Parking Functions with Fixed Ascent and Descent Sets

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Our Research Focus

¦\0h0I Statistics in parking functions and unit interval parking functions — ascents, descents, ties.

This research is inspired by

1. Billey et al. (2013): studied permutations with a given peak set
2. Schumacher (2018): enumerated parking functions with c descents and _ ties

Definitions

Given a parking function $\alpha = (9_1, 9_2, \ldots, 9_j)$:

- An 9 HOj§ is defined as an index _ such that $9_{\_} < 9_{\_+1}$.
- A LO HOj§ is defined as an index _ such that $9_{\_} > 9_{\_+1}$.
- A §_0 is defined as an index _ such that $9_{\_} = 9_{\_+1}$.

N²9 h qe0I

$$\alpha = \text{o¨}2441$$
$$\alpha = 1\text{¨©}4\text{Vo}$$
$$\alpha = 132\text{VV}1$$

Definitions

Given a parking function $\alpha = (9_1, 9_2, \ldots, 9_j)$:

- The 9 HOj§ O§ is defined as the set ^ with all ascents of α.
- The LO HOj§ O§ is defined as the set ^ with all descents of α.
- The §_O O§ is defined as the set ^ with all ties of α.
- A 9 HOj§¤LO HOj§¤§_O «@ O§ is defined as any subset of α's ascent/descent/tie set.

N²9hq℮0I $\alpha = 132441$

- Ascent set: $\{1, 3\}$
- Descent set: $\{2, 5\}$
- Tie set: $\{4\}$

Ge9 _H9e p9}c_jX S«jH§_nj

Established Results

b jn° j Nj«h0}9§_nj I

- Parking functions: $(j+1)^{j-1}$ (Riordan, 1969)
- Weakly increasing/decreasing parking functions: Catalan (Stanley, 1999)
- Parking functions with c ties (Yan, 2015)
- Parking functions with c descents and _ ties (Schumacher, 2018)

K0 H0j§]8 H0j§ ~´h h0§}´ ¦\0n}0h

Introduction to the Descent-Ascent Symmetry Theorem

K0»j_§_nj: Given a set $\wedge = \{\_1, \ldots, \_c\}$ $[j - 1]$, let $\wedge^{-1} := \{j - \_1, \ldots, j - \_c\}$.

Introduction to the Descent-Ascent Symmetry Theorem

K0»j_§_nj: Given a set $\wedge = \{\_1, \ldots, \_c\}$ $[j-1]$, let $\wedge^{-1} := \{j - \_1, \ldots, j - \_c\}$.

N²9hq℮0: $j = 5$, $\wedge = \{1, 2\}$ $\wedge^{-1} =$

Introduction to the Descent-Ascent Symmetry Theorem

K0»j_§_nj: Given a set $\wedge = \{\_1, \dots, \_c\}$ $[j-1]$, let $\wedge^{-1} := \{j - \_1, \dots, j - \_c\}$.

N²9hqȩ0: $j = 5$, $\wedge = \{1, 2\}$ $\wedge^{-1} = \{3, 4\}$.

Introduction to the Descent-Ascent Symmetry Theorem

K0»j_§_nj: Given a set $\wedge = \{\_1, \ldots, \_c\}$ $[j-1]$, let $\wedge^{-1} := \{j - \_1, \ldots, j - \_c\}$.

N²9hqℯ0: $j = 5$, $\wedge = \{1, 2\}$ $\wedge^{-1} = \{3, 4\}$.

	j		
$\wedge$	3	4	5
$\{1\}$	5	21	84
$\{2\}$	5	31	154
$\{3\}$		21	154
$\{4\}$			84

	j		
$\wedge$	3	4	5
$\{1,2\}$	1	9	56
$\{1,3\}$		19	161
$\{1,4\}$			126
$\{2,3\}$		9	91
$\{2,4\}$			161
$\{3,4\}$			56

	j	
$\wedge$	4	5
$\{1,2,3\}$	1	14
$\{1,2,4\}$		49
$\{1,3,4\}$		49
$\{2,3,4\}$		14
$\{1,2,3,4\}$		1

Table: The number of parking functions of length j with the descent/ascent sets $\wedge$

Descent-Ascent Symmetry Theorem (JES)

PFs with descent set $\wedge$

PFs with descent set $\wedge^{-1}$

PFs with ascent set $\wedge$

PFs with ascent set $\wedge^{-1}$

Descent-Ascent Inverse Equality

Theorem (Descent-Ascent Symmetry Theorem — JES)

¦ \0 j«h@0} nT q9}c_jX T«jH§_nj °_§\ L0 H0j§ 0§ ^ $= \{\_1, \_2, \ldots\}$ _ 0x«9e §n §\0 j«h@0} nT q9}c_jX T«jH§_nj °_§\ 9 H0j§ 0§ $^{-1} = \{j - _1, j - _2, \ldots\}$u

