Parking Functions with Fixed Ascent and Descent Sets

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Our Research Focus

**Theme:** Statistics in parking functions and unit interval parking functions — ascents, descents, ties.

This research is inspired by

1. Billey et al. (2013): studied permutations with a given peak set
2. Schumacher (2018): enumerated parking functions with $k$ descents and $i$ ties
Definitions

Given a parking function $\alpha = (a_1, a_2, \ldots, a_n)$:

- An **ascent** is defined as an index $i$ such that $a_i < a_{i+1}$.
- A **descent** is defined as an index $i$ such that $a_i > a_{i+1}$.
- A **tie** is defined as an index $i$ such that $a_i = a_{i+1}$.

**Example:**

$\alpha = 132441$

$\alpha = 132441$

$\alpha = 132441$
Definitions

Given a parking function $\alpha = (a_1, a_2, \ldots, a_n)$:

- The **ascent set** is defined as the set $I$ with all ascents of $\alpha$.
- The **descent set** is defined as the set $I$ with all descents of $\alpha$.
- The **tie set** is defined as the set $I$ with all ties of $\alpha$.
- A **ascent/descent/tie subset** is defined as any subset of $\alpha$’s ascent/descent/tie set.

**Example:** $\alpha = 132441$

- Ascent set: $\{1, 3\}$
- Descent set: $\{2, 5\}$
- Tie set: $\{4\}$
Classical Parking Functions
Established Results

**Known Enumerations:**

- Parking functions: \((n + 1)^{n-1}\) (Riordan, 1969)
- Weakly increasing/decreasing parking functions: Catalan (Stanley, 1999)
- Parking functions with \(k\) ties (Yan, 2015)
- Parking functions with \(k\) descents and \(i\) ties (Schumacher, 2018)
Descent-Ascent Symmetry Theorem
Introduction to the Descent-Ascent Symmetry Theorem

**Definition:** Given a set \( I = \{i_1, \ldots, i_k\} \subseteq [n-1], \) let \( I^{-1} := \{n-i_1, \ldots, n-i_k\}. \)
Introduction to the Descent-Ascent Symmetry Theorem

**Definition:** Given a set $I = \{i_1, \ldots, i_k\} \subseteq [n-1]$, let $I^{-1} := \{n-i_1, \ldots, n-i_k\}$.

**Example:** $n = 5$, $I = \{1, 2\} \Rightarrow I^{-1} =$
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<thead>
<tr>
<th>$l$</th>
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<td>161</td>
</tr>
<tr>
<td>${3,4}$</td>
<td>56</td>
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</tbody>
</table>

**Table:** The number of parking functions of length $n$ with the descent/ascent sets $l$.
Descent-Ascent Symmetry Theorem (JES)

- PFs with descent set $I$
- PFs with descent set $I^{-1}$
- PFs with ascent set $I$
- PFs with ascent set $I^{-1}$
Descent-Ascent Inverse Equality

**Theorem (Descent-Ascent Symmetry Theorem — JES)**

The number of parking functions with descent set \( I = \{i_1, i_2, \ldots \} \) is equal to the number of parking functions with ascent set \( I^{-1} = \{n - i_1, n - i_2, \ldots \} \).

Proof idea: bijection with reverse

\[
\alpha = (a_1, a_2, \ldots, a_n) = 142345 \\
\uparrow \\
\alpha' = (a_n, a_{n-1}, \ldots, a_1) = 543241
\]

For each descent \( i \in I \), \( n - i \) is in the ascent set of \( \alpha' \).
Descent-Ascent Symmetry Theorem (JES)

PFs with descent set $I$

PFs with ascent set $I^{-1}$

PFs with descent set $I$

PFs with ascent set $I^{-1}$
Descent-Ascent Equality

**Theorem (JES)**

*The number of parking functions with ascent set $I$ is equal to the number of parking functions with descent set $I$.*

Proven by strong induction.
Descent-Ascent Symmetry Theorem (JES)

PFs with
descent set $I$

PFs with
descent set $I^{-1}$

PFs with
ascent set $I$

PFs with
ascent set $I^{-1}$
Descent-Ascent Symmetry Theorem (JES)
Open Exercise!!

