Parking Functions with Fixed Ascent and Descent Sets

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Our Research Focus

Theme: Statistics in parking functions and unit interval parking functions — ascents, descents, ties.

This research is inspired by

- Silley et al. (2013): studied permutations with a given peak set
- Schumacher (2018): enumerated parking functions with k descents and i ties

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Definitions

Given a parking function $\alpha = (a_1, a_2, \dots, a_n)$:

- An ascent is defined as an index *i* such that $a_i < a_{i+1}$.
- A **descent** is defined as an index *i* such that $a_i > a_{i+1}$.
- A tie is defined as an index *i* such that $a_i = a_{i+1}$.

Example:

 $\alpha = 132441$ $\alpha = 132441$ $\alpha = 132441$

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Definitions

Given a parking function $\alpha = (a_1, a_2, \dots, a_n)$:

- The ascent set is defined as the set I with all ascents of α .
- The descent set is defined as the set I with all descents of α .
- The tie set is defined as the set *I* with all ties of *α*.
- A ascent/descent/tie subset is defined as any subset of α's ascent/descent/tie set.

Example: $\alpha = 132441$

- Ascent set: {1,3}
- Descent set: {2,5}
- Tie set: {4}

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Classical Parking Functions

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Established Results

Known Enumerations:

- Parking functions: $(n+1)^{n-1}$ (Riordan, 1969)
- Weakly increasing/decreasing parking functions: Catalan (Stanley, 1999)
- Parking functions with k ties (Yan, 2015)
- Parking functions with k descents and i ties (Schumacher, 2018)

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Descent-Ascent Symmetry Theorem

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Introduction to the Descent-Ascent Symmetry Theorem Definition: Given a set $I = \{i_1, ..., i_k\} \subseteq [n-1]$, let $I^{-1} := \{n - i_1, ..., n - i_k\}$.

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Introduction to the Descent-Ascent Symmetry Theorem **Definition**: Given a set $I = \{i_1, ..., i_k\} \subseteq [n-1]$, let $I^{-1} := \{n - i_1, ..., n - i_k\}$.

Example: $n = 5, I = \{1, 2\} \Rightarrow I^{-1} =$

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Parking Functions

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	n		1	3	3 4 5			n		
1	3	4	5	<u>∫1 2</u>]	1	0	56	Ι	4	5
<i>'</i>		т 01	04	(1,2)	1	10	161	{1,2,3}	1	14
{1}	5	21	84	$\{1,3\}$		19	101	$\{1, 2, 4\}$		49
{2}	5	31	154	$\{1,4\}$			126	(1,2,1)		40
{3}		21	154	$\{2,3\}$		9	91	$\{1, 3, 4\}$		49
			04	(-, 0)		-	161	{2,3,4}		14
{4}			04	{2,4}			101	$\{1234\}$		1
				{3,4}			56	[1,2,3,1]		-

Table: The number of parking functions of length n with the descent/ascent sets I

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Parking Functions

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Descent-Ascent Symmetry Theorem (JES)

PFs with descent set *I*

PFs with descent set I^{-1}



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Descent-Ascent Inverse Equality

Theorem (Descent-Ascent Symmetry Theorem — JES)

The number of parking functions with descent set $I = \{i_1, i_2, ...\}$ is equal to the number of parking functions with ascent set $I^{-1} = \{n - i_1, n - i_2, ...\}.$

Proof idea: bijection with reverse

$$\alpha = (a_1, a_2, \dots, a_n) = 142345$$
$$\Leftrightarrow$$
$$\alpha' = (a_n, a_{n-1}, \dots, a_1) = 543241$$

For each **descent** $i \in I$, n-i is in the **ascent** set of α' .

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Parking Functions

Unit Interval Parking Functions

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Descent-Ascent Symmetry Theorem (JES)



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Unit Interval Parking Functions

Descent-Ascent Equality

Theorem (JES)

The number of parking functions with ascent set I is equal to the number of parking functions with descent set I.

Proven by strong induction.

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Parking Functions

Unit Interval Parking Functions

Descent-Ascent Symmetry Theorem (JES)



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Open Exercise!!

Exercise

The bijection between parking functions with descent set I and parking functions with descent set I^{-1} or a direct proof that these sets of parking functions are equinumerous.

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Connection to Narayana Numbers!

Connection to Narayana Numbers!