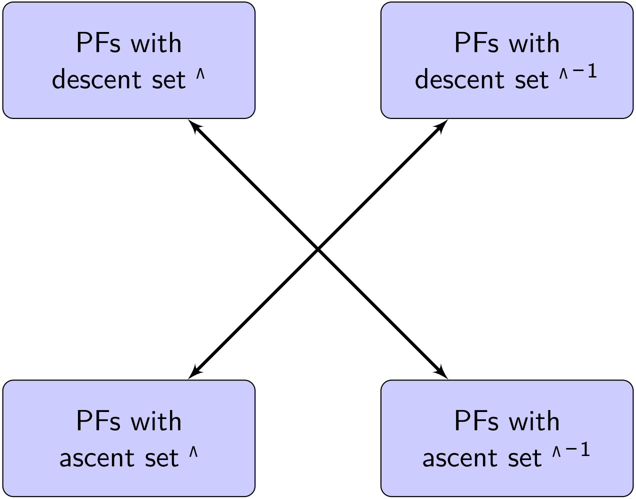
Proof idea: bijection with reverse

$$\alpha = (9_1, 9_2, \ldots, 9_j) = 142345$$

$$\alpha = (9_j, 9_{j-1}, \ldots, 9_1) = 543241$$

For each L0 H0j§ _ ^, $j - \_$ is in the 9 H0j§ set of α .

Descent-Ascent Symmetry Theorem (JES)



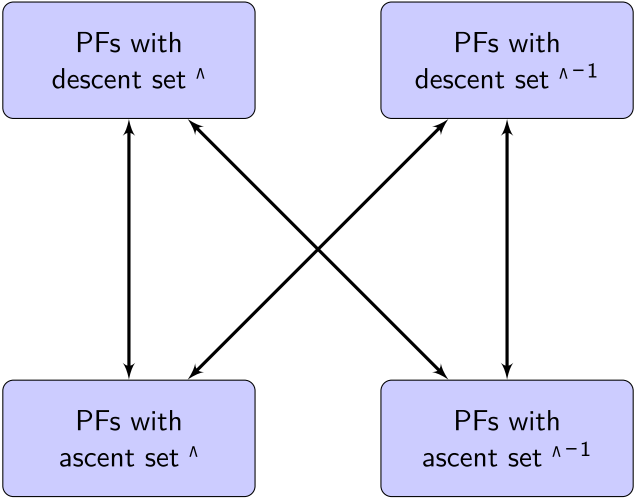
Descent-Ascent Equality

Theorem (JES)

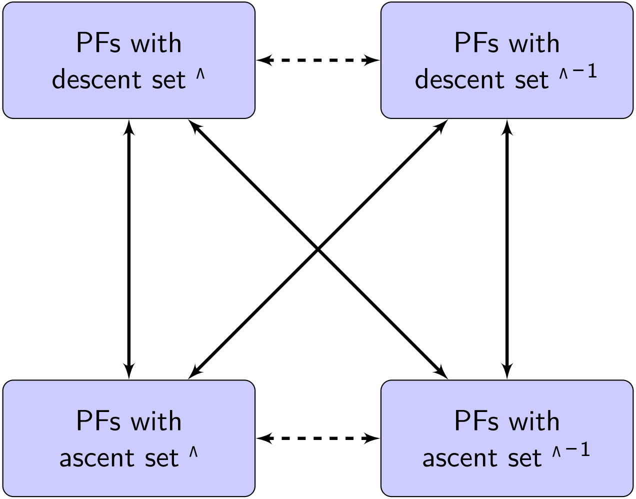
¦ \0 j«h@0} nT q9}c_jX T«jH§_nj °_§\ 9 H0j§ 0§ ^ _ 0x«9e §n §\0 j«h@0} nT q9}c_jX T«jH§_nj °_§\ L0 H0j§ 0§ ^u

Proven by strong induction.

Descent-Ascent Symmetry Theorem (JES)



Descent-Ascent Symmetry Theorem (JES)



Open Exercise!!

Exercise

The bijection between parking functions with descent set $^{\wedge}$ and parking functions with descent set $^{\wedge -1}$ or a direct proof that these sets of parking functions are equinumerous.

Gnjj0H§_nj §n i9}9´9j9 i«h@0} R

Gnjj0H§_nj §n i9}9´9j9 i«h@0} R

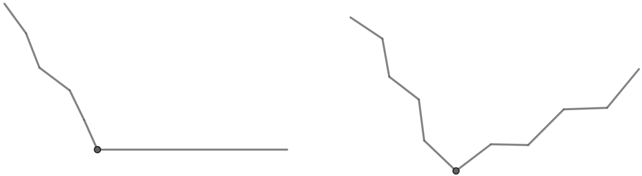
Play

Parking Functions with Descent Set $\{1,\ldots,c\}$

Theorem (JES)