Exercise

The bijection between parking functions with descent set $I$ and parking functions with descent set $I^{-1}$ or a direct proof that these sets of parking functions are equinumerous.
Connection to Narayana Numbers!
Connection to Narayana Numbers!
Parking Functions with Descent Set \{1, \ldots, k\}

**Theorem (JES)**

The number of parking functions of length \(n\) with descent set \(\{1, \ldots, k\}\) is

\[
\sum_{i=1}^{n} \frac{1}{n} \binom{n}{i} \binom{n}{i-1} \binom{i-1}{k}.
\]
Parking Functions with Descent Set \( \{1, \ldots, k\} \)

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**Proof Sketch:**

- Weakly increasing parking functions of length \( n \) with \( i \) distinct values — Narayana numbers (Stanley, 1999).
Parking Functions with Descent Set \{1, \ldots, k\}

**Theorem (JES)**

The number of parking functions of length \(n\) with descent set \(\{1, \ldots, k\}\) is

\[
\sum_{i=1}^{n} \frac{1}{n} \binom{n}{i} \binom{n}{i-1} \binom{i-1}{k}.
\]

**Proof Sketch:**

- Weakly increasing parking functions of length \(n\) with \(i\) distinct values — Narayana numbers (Stanley, 1999).
- Choose any \(k\) of the \(i-1\) distinct values greater than 1 to move to the front in decreasing order.
Conjecture

Parking functions of length $n$ with descent subset $\{1, \ldots, \hat{k}, \ldots, n-1\}$ are counted by the Narayana number

$$N(n+1, k+1) = \frac{1}{n+1} \binom{n+1}{k+1} \binom{n+1}{k}.$$
Parking functions with given ascent/descent/tie subsets
Parking Functions with Tie Subsets

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<th>$m$</th>
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**Table:** The number of parking functions of length $n$ with tie subset of size $m$
Parking Functions with Tie Subsets

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<th>(n)</th>
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**Table:** The number of parking functions of length \(n\) with tie subset of size \(m\)
Parking Functions with Tie Subsets

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<th>$n$</th>
<th>1</th>
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</tbody>
</table>

Table: The number of parking functions of length $n$ with tie subset of size $m$

The entries of this table are given by $(n + 1)^{n-1-m}$. 
Parking Functions with Given Tie Subsets

**Theorem (JES)**

For any $I \subseteq [n-1]$, let $T(n, I)$ be the set of parking functions of length $n$ whose tie set contains $I$. Then,

$$|T(n, I)| = (n + 1)^{n-1-|I|}.$$
Theorem (JES)

For any $I \subseteq [n-1]$, let $T(n, I)$ be the set of parking functions of length $n$ whose tie set contains $I$. Then,

$$|T(n, I)| = (n + 1)^{n-1-|I|}.$$

Proof using Prüfer Code:

- The Prüfer Code of parking function $\alpha = (a_1, \ldots, a_n)$ is $\rho = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1})$.
- A “0” in the Prüfer Code means a tie in the parking function.
- Each entry of the Prüfer Code has $n + 1$ possible values.
Parking Functions with Consecutive Descent/Ascent Subsets

**Theorem (JES)**

For any $I = \{i, i+1, \ldots, i+m-1\} \subseteq [n-1]$, the number of parking functions whose descent set contains $I$ is

$$D(n, I) = \binom{n+1}{|I|+1}(n+1)^{n-|I|-2}.$$
Parking Functions Starting with $k$ Distinct Values

**Theorem (JES)**

The number of parking functions of length $n$ that start with $k$ distinct values is

$$k! \cdot |D(n, [k - 1])|.$$
Unit Interval Parking Functions
Definitions

- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If car $i$ parks in spot $s_i$, we call $s_i - a_i$ the displacement of car $i$. 

### Example
Consider $\alpha = (1, 1, 2, 3) \in \text{UPF}_4$. 

\begin{table}
\begin{tabular}{cccc}
\hline
C1 & C2 & C3 & C4 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{tabular}
\end{table}
Definitions

- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If car $i$ parks in spot $s_i$, we call $s_i - a_i$ the displacement of car $i$.

- A unit interval parking function is a parking function $\alpha = (a_1, a_2, \ldots, a_n)$ where the individual displacement of each car is at most 1. We refer to the set of unit interval parking functions of length $n$ as $\text{UPF}_n$. 
Definitions

- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If car $i$ parks in spot $s_i$, we call $s_i - a_i$ the displacement of car $i$.

- A **unit interval parking function** is a parking function $\alpha = (a_1, a_2, \ldots, a_n)$ where the individual displacement of each car is at most 1. We refer to the set of unit interval parking functions of length $n$ as $\text{UPF}_n$.