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Theorem (JES)

The number of parking functions of length n with descent set $\{1, \ldots, k\}$ is

$$\sum_{i=1}^{n} \frac{1}{n} \binom{n}{i} \binom{n}{i-1} \binom{i-1}{k}.$$



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Proof Sketch:

- Weakly increasing parking functions of length *n* with *i* distinct values
 - Narayana numbers (Stanley, 1999).

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Proof Sketch:

- Weakly increasing parking functions of length *n* with *i* distinct values Narayana numbers (Stanley, 1999).
- Choose any k of the *i*-1 distinct values greater than 1 to move to the front in decreasing order.

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Conjecture

Conjecture

Parking functions of length *n* with descent subset $\{1, ..., \hat{k}, ..., n-1\}$ are counted by the Narayana number

$$N(n+1,k+1) = \frac{1}{n+1} \binom{n+1}{k+1} \binom{n+1}{k}.$$

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Parking functions with given ascent/descent/tie subsets

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Parking Functions with Tie Subsets

n m	1	2	3	4	5	6
2	1					
3	4	1				
4	25	5	1			
5	216	36	6	1		
6	2401	343	49	7	1	
7	32768	4096	512	64	8	1

Table: The number of parking functions of length n with tie subset of size m

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Table: The number of parking functions of length n with tie subset of size m

The entries of this table are given by $(n+1)^{n-1-m}$.

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Parking Functions with Given Tie Subsets

Theorem (JES)

For any $I \subseteq [n-1]$, let T(n, I) be the set of parking functions of length n whose tie set contains I. Then,

 $|T(n, I)| = (n+1)^{n-1-|I|}.$

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Proof using Prüfer Code:

- The **Prüfer Code** of parking function $\alpha = (a_1, \ldots, a_n)$ is $\rho = (a_2 a_1, a_3 a_2, \ldots, a_n a_{n-1}).$
- A "0" in the Prüfer Code means a tie in the parking function.
- Each entry of the Prüfer Code has n+1 possible values.

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Parking Functions with Consecutive Descent/Ascent Subsets

Theorem (JES)

For any $I = \{i, i+1, ..., i+m-1\} \subseteq [n-1]$, the number of parking functions whose descent set contains I is

$$D(n,l) = \binom{n+1}{|l|+1} (n+1)^{n-|l|-2}.$$

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Parking Functions Starting with k Distinct Values

Theorem (JES)

The number of parking functions of length n that start with k distinct values is

 $k! \cdot |D(n, [k-1])|.$

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Unit Interval Parking Functions

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Definitions

Let α = (a₁,..., a_n) be a parking function. If car *i* parks in spot s_i, we call s_i - a_i the **displacement** of car *i*.

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- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If car *i* parks in spot s_i , we call $s_i a_i$ the **displacement** of car *i*.
- A unit interval parking function is a parking function
 α = (a₁, a₂,..., a_n) where the individual displacement of each car is at
 most 1. We refer to the set of unit interval parking functions of
 length n as UPF_n.

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 α = (a₁, a₂,..., a_n) where the individual displacement of each car is at
 most 1. We refer to the set of unit interval parking functions of
 length n as UPF_n.

Example: Consider $\alpha = (1, 1, 2, 3) \in \mathsf{UPF}_4$.

C1	C2	C3	C4	
æ	æ	æ	æ	
1	2	3	4	

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Conclusions 0000

IS IT A UNIT INTERVAL?

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IS IT A UNIT INTERVAL?

• $\alpha = (1, 1, 2)$

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IS IT A UNIT INTERVAL?

• $\alpha = (1, 1, 2)$ YES!! 😄

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IS IT A UNIT INTERVAL?

• $\alpha = (1,1,2)$ YES!! \bigcirc • $\alpha = (2,1,1)$

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IS IT A UNIT INTERVAL?

•
$$\alpha = (1,1,2)$$
 YES!! \cong
• $\alpha = (2,1,1)$ NO!! •

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Conclusions

IS IT A UNIT INTERVAL?

•
$$\alpha = (1, 1, 2)$$
 YES!!

•
$$\alpha = (2, 1, 1)$$
 NO!! •

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$$\alpha = (1, 2, 1, 3)$$

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Unit Interval Parking Functions

Conclusions 0000

IS IT A UNIT INTERVAL?

•
$$\alpha = (1, 1, 2)$$
 YES!! 😄

•
$$\alpha = (2, 1, 1)$$
 NO!! •

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Unit Interval Parking Functions

Conclusions 0000

IS IT A UNIT INTERVAL?

