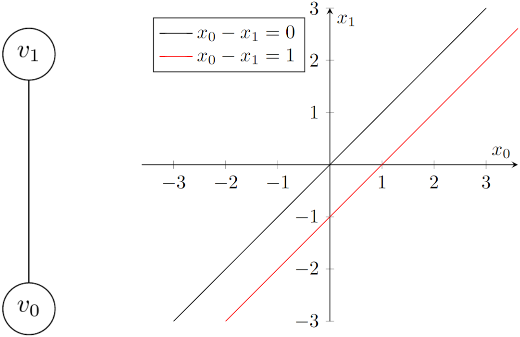




A *hyperplane arrangement* is a finite collection of hyperplanes. The hyperplane arrangement seperates the vector space into disjoint *chambers* or *regions*.

G-Shi Arrangement

Given a graph G, the G-Shi arrangement includes the hyperplanes $x_i - x_j = 0$ and $x_i - x_j = 1$ for each edge $\{i, j\}$ where $i < j$.



If G has n vertices and m edges, the G-Shi arrangement exists in n dimensions and has $2m$ hyperplanes.

G-Parking Functions

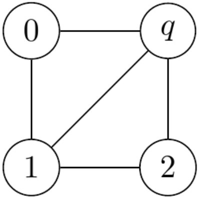
Let G be a graph on $n+1$ vertices: $\{0, 1, \dots, n-1\}$ and a sink q.

Definition

A G-parking function is an n-tuple $(a_0, a_1, ... a_{n-1})$ such that for any non-empty subset $S \subseteq \{0, \dots, n-1\}$, there exists $v \in S$ such that $a_v < \mathrm{outdeg}_S(v)$.

This is also known as a *superstable configuration* in chip-firing.

An Example of G-Parking Functions



$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{0,1,2\}$
$a_0 < 2$	$a_1 < 3$	$a_2 < 2$	$a_0 < 1$ or $a_1 < 2$	$a_0 < 2$ or $a_2 < 2$	$a_1 < 2$ or $a_2 < 1$	$a_0 < 1$ or $a_1 < 1$ or $a_2 < 1$

These constraints yield the G-parking functions $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, and $(0,2,0)$

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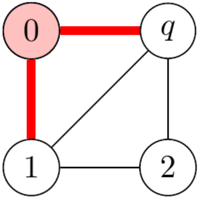

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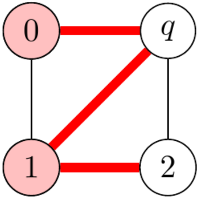

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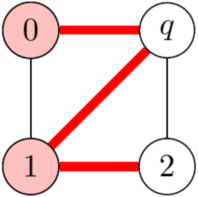

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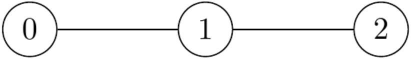
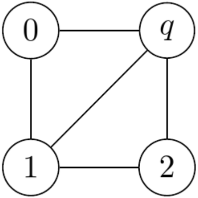
Appending Sinks: $G_\bullet$

Let $G = (V, E)$ be a graph. Then, $G_\bullet$ is the graph obtained from G by adding a vertex q, which we call the *sink*, and an edge between q and each $v \in V$. For example, let $G = P_3$.

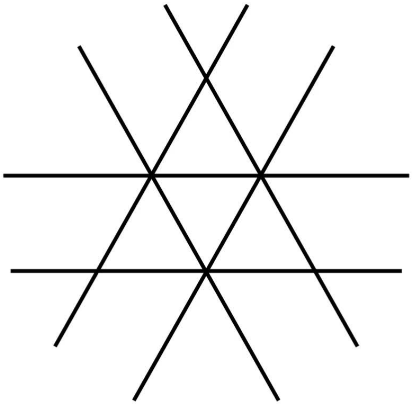


Then, $G_\bullet$ is the graph below.