¦ \0 j«h@0} nT q9}c_jX T«jH§_nj nT e0jX§\ j °_§\ L0 H0j§ 0§ $\{1,\ldots,c\}$ _

$$\sum_{\_=1} \frac{1}{j} \quad j \quad \_ \quad j \quad \_ -1 \quad \_ -1 \quad c \quad .$$



S `FBM; 6mM+iBQMb fR:?; F2b+2Mi a

[redacted]

h?2 MmK#2` Q7 T `FBM; 7mM+iBQMb Q7 R:H;2Mi

$$X^M \begin{pmatrix} R & M \\ M & B \end{pmatrix} \begin{pmatrix} M \\ B & R \end{pmatrix} \begin{pmatrix} B & R \\ F \end{pmatrix} :$$

B R

S `FBM; 6mM+iBQMb rBi? ;Q .2b+2Mi a

[redacted]

h?2 MmK#2` Q7 T `FBM; 7mM+iBQMb Q7 R:H;2FM;iQ

$$X^M \frac{R \; M \quad M}{M \; B \quad B \; R} \; \frac{B \; R}{F} :$$

$B \; R$

S`QQ7 aF2i+?,

q2 FHv BM+`2 bBM; T `FBM; M7m BM+iBQMb Q7 p

L ` v M MmK#2`b Uai MH2v- RNNNVX

Parking Functions with Descent Set $\{1,\ldots,c\}$

Theorem (JES)

¦ \0 j«h@0} nT q9}c_jX T«jH§_nj nT e0jX§\ j °_§\ L0 H0j§ 0§ $\{1,\ldots,c\}$ _

$$\sum_{\_=1}^{j}\left[\frac{1}{j}\binom{j}{\_}\binom{j}{\_-1}\right]\binom{\_-1}{c}.$$

p}nnT ~c0§H\I

- Weakly increasing parking functions of length j with _ distinct values — Narayana numbers (Stanley, 1999).
- Choose any c of the $\_-1$ distinct values greater than 1 to move to the front in decreasing order.

*QMD2+im`2

[redacted]

S `FBM; 7mM+iBQMMbiQ7/2Hb2+M2,MfR?R:m;#Fb:2;iM Rg `2 +QmMi2/ #v i?2 L ` v M MmK#2`

$$L(M+R,F+R) = \frac{R}{M+R} \frac{M+R}{F+R} \frac{M+R}{F}.$$

S `FBM; 7mM+iBQMb b+2Mif/2b+2MifiB2

S `FBM; 6mM+iBQMb rBi? hB2 am#b

a a a a a K M a a	R	k	j	9	8	e
k	R					
j	9	R				
9	k 8	8	R			
8	k R e	j e	e	R		
e	k 9 y	R j 9 j	9 N	d	R	
d	j k d e	3 9 y N	e8 R	k e 9	3	R

h #HH2; ?2 MmK#2` Q7 T `FBM; 7mMM+iBQMb...

S `FBM; 6mM+iBQMb rBi? hB2 am#b

a a a a a a M K	R	k	j	9	8	e
k	R					
j	9	R				
9	k8	8	R			
8	kRe	je	e	R		
e	k9yR	j9j	9N	d	R	
d	jkde	39yN	e8R	ke9	3	R

h #HH2; h2? 2 MmK#2` Q7 T `FBM; 7mMM+iBQMb bQm#b#2iMQ

Parking Functions with Tie Subsets

j \ h	1	2	3	4	5	6
2	1					
3	4	1				
4	25	5	1			
5	216	36	6	1		
6	2401	343	49	7	1	
7	32768	4096	512	64	8	1

Table: The number of parking functions of length j with tie subset of size h

The entries of this table are given by $(j+1)^{j-1-h}$.

S `FBM; 6mM+iBQMb rBi? :Bp2M hB2

[redacted]

6Q` M v [M A R- H2(i M A)# 2 i?2 b2i Q7 T `FBM; 7mM+
r?Qb2 iB2 b2i +QMi BMb A X h?2M-

$$jh(M,A) = (M+R)^{M R j A}.$$

Parking Functions with Given Tie Subsets

Theorem (JES)

Sn} 9j ´ ^ [j − 1]J ê0§ ¦ (j,^) @0 §\0 0§ nT q9}c_jX T«jH§_nj nT ê0jX§\ j ° \n 0 §_0 0§ Hnj§9_j ^u ¦ \0jJ

$$|¦(j,^)| = (j+1)^{j-1-|^|}.$$

p}nnT « _jX p}ʁT0} GnL0I

- The p}ʁT0} GnL0 of parking function $\alpha = (9_1, \ldots, 9_j)$ is $\rho = (9_2 - 9_1, 9_3 - 9_2, \ldots, 9_j - 9_{j-1})$.
- A "0" in the Prüfer Code means a tie in the parking function.
- Each entry of the Prüfer Code has $j+1$ possible values.