•
$$\alpha = (1, 1, 2)$$
 YES!! 😄

•
$$\alpha = (2, 1, 1)$$
 NO!! •

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$$\alpha = (4, 1, 1, 2)$$

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Unit Interval Parking Functions

Conclusions 0000

IS IT A UNIT INTERVAL?

- $\alpha = (1, 1, 2)$ YES!! 😄
- α = (2,1,1) NO!! •
- α = (1,2,1,3) NO!! [™]
- $\alpha = (4, 1, 1, 2)$ YES!! 👑

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Fubini Numbers!



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Number of Unit Interval Parking Functions

п	1	2	3	4	5	6
$ PF_n $	1	3	16	125	1296	16807
$ UPF_n $	1	3	13	75	541	4683

Table: The number of unit interval parking functions of length n

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Number of Unit Interval Parking Functions

п	1	2	3	4	5	6
$ PF_n $	1	3	16	125	1296	16807
UPF _n	1	3	13	75	541	<mark>4683</mark>

Table: The number of unit interval parking functions of length n

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• A **Fubini ranking** is a sequence $\beta = (b_1, \ldots, b_n)$ that represents one way *n* players can rank in a competition, allowing ties. We denote FR_n as the set of all Fubini rankings of length *n*. (Hadaway, 2022)

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- The **Fubini numbers** are defined as $Fb_n = |FR_n|$.

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- The **Fubini numbers** are defined as $Fb_n = |FR_n|$.
- Note: The *n*-tuple β = (b₁,..., b_n) ∈ [n]ⁿ is a Fubini ranking of length n if the following holds: For all i ∈ [n], if k entries of β are equal to i, then the next largest value in β is i + k.

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IS IT A FUBINI RANKING?

β = (1,1,2)

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Unit Interval Parking Functions

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IS IT A FUBINI RANKING?

• $\beta = (1, 1, 2)$ NO!! 😠

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Conclusions

IS IT A FUBINI RANKING?

• $\beta = (1,1,2)$ NO!! • $\beta = (1,2,2)$

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Conclusions 0000

IS IT A FUBINI RANKING?

• $\beta = (1,1,2)$ NO!! • $\beta = (1,2,2)$ YES!!

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IS IT A FUBINI RANKING?

• $\beta = (1,1,2)$ NO!! • $\beta = (1,2,2)$ YES!!

• $\beta = (1, 1, 1, 3)$

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Conclusions 0000

IS IT A FUBINI RANKING?



• $\beta = (1, 1, 1, 3)$ NO!! 😢

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Conclusions 0000

IS IT A FUBINI RANKING?



- $\beta = (1, 2, 2)$ YES!! 🔆
- $\beta = (1, 1, 1, 3)$ NO!! 😢
- β = (4,1,1,3)

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Conclusions 0000

IS IT A FUBINI RANKING?

- $\beta = (1, 1, 2)$ NO!! 😠
- $\beta = (1, 2, 2)$ YES!! 🔆
- $\beta = (1, 1, 1, 3)$ NO!! 😢
- $\beta = (4, 1, 1, 3)$ YES!! Υ

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• An **r-Fubini ranking** is a Fubini ranking that begins with r distinct values. We denote FR_n^r as the set of all Fubini rankings of length n. *Note:* $FR_n^i \subset FR_n^j$ for i > j.

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- The **r-Fubini numbers** are defined as $Fb_n^r = |FR_n^r|$.

Example: $\alpha = (1,1,3)$ is a 1-Fubini ranking and $\beta = (1,2,3,4)$ is a 4-Fubini ranking.

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Established Results

Unit interval parking functions are in bijection with:

- Fubini rankings (Hadaway, 2022).
- Dyck paths with height at most 2 (Baril, Kirgizov **and** Petrossian, 2018).

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Fubini Ranking Observations

Lemma (MSRI-UP 2021, JES 2022)

Fubini rankings are permutation invariant.

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Lemma (MSRI-UP 2021, JES 2022)

Fubini rankings are a subset of parking functions.

Furthermore, the k^{th} occurrence of any i in $\beta \in FR_n$ parks in spot i + k - 1.

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Fubini rankings are a subset of parking functions.