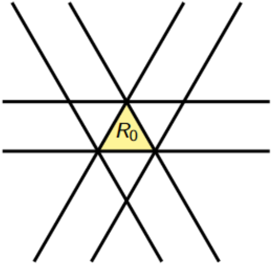
Shi Adjacency Graph

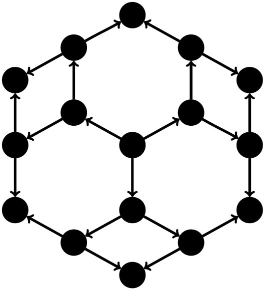
The Shi adjacency graph is defined from the G-Shi arrangement by letting each region be a vertex and placing an edge between bordering regions.



Directing the Shi Adjacency Graph

First, we label the region given by $x_0 > x_1 > \cdots > x_{n-1}$ and $x_0 - x_{n-1} < 1$ by R_0, and call it the *base region*. Then, we direct the edges away from R_0.

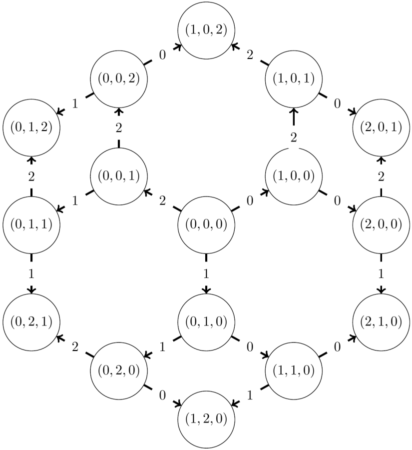




Above is the example for $G = K_3$.

Pak-Stanley Labels

In the Pak-Stanley Algorithm, we label each edge $x_i - x_j = 0$ with i and each edge $x_i - x_j = 1$ with j. Then, starting from R_0, we label the vertices by increasing coordinate i along edges labeled i.



Pak-Stanley Labels

The set of Pak-Stanley labels for the regions of the G-Shi arrangement are called *the Pak-Stanley labels for G*; these are the same as the $G_\bullet$-parking functions (Hopkins-Perkinson).

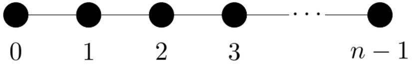
Theorem (Hopkins-Perkinson, Corollary 2.8)

Every $G_\bullet$-parking function occurs as a label in the Pak-Stanley algorithm on the G-Shi arrangement.

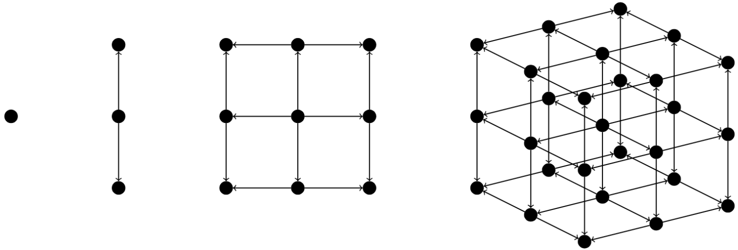
Furthermore, for complete graphs, each $(K_n)_\bullet$-parking function appears exactly once. However, this is not the case for graphs which not are not complete.

An Example: Path Graphs

We started by studying a simple family of graphs: path graphs.



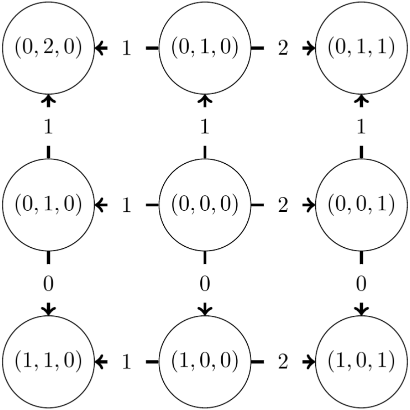
The Shi adjacency digraph of P_1, P_2, P_3, and P_4:



These graphs look like subdivided $(n-1)$-dimensional cubes.

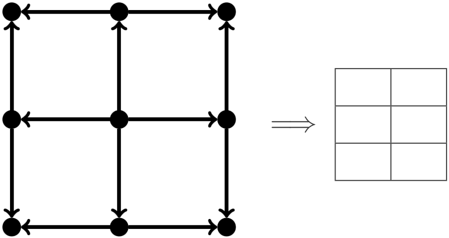
Repetitions in the Labels for Path Graphs

The Pak-Stanley labels on the Shi adjacency graph for P_3.