Parking Functions with Consecutive Descent/Ascent Subsets

Theorem (JES)

Sn} 9j ´ ^ = {_, _ + 1, . . . , _ + h − 1} [j − 1]J §\0 j«h@0} nT q9}c_jX T«jH§_nj ° \n 0 L0 H0j§ 0§ Hnj§9_j ^ _

$$K(j,\wedge) = \binom{j+1}{|\wedge|+1}(j+1)^{j-|\wedge|-2}.$$

S `FBM; 6mM+iBQMb. aBib ìBBM+;i rdBiHm 2

[redacted]

h?2 MmK#2` Q7 T `FBM; 7mM+iBQMb Q7 H2M;i? p Hm2b Bb

F j.(M[F R)j:

lMBi AMi2`p H S `FBM

. 2}MBiBQMb

G2i=(R::::; M) #2 T `FBM; 7mM+iBQM XBM7b+T2Q

+ HbH B i?2/BbTH +2QK2MiX`

.2}MBiBQMb

G2i=(R::::; M) #2 T `FBM; 7mM+iBQMb XB AM7Q`K b +2`iB +HbH B i?2/BbTH +2K2Mi X

mMBi BMi2`p H T `FBM; 7mM+iBQM =(R k::::; M) r?2`2 i?2 BM/BpB/m H /BbTH +2K2Mi KQbi RX q2 `272` iQ i?2 b2i Q7 mMBi BMi2`p H2MM;i?b lS6M X

.2}MBiBQMb

G2i =(R;:::; M) #2 T `FBM; 7mM+iBQMXBiBb ++HH2/ BbTH +2K2Mi BbTH +2K2Mi i?2 /BbTH +2K2MiX

mMBi BMi2`p H T `FBM; 7mM+iBQM =(R k;:::; M) r?2`2 i?2 BM/BpB/m H /BbTH +2 KQbi RX q2 `272` iQ i?2 b2i Q7 mMBi BMi2`p H 2MM;M ?b IS6MX

1t KTH2Qb *QMbB/2` =(R;R;k;j) 2 IS69X

*R	*k	*j	*9
●	●	●	●
R	k	j	9

Aa Ah lLAh ALh1_o G\

Aa Ah lLAh ALh1_o G\

$$=(\mathbb{R}, \mathbb{R}, k)$$

Aa Ah lLAh ALh1_o G\

=(R, R, k) u1a55

Aa Ah lLAh ALh1_o G\

=(RRk) u1a55

=(kRR)

Aa Ah lLAh ALh1_o G\

=(R, R, k) u1a55

=(k, R, R) LP55

Aa Ah lLAh ALh1_o G\

=(ŖŖķ) u1a55

=(ķŖŖ) LP55

=(Ŗķ Ŗj)

Aa Ah lLAh ALh1_o G\

=(RRk) u1a55

=(kRR) LP55

=(RkRj) LP55

Aa Ah lLAh ALh1_o G\

=(R, R, k) u1a55

=(k, R, R) LP55

=(R, k, R, j) LP55

=(9, R, R, k)

IS IT A UNIT INTERVAL?

- $\alpha = (1, 1, 2)$ YES!!
- $\alpha = (2, 1, 1)$ NO!!
- $\alpha = (1, 2, 1, 3)$ NO!!
- $\alpha = (4, 1, 1, 2)$ YES!!

6m#BMB LmK#2`b

S H v

LmK#2` Q7 lMBi AMi2`p H S `FBM; 6

M	R	k	j	9	8	e
jS6jM	R	j	Re	Rk	8RkNe	Re3yd
jlS6jM	R	j	Rj	d8	89R	9e3j

h #Hh2?;2 MmK#2` Q7 mMBi BMi2`p H T `FBM; 7r

LmK#2` Q7 lMBi AMi2`p H S `FBM; 6

M	R	k	j	9	8	e
jS6jM	R	j	Re	Rk	8RkN	eRe3yd
jlS6jM	R	j	Rj	d8	89R	9e3j

h #Hh2;?2 MmK#2` Q7 mMBi BMi2`p H T `FBM; 7r

.2}MBiBQMb

6m#BMB `BMbFBbM2;[m2(M#$_R$2:;#$_M$) i? i `2T`2b2Mi
r vMTH v2`b + M ` MF BM +QKT2iBiBQM- H
6$_M$ b i?2 b2i Q7 HH 6m#BMBMX MUF>B/Mr, bv-Qk7yHk

.2}MBiBQMb

6m#BMB `BMbF BbM2;[m2(M#R2:;#M) i? i `2T`2b2Mi
r vMTH v2`b + M ` MF BM +QKT2iBiBQM- H
6_M b i?2 b2i Q7 HH 6m#BMBMX MUF>B/Mr, bv-Qk7ykk
h?2 6m#BMB Mm`K2#/2}bM2/ M+b j6#_M iX