Furthermore, the k^{th} occurrence of any i in $\beta \in FR_n$ parks in spot i + k - 1. Ex:

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Introductions

Fubini Ranking to Unit Interval PF Bijection

Definition

Define ϕ : FR_n \mapsto UPF_n as follows: Given $\beta = (b_1, \dots, b_n) \in$ FR_n, $\phi(\beta) = (a_1, \dots, a_n) \in$ UPF_n, where

 $a_{i} = \begin{cases} b_{i} & \text{if index } i \text{ has the first or second occurrence of } b_{i} \\ b_{i} + k - 2 & \text{if index } i \text{ has the } k^{\text{th}} \text{ occurrence of } b_{i} (k > 2) \end{cases}$

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Ex:

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Ex:

$$\beta$$
 $\phi(\beta)$
3 1 4 8 4 4 1 4 3 1 4 8 5 6 1

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Introductions

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 $a_i = \begin{cases} b_i & \text{if index } i \text{ has the first or second occurrence of } b_i \\ b_i + k - 2 & \text{if index } i \text{ has the } k^{\text{th}} \text{ occurrence of } b_i (k > 2) \end{cases}$

Ex:

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$egin{array}{c|c} eta \in \mathsf{FR}_5 & \phi(eta) \in \mathsf{UPF}_5 \ \hline 12345 & & & \ 11111 & & \ 14141 & & \ 31415 & & & \ \end{array}$

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$\beta \in FR_5$	$\phi(eta) \in UPF_5$
12345	12345
11111	11234
14141	
31415	

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$\beta \in FR_{5}$	$\phi(eta) \in UPF_5$
12345	12345
11111	11234
14141	14142
31415	

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$\beta \in FR_{5}$	$\phi(eta) \in UPF_5$
12345	12345
11111	11234
14141	14142
31415	31415

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UPF_n Enumeration

Theorem (JES)

The number of unit interval parking functions of length n is Fb_n.

Proof Sketch: We proved ϕ is a bijection from FR_n to UPF_n, so

 $|\mathsf{UPF}_n| = |\mathsf{FR}_n| = \mathsf{Fb}_n.$

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Weakly Increasing/Decreasing Unit Interval Parking Functions

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n	WIUPF _n
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Table: The number of weakly increasing unit interval parking functions of length n

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п	WIUPF _n
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Table: The number of weakly increasing unit interval parking functions of length n

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Theorem (JES)

The number of weakly increasing unit interval parking functions of length n is 2^{n-1} .

Weaking increasing unit interval parking functions:

- n = 1: 1
- *n* = 2: 11, 12
- *n* = 3: 112 , 113, 122, 123
- *n* = 4: 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234
- n = 5: 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345

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Theorem (JES)

The number of weakly increasing unit interval parking functions of length n is 2^{n-1} .

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$$n = 1: 1$$

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- *n* = 3: 112 , 113, 122, 123
- *n* = 4: 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234
- n = 5: 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345

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Theorem (JES)

The number of weakly increasing unit interval parking functions of length n is 2^{n-1} .

Weaking increasing unit interval parking functions:

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$$n = 1: 1$$

•
$$n = 3$$
: 112, 113, 122, 123

• *n* = 4: 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234

n = 5: 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345

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Connections to Fubini Rankings

n	WIUPF _n	WIFR _n	
1	1	1	
2	11, 12	11, 12	
3	112 , 113, 122, 123	111 , 113, 122, 123	
4	1123 , 1124 , 1133, 1134,	1111 , 1114 , 1222 , 1133,	
	1223 , 1224, 1233, 1234	1134, 1224, 1233, 1234	

Table: WIUPF_n and WIFR_n

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п	WDUPF _n
1	1
2	2
3	3
4	5
5	8
6	13
7	21

Table: Weakly decreasing unit interval parking functions of length n

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п	WDUPF _n
1	1
2	2
3	3
4	5
5	8
6	13
7	21

Table: Weakly decreasing unit interval parking functions of length n

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Theorem (JES)

The number of weakly decreasing unit interval parking functions of length n is F_{n+1} . (F_n is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Weaking decreasing unit interval parking functions:

- n = 1: 1
- *n* = 2: 11, 21
- *n* = 3: 221, 311, 321
- *n* = 4: 3311, 3321, 4221, 4311, 4321
- *n* = 5: 44221, 44311, 44321, 53311, 53321, 54221, 54311, 54321

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• n = 5: 44221, 44311, 44321, 53311, 53321, 54221, 54311, 54321

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Weaking decreasing unit interval parking functions:

•
$$n = 1: 1$$

•
$$n = 3$$
: 221, 311, 321

•
$$n = 4$$
: 3311, 3321, 4221, 4311, 4321

• *n* = 5: 44221, 44311, 44321, 53311, 53321, 54221, 54311, 54321

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Connections to Fubini Rankings

n	WDUPF _n	WDFR _n
1	1	1
2	11, 21	11, 21
3	221, 311, 321	111, 221, 311, 321
4	3311, 3321, 4221, 4311,	1111,4111,2221 , 3311,
	4321	3321, 4221, 4311, 4321

Table: WDUPF_n and WDFR_n

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r-Fubini Numbers

Eva, Susan, Juliet

Stats on PFs

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r-Fubini Rankings

Definition

The *n*-tuple $\beta = (b_1, \dots, b_n) \in [n]^n$ is an **r-Fubini ranking** of length *n* if β is a Fubini ranking starting with *r* distinct values, we denote FR_n^r .