**Example:** Consider $\alpha = (1, 1, 2, 3) \in \text{UPF}_4$. 

<table>
<thead>
<tr>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
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<tr>
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<td>3</td>
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</tr>
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IS IT A UNIT INTERVAL?

\[ \alpha = (1,1,2) \]
YES!!

\[ \alpha = (2,1,1) \]
NO!!

\[ \alpha = (1,2,1,3) \]
NO!!

\[ \alpha = (4,1,1,2) \]
YES!!
IS IT A UNIT INTERVAL?

- $\alpha = (1, 1, 2)$
IS IT A UNIT INTERVAL?

- $\alpha = (1,1,2)$  YES!! 😊
IS IT A UNIT INTERVAL?

- $\alpha = (1,1,2)$  YES!! 😊
- $\alpha = (2,1,1)$
IS IT A UNIT INTERVAL?

- $\alpha = (1,1,2)$ YES!! 😊
- $\alpha = (2,1,1)$ NO!! 👻
IS IT A UNIT INTERVAL?

- $\alpha = (1, 1, 2)$  YES!! 😊
- $\alpha = (2, 1, 1)$  NO!! 👻
- $\alpha = (1, 2, 1, 3)$
IS IT A UNIT INTERVAL?

- $\alpha = (1,1,2)$ YES!! 😊
- $\alpha = (2,1,1)$ NO!! 💀
- $\alpha = (1,2,1,3)$ NO!! 😬
IS IT A UNIT INTERVAL?

- $\alpha = (1, 1, 2)$ YES!! 😊
- $\alpha = (2, 1, 1)$ NO!! 👻
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- $\alpha = (4, 1, 1, 2)$
IS IT A UNIT INTERVAL?

- $\alpha = (1,1,2)$ YES!! 😊
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- $\alpha = (1,2,1,3)$ NO!! 👻
- $\alpha = (4,1,1,2)$ YES!! 👑
Fubini Numbers!
Number of Unit Interval Parking Functions

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<th>$n$</th>
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<td>75</td>
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**Table:** The number of unit interval parking functions of length $n$
Number of Unit Interval Parking Functions

| $n$ | $|PF_n|$ | $|UPF_n|$ |
|-----|---------|---------|
| 1   | 1       | 1       |
| 2   | 3       | 3       |
| 3   | 16      | 13      |
| 4   | 125     | 75      |
| 5   | 1296    | 541     |
| 6   | 16807   | 4683    |

Table: The number of unit interval parking functions of length $n$
Definitions

- A **Fubini ranking** is a sequence $\beta = (b_1, \ldots, b_n)$ that represents one way $n$ players can rank in a competition, allowing ties. We denote $\text{FR}_n$ as the set of all Fubini rankings of length $n$. (Hadaway, 2022)
Definitions

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- The **Fubini numbers** are defined as $\text{Fb}_n = |\text{FR}_n|$. 
Definitions

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- The **Fubini numbers** are defined as \( \text{Fb}_n = |\text{FR}_n| \).
- **Note:** The \( n \)-tuple \( \beta = (b_1, \ldots, b_n) \in [n]^n \) is a **Fubini ranking** of length \( n \) if the following holds: For all \( i \in [n] \), if \( k \) entries of \( \beta \) are equal to \( i \), then the next largest value in \( \beta \) is \( i + k \).
IS IT A FUBINI RANKING?
IS IT A FUBINI RANKING?

- $\beta = (1,1,2)$
IS IT A FUBINI RANKING?

\[ \beta = (1,1,2) \quad \text{NO!!} \quad 😞 \]
IS IT A FUBINI RANKING?

- $\beta = (1,1,2)$ NO!! 😞
- $\beta = (1,2,2)$
IS IT A FUBINI RANKING?

- \( \beta = (1,1,2) \) NO!! 😞
- \( \beta = (1,2,2) \) YES!! 🌸
IS IT A FUBINI RANKING?

- $\beta = (1, 1, 2)$ NO!! 😞
- $\beta = (1, 2, 2)$ YES!! 🌸
- $\beta = (1, 1, 1, 3)$
IS IT A FUBINI RANKING?

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- $\beta = (1, 2, 2)$ YES!! 🌸
- $\beta = (1, 1, 1, 3)$ NO!! 😞
IS IT A FUBINI RANKING?