.2}MBiBQMb

6m#BMB `BM; bM2[m2M+2 (#R, :::, #M) i? i `2T`2b2Mi r vMTH v2`b + M ` MF BM +QKT2iBiBQM- H
6_M b i?2 b2i Q7 HH 6m#BMB `BM; bM2[m2M+2b Q7 H2M;i? M X
h?2 6m#BMB Mm`K# 2` }bM2/ b j6#_M X
LQi2 i?i 7Q` M@im TH2 (#R, :::, #M) 2 [M]M Bb 6m#BMB `BM; bM2[m2M+2 B7 M/ QMHv B7
H2M;i? 7Q` i?2 7QHHQrBM; 2[mBH27 r2 ?p2 bQ`i2/ M2ti H `;2bi #B
2[m HBiQ? 2M i?2 M2ti H `;2bi bm##2 BM

Aa Ah 6l"ALA _ LEAL:\

Aa Ah 6l"ALA _ LEAL:\

=(R, R, k)

Aa Ah 6l"ALA _ LEAL:\

=(R, R, k) LP55

Aa Ah 6l"ALA _ LEAL:\

=(R, R, k) LP55

=(R, k, k)

Aa Ah 6l"ALA _ LEAL:\

=(R, R, k) LP55

=(R, k, k) u1a55

Aa Ah 6l"ALA _ LEAL:\

=(R,R,k) LP55

=(R,k,k) u1a55

=(R,R,Rj)

Aa Ah 6l"ALA _ LEAL:\

=(ŖŖķ) LP55

=(Ŗķ ķ) u1a55

=(ŖŖŖj) LP55

Aa Ah 6l"ALA _ LEAL:\

=(ŖŖķ) LP55

=(Ŗķ ķ) u1a55

=(ŖŖŖj) LP55

=(9ŖŖj)

Aa Ah 6l"ALA _ LEAL:\

=(R̦R̦k̦) LP55

=(R̦k̦k̦) u1a55

=(R̦R̦R̦j) LP55

=(9̦R̦R̦j) u1a55

.2}MBiBQMb

M`@6m#BMBB`bMF6mM#BMB ` MFBM; î?/Bib#i2;2`Mb
p Hm2bX q2 /2`MQi2 i?2 b2i Q7 HH 6m#BMBB`bMF6mM#BMB ` MFBM; 6mM+iBQMb X M
LQi2 6, $\frac{B}{M}$ 6 $\frac{D}{M}$ 7Q` B> DX

.2}MBiBQMb

M`@ 6m#BMBB`bMF 6mM#BMB ` MFBM; î?/Bib#i2;MB
p Hm2bX q2 /2`M bQii?2 6b_2i Q7 HH 6m#BMBMX M
LQi2 6, B_M 6 D_M 7QB` DX

h?2`@ 6m#BMB M`2 mK2#}2M`2 / `_M + j6#_M`i X

.2}MBiBQMb

M`@6m#BMBB`bMF6mM#;BMB ` MFBM; î?/BibMi2B;M
p Hm2bX q2 /2`MbQii?226b_2i Q7 HH 6m#BMBMX M
LQi26,B_M 6 D_M 7QB> DX

h?2`@6m#BMB M`2nK2#}2M`2/ `b j6#_MiX

1t KTH2=,(R,R,j) Bb R@6m#BMB =(MR,R,j) 9 ;BbM /
9@6m#BMB ` MFBM;X

Established Results

ª j_§ _j§O}®9e q9}c_jX T«jH§_nj 9}O _j @_aOH§_nj °_§\I

- Fubini rankings (Hadaway, 2022).
- Dyck paths with height at most 2 (Baril, Kirgizov 9jL Petrossian, 2018).

6m#BMB _ MFBM; P#b2`p iBQMb

[redacted]

6m#BMB ` MFBM;b `2 T2`Kmi iBQM BMp `B M [redacted]

6m#BMB _ MFBM; P#b2`p iBQMb

[redacted]

6m#BMB ` MFBM;b `2 T2`Kmi iBQM BMp `B M

[redacted]

6m#BMB ` MFBM;b `2 bm#b2i Q7 T `FBM; 7m

6m`i?2`KQ`2-Qi?2`m``2M+B2b Q7 6MT `Fb BMbFT QRiX

Fubini Ranking Observations

Lemma (MSRI-UP 2021, JES 2022)

S«@_j_ }9jc_jX 9}0 q0}h«§9§_nj _j®9}_9j§u

Lemma (MSRI-UP 2021, JES 2022)

S«@_j_ }9jc_jX 9}0 9 «@ 0§ nT q9}c_jX T«jH§_nj u

Furthermore, the c^{th} occurrence of any _ in β FR$_j$ parks in spot _ $+ c - 1$.
N²I

$$\begin{array}{cccccccc} 4 & 3 & 1 & 4 & 8 & 4 & 4 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 & 5 & 8 & 6 & 7 & 2 \end{array}$$

6m#BMB _ MFBM; iQ lMBi AMi2`pH

[redacted]