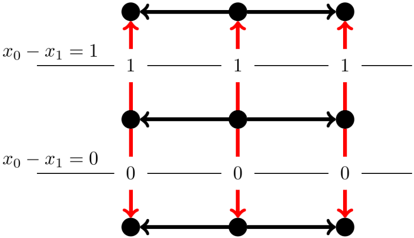


Three Rows Game

Creating the Three Rows Game



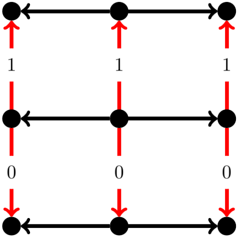
Creating the Three Rows Game



$\Longrightarrow$

$x_0 - x_1 < 0$	
$0 < x_0 - x_1 < 1$	
$x_0 - x_1 > 1$	

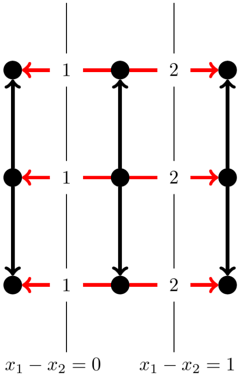
Creating the Three Rows Game



$\Longrightarrow$

0	
1	

Creating the Three Rows Game



$\Longrightarrow$

	$x_1 - x_2 < 0$
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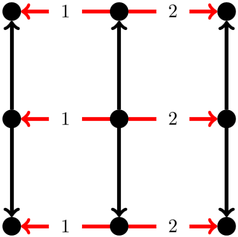
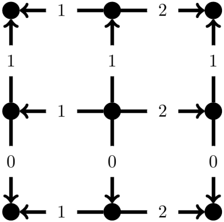
Creating the Three Rows Game



$$\Longrightarrow$$

	1
	2

Creating the Three Rows Game



$\Longrightarrow$

0	1
1	2

Three Rows Game on Paths

The Three Rows Game for P_n uses the following board:

0	1	2	$\cdots$	$n-3$	$n-2$
1	2	3	$\cdots$	$n-2$	$n-1$

To play, pick one cell in each column and keep track of the values selected.

Example

0	**1**	**2**	3	**4**
(selected)				
1	2	3	**4**	5

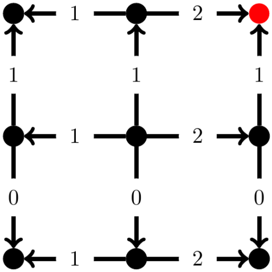
$$\Longrightarrow \{1, 2, 4, 4\}$$

Histories

The vertices in the Shi adjacency digraph correspond to *histories* in the Three Rows Game.

Example

What history corresponds to this vertex?



$\Longrightarrow$

0	1
1	2

?

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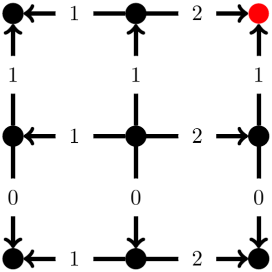


$\Longrightarrow$

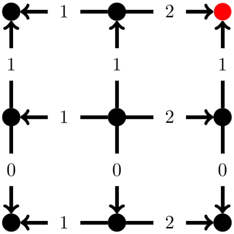
0	1
1	2

Outcomes

The *outcomes* of the Three Rows Game correspond to the Pak-Stanley labels.

Example

What is the Pak-Stanley label for this vertex?



$\Longrightarrow$

0	1
1	2

$$\Longrightarrow \{1,2\} \Longrightarrow (0,1,1)$$

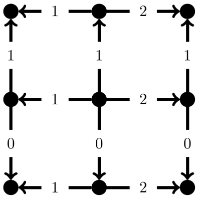
Outcomes

Question

Given an outcome of the Three Rows Game (or Pak-Stanley label), can we determine which vertex it came from?

Example

Which vertex does the outcome $\{1\}$ correspond to?