Example: $\beta = (1, 2, 2) \in FR_3^2$

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r-Fubini Bijection

Theorem

The number unit interval parking functions of length n starting with r distinct values is Fb_n^r .

Proof Sketch: The bijection ϕ : FR_n \mapsto UPF_n maintains the maximal number of distinct starting values.

$\beta \in FR_4$	$\phi(\beta) \in UPF_4$
1234	1234
123 3	123 3
14 11	14 12
12 22	12 23

So $|\mathsf{UPF}_n^r| = |\mathsf{FR}_n^r| = \mathsf{Fb}_n^r$.

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Bijection to Unit Interval Parking Functions with Certain Tie Subsets

Theorem (JES)

Each of the following is equal to Fb_n^r :

- $\bigcirc UPF_n^r$
- **2** The number of unit interval parking functions of length n + r 1 with tie subset $\{2, 4, ..., 2r 2\}$.
- Solution The number of unit interval parking functions of length n + r with tie subset {1,3,...,2r−1}.

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Add Tie Algorithm

Question: Given a $k \in [n]$, $\alpha \in UPF_n^k$, can we find a unique $\alpha' \in UPF_{n+1}^k$ with an added tie at index k?

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Add Tie Algorithm

Definition (JES)

Define $\lambda(\alpha, k) : \mathsf{UPF}_n^k, [n] \mapsto \mathsf{UPF}_{n+1}^k$ as $\lambda(\alpha, k) = (a'_1, \dots, a'_n, a'_{n+1})$, where

$$a'_{i} = \begin{cases} a_{i} & \text{if } i < k \text{ and } a_{i} < a_{k}, \text{ or } i = k, \text{ or } i = k+1 \\ a_{i}+1 & \text{if } i < k \text{ and } a_{i} \ge a_{k} \\ a_{i-1} & \text{if } i > k+1 \text{ and } a_{i-1} < a_{k} \\ a_{i-1}+1 & \text{if } i > k+1 \text{ and } a_{i-1} \ge a_{k} \end{cases}.$$

For any $\alpha \in \mathsf{UPF}_n^k$, $\lambda(\alpha, k) \in \mathsf{UPF}_{n+1}^k$.

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Add Tie Algorithm

Definition (JES)

Define $\lambda(\alpha, k) : \mathsf{UPF}_n^k, [n] \mapsto \mathsf{UPF}_{n+1}^k$ as $\lambda(\alpha, k) = (a'_1, \dots, a'_n, a'_{n+1})$, where

$$a'_{i} = \begin{cases} a_{i} & \text{if } i < k \text{ and } a_{i} < a_{k}, \text{ or } i = k, \text{ or } i = k+1 \\ a_{i}+1 & \text{if } i < k \text{ and } a_{i} \ge a_{k} \\ a_{i-1} & \text{if } i > k+1 \text{ and } a_{i-1} < a_{k} \\ a_{i-1}+1 & \text{if } i > k+1 \text{ and } a_{i-1} \ge a_{k} \end{cases}$$

For any $\alpha \in \mathsf{UPF}_n^k$, $\lambda(\alpha, k) \in \mathsf{UPF}_{n+1}^k$.

Example: $\alpha = (5,1,3,3,2) \in \mathsf{UPF}_5^3 \mapsto \lambda(\alpha,3) = (6,1,3,3,4,2) \in \mathsf{UPF}_6^3$

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Bijections Summary



Eva, Susan, Juliet

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Future Directions

- Finding a bijection between descent set and inverse descent set
- Using inclusion/exclusion principle to get ascent/descent/tie set formulas from subset formulas
- Enumerating parking function with a given peak or valley set
- Analyzing these statistics for ℓ -interval parking functions

Thanks for listening!

Eva, Susan, Juliet

Summer@ICERM 2022

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Introductions 0000 Unit Interval Parking Functions

Sequence Music Sound example



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