. 2}M2: 6 _M 7! lS 6M b 7QHHQrb= (: #B:p:2; #M) 2 6 _M

( ) = (R:::; M) 2 lS 6M r?2`2

B = { #B B7 BM?2b i?2 }`bi Q` b2+QM/ #B++m ; #B+ F k B7 BM?/2t i?2 F2i? Q++m``2M+2 #B(+F2 k) Q7 }

6m#BMB _ MFBM; iQ lMBi AMi2`p H

[redacted]

.2}M2: 6_M 7! lS 6M b 7QHHQrb= (: B R p:2;M) 2 6_M- ( )=(R:::; M) 2 lS 6M- r?2`2

B= #B B7 BMi2`BQ` b2+QM/ Q++m`
#B+ F k B7 BMi2`BQ` Q++m``2M+2 Q7

1t,

9 j R 9 39 9 R

6m#BMB _ MFBM; iQ lMBi AMi2`p H

[redacted]

. 2}M2: 6 _M7! lS6M b 7QHHQr b= (:BR p:2; M) 2 6 _M-
( ) = (R:::; M) 2 lS6M- r?2`2

$$B = \begin{cases} \#_B & B7 BMB?2 b i?2 }`bi Q` b2+QM/ @+ + \\ \#_B + F k & B7 BM/2t i?2 i?2 Q++m``2M+2 Q7 \end{cases}$$

1t,

( )

9 j R 9 39 9 R 9 j R 9 38 e R

6m#BMB _ MFBM; iQ IMBi AMi2`p H

[redacted]

.2}M2: 6_M7! IS6M b 7QHHQrb=(:BR;p:2;M) 2 6_M-
( )=(R:::; M) 2 IS6M- r?2`2

(#B B7 BM?2b i?2 }`bi Q` b2+QM/ @+ + m
B= #B+ F k B7 BM/ 2t i?2 Q++m``2M+2 Q7
i?2 #B(F2 k)

1t,

( )

9 j R 9 39 9 R 9 j R 9 38 e R
\# \# \# \# \# \# \# \# \# \# \# \# \# \# \# \#
9 j R 8 3e d k 9 j R 8 3e d k

6m#BMB "BD2+iBQM 1t2`+Bb2b

2 6_8	( )2 lS6 8
Rkj98	
RRRRR	
R9R9R	
jR9R	8

6m#BMB "BD2+iBQM 1t2`+Bb2b

2 6_8	( )2 lS6_8
Rkj98	Rkj98
RRRRR	
R9R9R	
jR9R8	

6m#BMB "BD2+iBQM 1t2`+Bb2b

2 6_8	( )2 lS6_8
Rkj98	Rkj98
RRRRR	RRkj9
R9R9R	
jR9R8	

6m#BMB "BD2+iBQM 1t2`+Bb2b

2 6_8	( )2 lS6_8
Rkj98	Rkj98
RRRRR	RRkj9
R9R9R	R9R9k
jR9R8	

6m#BMB "BD2+iBQM 1t2`+Bb2b

$2\ 6_8$	$()2\ lS6_8$
Rkj98	Rkj98
RRRRR	RRkj9
R9R9R	R9R9k
jR9R8	jR9R8

lS6$_M$ 1MmK2` iBQM

[redacted]

h?2 MmK#2` Q7 mMBi BMi2`p H T `FBM;X7mM+

S`QQ7 aF2i+?, Q p 2B/b #BD2+iBQM$_M$ Q7 lS6$_M$ iQ Kb O_

$$j \, lS6_M i = j \, 6\_M \, i = 6\#_M$$

¯09ce´ ^jH}09 _jX¤K0H}09 _jX ª j_§
^j§0}®9e p9}c_jX S«jH§_nj

Weakly Increasing Unit Interval Parking Functions

j	$\lvert \text{WIUPF}_j \rvert$
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Table: The number of weakly increasing unit interval parking functions of length j

Weakly Increasing Unit Interval Parking Functions

j	$\lvert \mathrm{WIUPF}_j \rvert$
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Table: The number of weakly increasing unit interval parking functions of length j

2^{j-1} Play

q2 FHv AM+`2 bBM; IMBi AMi2`p H S

[redacted]

h?2 MmK#2` Q7 r2 FHv BM+`2 bBM; mMBi BMi2
M Bk^{M}b ^{R}X

q2 FBM; BM+`2 bBM; mMBi BMi2`p H T `FBM; 7

M= R, R

M= k, RR- Rk

M= j, RR-k RRj Rkk- Rkj

M= 9, RRkj- RRk9j- RRjk9kj- Rkk9- Rkjj- Rkj9

M= 8, RRkj9- RRk99- RRjj9- RRj99- Rkkj9- Rk
RRkj8- RRk98- RRjj8- RRj98- Rkkj8- Rkk98-

q2 FHv AM+`2 bBM; lMBi AMi2`p H S

[redacted]