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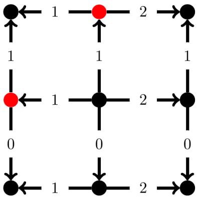

There are multiple solutions! This corresponds to the fact that the label $(0, 1, 0)$ appears twice, as discussed previously.

Repetitions

Counting Repetitions

Question

For a specific G-parking function, how many times will it appear in the P_n-Shi arrangement?

0	1	2	3
1	2	3	4

How many histories yield the following outcomes:

$$\{0, 2, 2\}?$$

Counting Repetitions

Question

For a specific G-parking function, how many times will it appear in the P_n-Shi arrangement?

0	1	2	3
1	2	3	4

How many histories yield the following outcomes:

$$\{0, 2, 2\}? \longrightarrow 1$$

0	1	2	3
1	2	3	4

Counting Repetitions

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For a specific G-parking function, how many times will it appear in the P_n-Shi arrangement?

0	1	2	3
1	2	3	4

How many histories yield the following outcomes:

$$\{1, 2, 4\}?$$

Counting Repetitions

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For a specific G-parking function, how many times will it appear in the P_n-Shi arrangement?

0	1	2	3
1	2	3	4

How many histories yield the following outcomes:

$$\{1, 2, 4\}? \quad \longrightarrow 3$$

0	1	2	3
1	2	3	4

0	1	2	3
1	2	3	4

0	1	2	3
1	2	3	4

Path Repetition Theorem

Definition

A *run* of length n is a subsequence of the Pak-Stanley label of the form $0, 1, 1, \ldots, 1, 1, 0$ where there are n 1's.

Theorem (C. Bennett, A. Mock, R. Truax)

Let $\mathbf{p}$ be a Pak-Stanley label on P_n. If the length of a run r is denoted $l(r)$, then the number of vertices in $\Gamma\mathscr{S}(P_n)$ with label $\mathbf{p}$ is

$$\prod_{\text{runs } r \text{ in } \mathbf{p}} (l(r)+1).$$

Using the Path Repetition Theorem

Question

How many times will the Pak-Stanley label $(0, 1, 1, 1, 0, 2, 0)$ appear in the P_7-Shi arrangement?

Using the Path Repetition Theorem

Question

How many times will the Pak-Stanley label $(0, 1, 1, 1, 0, 2, 0)$ appear in the P_7-Shi arrangement?

Solution

There is one run of length 3:

$$(0, {\color{red}1}, {\color{red}1}, {\color{red}1}, 0, 2, 0)$$

so the label appears $3 + 1 = \mathbf{4}$ times.

Using the Path Repetition Theorem

Question

How many times will the Pak-Stanley label $(0, 1, 1, 0, 1, 1, 0)$ appear in the P_7-Shi arrangement?

Using the Path Repetition Theorem

Question

How many times will the Pak-Stanley label $(0, 1, 1, 0, 1, 1, 0)$ appear in the P_7-Shi arrangement?

Solution

There are two runs of length 2:

$$(0, 1, 1, 0, 1, 1, 0)$$

so the label appears $(2 + 1)(2 + 1) = \mathbf{9}$ times.

Results from the Three Rows Game

Fact (C. Bennett, A. Mock, R. Truax)

The number of regions in the P_n-Shi arrangement is equal to 3^{n-1}.

Explanation

Recall that each region corresponds to a history in the Three Rows Game. For the P_n-Shi arrangement, there are $n-1$ columns.

0	1	$\cdots$	$n-2$
1	2	$\cdots$	$n-1$

We choose one cell in each column, so there are 3^{n-1} histories.

Results from the Three Rows Game

Fact (C. Bennett, A. Mock, R. Truax)

The number of sinks in the Shi adjacency digraph of P_n is equal to 2^{n-1}.

Explanation

A *sink* is a vertex in the Shi adjacency digraph with outdegree 0. For paths (and indeed trees), this corresponds to having no blanks in the history:

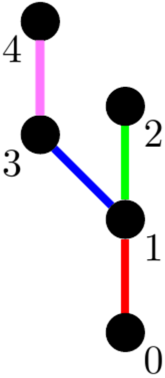
0	1	$\cdots$	$n-2$
1	2	$\cdots$	$n-1$

There are $n-1$ columns, and two choices for each. Thus, there are 2^{n-1} such histories.