h?2 MmK#2` Q7 r2 FHv BM+`2 bBM; mMBi BMi2
M Bk^M ^{R}X

q2 FBM; BM+`2 bBM; mMBi BMi2`p H T `FBM; 7

- M= R , R
- M= k , RR- Rk
- M= j , RR- k RRj- Rkk- Rkj
- M= 9 , RRkj- RRk9- RRjj- RRj9- Rkkj- Rkk9- Rkjj- Rkj9
- M= 8 , RRkj9- RRk99- RRjj9- RRj99- Rkkj9- R
  RRkj8- RRk98- RRjj8- RRj98- Rkkj8- Rkk98-

q2 FHv AM+`2 bBM; IMBi AMi2`p H S

[redacted]

h?2 MmK#2` Q7 r2 FHv BM+`2 bBM; mMBi BMi2
M Bkb ^M RX

q2 FBM; BM+`2 bBM; mMBi BMi2`p H T `FBM; 7

M= R, R

M= k, RR- Rk

M= j, RR RR-jRkk- Rkj

M= 9 RRkj- R RRjj- R-RR9kkj- Rkk9- Rkjj- Rkj9

M= 8, RRkj9- RRk99- RRjj9- RRj99- Rkkj9- R
RRkj8- RRk98- RRjj8- RRj98- Rkkj8- Rkk98-

Connections to Fubini Rankings

j	WIUPF_j	WIFR_j
1	1	1
2	11, 12	11, 12
3	112, 113, 122, 123	111, 113, 122, 123
4	1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234	1111, 1114, 1222, 1133, 1134, 1224, 1233, 1234

Table: WIUPF_j and WIFR_j

Weakly Decreasing Unit Interval Parking Functions

j	$\lvert \mathrm{WDUPF}_j \rvert$
1	1
2	2
3	3
4	5
5	8
6	13
7	21

Table: Weakly decreasing unit interval parking functions of length j

Weakly Decreasing Unit Interval Parking Functions

j	$\lvert \mathrm{WDUPF}_j \rvert$
1	1
2	2
3	3
4	5
5	8
6	13
7	21

Table: Weakly decreasing unit interval parking functions of length j

Fibonacci Play

Weakly Decreasing Unit Interval Parking Functions

Theorem (JES)

¦\0 j«h@0} nT °09ce´ L0H}09 _jX «j_§ _j§0}®9e q9}c_jX T«jH§_nj nT e0jX§\ j _ S_{j+1}.
rS_j _ §\0 S_@nj9HH_ 0x«0jH0I $S_0 = 0$, $S_1 = 1$, $S_j = S_{j-1} + S_{j-2}$.s

¯09c_jX L0H}09 _jX «j_§ _j§0}®9e q9}c_jX T«jH§_nj I

- $j = 1$: 1
- $j = 2$: 11, 21
- $j = 3$: 221, 311, 321
- $j = 4$: 3311, 3321, 4221, 4311, 4321
- $j = 5$: 44221, 44311, 44321, 53311, 53321, 54221, 54311, 54321

q2 FHv .2+`2 bBM; lMBi AMi2`p H S

[redacted]

h?2 MmK#2` Q7 r2 FHv /2+`2 bBM; mMBi BMi2`p
M Bb $6_{M R}$
U6Bb i?2 6B#QM + +B=by2 6m=2RM6+2 ,6 $6_{M R}$ + $6_{M k}$: V

q2 FBM; /2+`2 bBM; mMBi BMi2`p H T `FBM; 7r

M= R, R

M= k, RR- kR

M= j, kkR- jRR- jkR

M= 9, jjRR- jjkR- 9kkR- 9jRR- 9jkR

M= 8, 99kkR- 99jRR- 89jkR- 8jjkR- 89kkR- 89j

q2 FHv .2+`2 bBM; lMBi AMi2`p H S

[redacted]

h?2 MmK#2` Q7 r2 FHv /2+`2 bBM; mMBi BMi2`p
M Bb$_M$ $_R$6
U$_M$6Bb i?2 6B#QM + + B$_y$=by2 6m[R=2RM6$_M$=2 ,66$_M$ $_R$+ 6$_M$ $_k$: V

q2 FBM; /2+`2 bBM; mMBi BMi2`p H T `FBM; 7r

M= R, R

M= k, RR- kR

M= j, kkR- jRR- jkR

M= 9 jjRR- jjkR- 9kkR- 9jRR- 9jkR

M= 8 99kkR- 99jRR 8jjRR- 8jjkR- 89kkR- 89jR

*QMM2+iBQMb iQ 6m#BMB _ MFBM;

M	q.lS6$_M$	q.6_$_M$
R	R	R
k	RR- kR	RR- kR
j	kkR- jRR- jkR	RR+RkkR- jRR- jkR
9	jjRR- jjkR- 9kkR- 9 9jkR	RR;9RR;kRk kR+jjRR- jjkR- 9kkR- 9jRR- 9

h #H2., q.lS6$_M$ M/ q.6$_M$

r]S«@_j_ i«h@0}

}-Fubini Rankings

Definition

The j-tuple $\beta = (@_1, \ldots, @_j)$ $[j]^j$ is an }]S«@_j_ }9jc_jX of length j if β is a Fubini ranking starting with } distinct values, we denote $\mathrm{FR}_j^{}}$.

N²9hqe0I $\beta = (1, 2, 2)$ FR_3^2

}-Fubini Bijection

Theorem

¦ \0 j«h@0} «j_§ _j§0}®9e q9}c_jX T«jH§_nj nT e0jX§\ j §9}§_jX °_§\ }
L_ §_jH§ ®9e«0 _ S@$^{}_{j}$ u

p}nnT ~c0§H\I The bijection $\varphi : \mathrm{FR}_j \to \mathrm{UPF}_j$ maintains the maximal number of distinct starting values.

β	FR_4	$\varphi(\beta)$	UPF_4
	1234		1234
	123 3		123 3
	14 11		14 12
	12 22		12 23

So $|\mathrm{UPF}^{}_j| = |\mathrm{FR}^{}_j| = \mathrm{Fb}^{}_j$.

"BD2+iBQM iQ lMBi AMi2`p H S `FBM hB2 am#b2ib

[redacted]

1 +? Q7 i?2 7QHHQrBM; Bb 2[m H iQ 6#

- lS6
- h?2 MmK#2` Q7 mMBi BMi2`p H T `FBM;b r7Bi? iB2 bm#b2i; k` kg:
- h?2 MmK#2` Q7 mMBi BMi2`p H T `FBM;b r7Bi? bm#b2i; k` Rg:

Add Tie Algorithm

w«0 §_njI Given a c $[j]$, α UPF_j^c, can we find a unique α UPF_{j+1}^c with an added tie at index c?

//hB2 H;Q`Bi?K

[redacted]

.2}M2 (; F): IS6$_M^F$ [M7! IS6$_{M+R}^F$ b (; F) = (0:::; 0: 0)- r?2

0 = B

B BB< F M/B< F- QB= F- QB= F+ R
B+ R BB7 F M/B F
B R BB> F+ R M/B R< F
B R+ R BB7 F+ R M/B R F :

6Q` M2v IS6$_M^F$ (; F) 2 IS6$_{M+R}^F$:

//hB2 H;Q`Bi?K

[redacted]

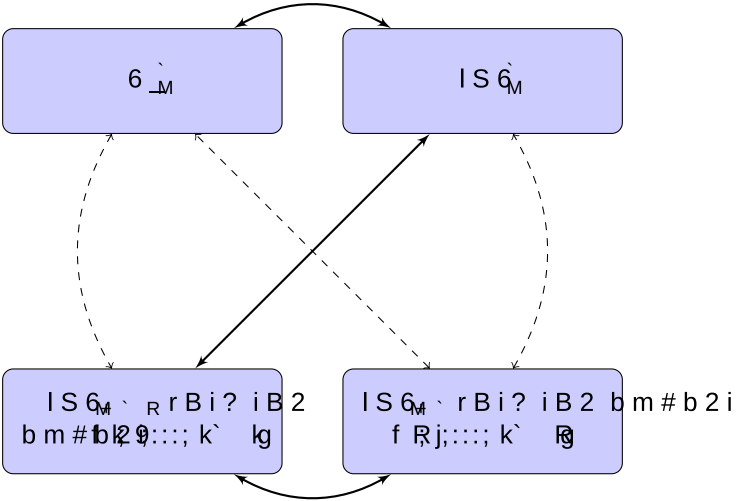
.2}M2 (; F) : IS6$_M^F$ [M7! IS6$_{M+R}^F$ b (; F) = (0$_R$: : : ; 0$_M$; 0$_{M+R}$) - r?2

0$_B$ = { B : BB< F M / B< F - QB= F - QB= F+ R ; B+ R : BB7 F M / B F ; B R : BB> F+ R M / B R< F ; B R+ R : BB7 F+ R M / B R F

6Q` M2v IS6$_M^F$ (; F) 2 IS6$_{M+R}^F$:

1t KTH2= (8; R; j; j; k) 2 IS6$_8^i$ 7! (; j) = (e; R; j; j; 9; k) 2 IS6$_e^i$

"BD2+iBQMb amKK `v



6mim`2 .B`2+iBQMb

6BM/BM; #BD2+iBQM #2ir22M /2b+2Mi b2i

lbBM; BM+HmbBQMf2t+HmbBQM T`BM+BTH2
7Q`KmH b 7`QK bm#b2i 7Q`KmH b

1MmK2` iBM; T `FBM; 7mM+iBQM rBi? ;Bp2

M HvxBM; i?2b2 b`i@iBMb2i2`p H 7Q``FBM; 7mM+

h? MFb 7Q` HBbi2

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Sequence Music

Sound example

NARAYANA Play

FUBINI Play

POWER OF 2 Play

FIBONACCI Play