Generalizing to Trees

The T-Three Rows Game

The T-Three Rows Game has a column $\begin{array}{|c|}\hline i \\ \hline \\ \hline j \\ \hline \end{array}$ for each edge $\{i, j\}$.



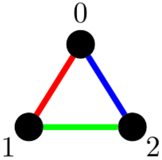
$\Longrightarrow$

0	1	1	3
1	2	3	4

Acyclicity and the Analogy

Acyclicity allows us to choose any sequence of moves.

$\hookrightarrow$ a cycle could introduce a contradiction.



0	1	0
1	2	2

$x_0 - x_1 < 0$
$x_1 - x_2 < 0$
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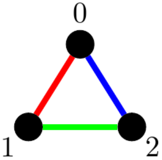

0	1	0
1	2	2

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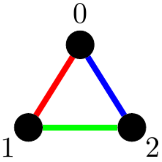
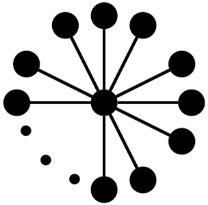


0	1	0
1	2	2

$$x_0 - x_2 < 0$$
$$x_0 - x_2 > 1$$

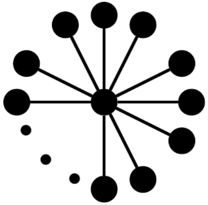
Since trees have no cycles, this issue doesn't arise. Thus, the Shi adjacency digraph still looks like a barycentrically subdivided $(n-1)$-cube, it has 3^{n-1} vertices and 2^{n-1} sinks, etc.

An Example: The Star Graph



0	0	0	$\cdots$	0	0
1	2	3	$\cdots$	$n-1$	n

An Example: The Star Graph

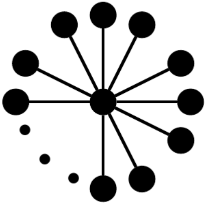


0	0	0	$\cdots$	0	0
1	2	3	$\cdots$	$n-1$	n

How many histories of the game give the outcome $\{1, 2, 4\}$?

0	0	0	0	0	0	0
1	2	3	4	5	6	7

An Example: The Star Graph



0	0	0	$\cdots$	0	0
1	2	3	$\cdots$	$n-1$	n

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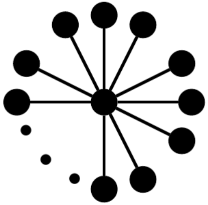
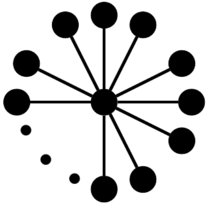


0	0	0	$\cdots$	0	0
1	2	3	$\cdots$	$n-1$	n

How many histories of the game give the outcome $\{0, 0, 1, 4, 5\}$?

0	0	0	0	0	0	0
1	2	3	4	5	6	7

An Example: The Star Graph



0	0	0	$\cdots$	0	0
1	2	3	$\cdots$	$n-1$	n

How many histories of the game give the outcome $\{0, 0, 1, 4, 5\}$?

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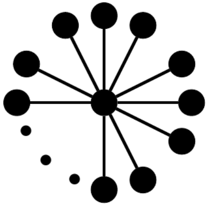

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$$\binom{4}{2} = 6$$

Classifying Repetitions in the Star Graph

Theorem (C. Bennett, A. Mock, R. Truax)

The number of times an outcome repeats in the S_n-Three Rows Game is

$$\binom{\overbrace{c_0}^{\#\text{ of } 0\text{s}} + \overbrace{c_\square}^{\#\text{ of } \square\text{s}}}{c_0}.$$

Corollary

The number of times a Pak-Stanley label $\mathbf{p} = (p_0, \ldots, p_n)$ repeats in the S_n-Shi arrangement is

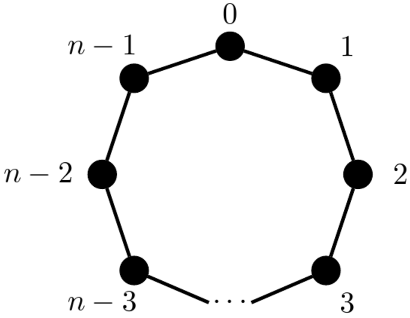
$$\binom{n - p_1 - \cdots - p_n}{p_0}.$$

Notice that any outcome which has no $\square$s appears precisely once.

Beyond Trees

Cycle Graphs

If G has a cycle, some histories of the G-Three Rows Game are *illegal* (they correspond to nonexistent regions). The simplest case is when G *is* a cycle: the cycle graph C_n:



Illegal Histories in Cycle Graphs

Here are two examples of illegal histories for $G = C_n$:

0	1	$\cdots$	$n-2$	0
1	2	$\cdots$	$n-1$	$n-1$

and

0	1	$\cdots$	$n-2$	0
1	2	$\cdots$	$n-1$	$n-1$

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To see why these histories are illegal, notice that the first $n-1$ choices ("path choices") become the inequalities

$x_0 - x_1 < 0$	$x_1 - x_2 < 0$	$\cdots$	$x_{n-2} - x_{n-1} < 0$

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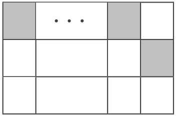
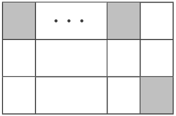
But the final choice ("cycle choice") becomes $0 < x_0 - x_{n-1} < 1$ and $x_0 - x_{n-1} > 1$ respectively (a contradiction).

A Classification of Illegal Histories

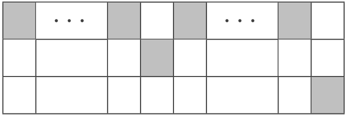
Case 1: All the path choices are T or M.

This case has $n+1$ illegal histories:

1 

and 

(2 illegal histories).

2 

($n-1$ illegal histories).

Case 2: All the path choices are M or B.

This case has $2^n - 1$ illegal histories: all the path choices are M or B and the cycle choice is M or T, except the history of all Ms.

Corollaries of the Classification

The classification yields the following results:

Theorem (C. Bennett, A. Mock, R. Truax)

There are $2^n + n$ illegal histories in the C_n-Three Rows Game.

Corollary

There are $3^n - 2^n - n$ legal histories in the C_n-Three Rows Game.

Corollary

There are $3^n - 2^n - n$ regions in the C_n-Shi arrangement.

It also gives a Cycle Repetition Theorem analogous to the Path Repetition Theorem (but more complicated).

Maximal Labels

A *maximal* $G_\bullet$-parking function is one that cannot be made any larger and remain a $G_\bullet$-parking function. For example, $(1,1,0,1,1)$ is a maximal $(P_5)_\bullet$-parking function.

Proposition

Suppose G has m edges. Then any maximal $G_\bullet$-parking function has coordinate sum m; conversely, any $G_\bullet$-parking function with coordinate sum m is maximal.

Corollary

An outcome of the G-Three Rows Game corresponds to a maximal $G_\bullet$-parking function if and only if the middle row is never used.

i_1	i_2	$\cdots$	i_m
j_1	j_2	$\cdots$	j_m

Uniqueness of Maximal Labels

Theorem (C. Bennett, A. Mock, R. Truax)

Any outcome of the G-Three Rows Game which never uses the middle row is the result of a unique legal history.

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Proof.

The idea is to choose a legal history $\mathbf{h}$ inducing a maximal outcome $\mathbf{o}$, and show that any other history $\mathbf{h}'$ inducing $\mathbf{o}$ is illegal.

Indeed, if $\mathbf{h}'$ is not illegal, then it would need to have infinitely many columns distinct from $\mathbf{h}$, which is impossible. $\square$

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Corollary

Any maximal $G_\bullet$-parking function appears uniquely.

Conclusion

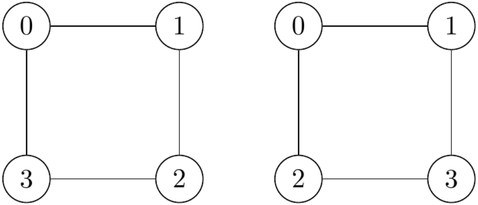
Open Problems: Counting Regions

Question

How many regions are in the G-Shi arrangement of any graph?

Example

The G-Shi arrangements of these graphs have different numbers of regions.



Open Problems: Counting Repetitions

Question

How many times does a given label repeat in the T-Shi arrangement of a tree graph T?

- We can find the answer using breadth-first search.
- Does a faster algorithm exist?

Question

How many times does a given label repeat in the G-Shi arrangement for a general graph G?

Open Problems: Sinks

Every vertex with a maximal Pak-Stanley label is a sink in the Shi adjacency digraph.

Question

Does every sink in the Shi adjacency digraph have a maximal Pak-Stanley label?

- We have shown that this is true for tree graphs.

Thank You!

Appendix

Uniqueness in the T-Three Rows Game

Theorem (C. Bennett, A. Mock, R. Truax)

Any outcome of the T-Three Rows Game with no $\square$s (i.e. with n numbers) is the result of a unique history.

Uniqueness in the T-Three Rows Game

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Any outcome of the T-Three Rows Game with no $\square$s (i.e. with n numbers) is the result of a unique history.

Proof.

First, we introduce the concept of *distance* between outcomes:

$$d(\mathbf{o}, \mathbf{o}') = \sum_{i=0}^{n} \left| \underbrace{m_{\mathbf{o}}(i)}_{\text{occurences of } i \text{ in } \mathbf{o}} - \underbrace{m_{\mathbf{o}'}(i)}_{\text{occurences of } i \text{ in } \mathbf{o}'} \right|.$$

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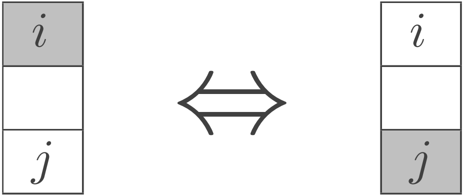
Since T is a tree, we can relabel the vertices so that every column (after the first) contains one old vertex and one new vertex.

Uniqueness in the T-Three Rows Game (Cont.)

Proof (Cont).

Suppose that $\mathbf{h}_1$ and $\mathbf{h}_2$ give the same outcome $\mathbf{o}$. Since our outcome has no □s, $\mathbf{h}_1$ can be transformed into $\mathbf{h}_2$ using "swaps":



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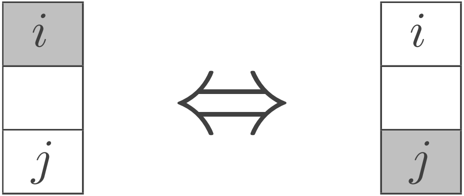
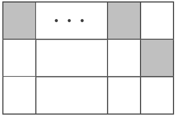
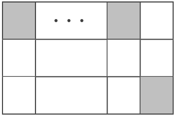
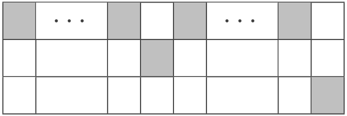


Now, the first swap increases the distance by 2. However, each swap afterwards doesn't decrease the distance (since there is one new and one old vertex). This gives a contradiction. □

Cycle Graphs

Remember that if there is a cycle, some histories are *illegal*. In the case $G = C_n$, we can count these illegal histories:

1.  and .
2. .
3. Any history where the first $n-1$ choices are all M or B and the last choice is M or T (except the history of all Ms).

Thus, the number of regions in the C_n-Shi arrangement is

$$3^n - 2 - (n-1) - (2^n - 1) = 3^n - 2^n - n.$$

Patterns

A *pattern* is a maximal sequence of weakly-increasing choices with exactly one middle choice.

These histories contain patterns:

0	1	2	3	4
1	2	3	4	5

0	1	2	3	4
1	2	3	4	5

These histories contain no nontrivial patterns:

0	1	2	3	4
1	2	3	4	5

0	1	2	3	4
1	2	3	4	5