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- $\beta = (1,2,2)$ YES!! 🌸
- $\beta = (1,1,1,3)$ NO!! 😞
- $\beta = (4,1,1,3)$
IS IT A FUBINI RANKING?

- $\beta = (1,1,2)$  NO!! 😞
- $\beta = (1,2,2)$  YES!! 🌸
- $\beta = (1,1,1,3)$  NO!! 😢
- $\beta = (4,1,1,3)$  YES!! 🏆
## Definitions

- **r-Fubini ranking** is a Fubini ranking that begins with \( r \) distinct values. We denote \( FR_n^r \) as the set of all Fubini rankings of length \( n \). 

  *Note:* \( FR_n^i \subset FR_n^j \) for \( i > j \).
Definitions

- **An r-Fubini ranking** is a Fubini ranking that begins with $r$ distinct values. We denote $\text{FR}_n^r$ as the set of all Fubini rankings of length $n$. Note: $\text{FR}_n^i \subset \text{FR}_n^j$ for $i > j$.
- **The r-Fubini numbers** are defined as $Fb_n^r = |\text{FR}_n^r|$. 

Definitions
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  _Note:_ \( \text{FR}^i_n \subset \text{FR}^j_n \) for \( i > j \).

- **The r-Fubini numbers** are defined as \( \text{Fb}^r_n = |\text{FR}^r_n| \).

**Example:** \( \alpha = (1,1,3) \) is a 1-Fubini ranking and \( \beta = (1,2,3,4) \) is a 4-Fubini ranking.
Established Results

Unit interval parking functions are in bijection with:

- Fubini rankings (Hadaway, 2022).
- Dyck paths with height at most 2 (Baril, Kirgizov and Petrossian, 2018).
Fubini Ranking Observations

Lemma (MSRI-UP 2021, JES 2022)

Fubini rankings are permutation invariant.
Fubini Ranking Observations

Lemma (MSRI-UP 2021, JES 2022)

Fubini rankings are permutation invariant.

Lemma (MSRI-UP 2021, JES 2022)

Fubini rankings are a subset of parking functions.

Furthermore, the $k^{\text{th}}$ occurrence of any $i$ in $\beta \in \text{FR}_n$ parks in spot $i + k - 1$. 

4 3 1 4 8 4 4 1
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
4 3 1 5 8 6 7 2
Fubini Ranking Observations

Lemma (MSRI-UP 2021, JES 2022)

*Fubini rankings are permutation invariant.*

Lemma (MSRI-UP 2021, JES 2022)

*Fubini rankings are a subset of parking functions.*

Furthermore, the $k^{th}$ occurrence of any $i$ in $\beta \in \text{FR}_n$ parks in spot $i + k - 1$.

Ex:

```
4 3 1 4 8 4 4 1
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
4 3 1 5 8 6 7 2
```
Fubini Ranking to Unit Interval PF Bijection

Definition

Define $\phi : FR_n \mapsto UPF_n$ as follows: Given $\beta = (b_1, \ldots, b_n) \in FR_n$, $\phi(\beta) = (a_1, \ldots, a_n) \in UPF_n$, where

$$a_i = \begin{cases} b_i & \text{if index } i \text{ has the first or second occurrence of } b_i \\ b_i + k - 2 & \text{if index } i \text{ has the } k^{\text{th}} \text{ occurrence of } b_i \ (k > 2) \end{cases}$$
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**Definition**

Define $\phi : FR_n \mapsto UPF_n$ as follows: Given $\beta = (b_1, \ldots, b_n) \in FR_n$, $\phi(\beta) = (a_1, \ldots, a_n) \in UPF_n$, where

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**Ex:**

$$\beta = \begin{pmatrix} 4 & 3 & 1 & 4 & 8 & 4 & 4 & 1 \end{pmatrix}$$
Fubini Ranking to Unit Interval PF Bijection

**Definition**

Define \( \phi : \text{FR}_n \mapsto \text{UPF}_n \) as follows: Given \( \beta = (b_1, \ldots, b_n) \in \text{FR}_n \), \( \phi(\beta) = (a_1, \ldots, a_n) \in \text{UPF}_n \), where

\[
a_i = \begin{cases} 
  b_i & \text{if index } i \text{ has the first or second occurrence of } b_i \\
  b_i + k - 2 & \text{if index } i \text{ has the } k^{th} \text{ occurrence of } b_i (k > 2)
\end{cases}
\]

**Ex:**

\[
\begin{align*}
\beta & \\
4 & 3 & 1 & 4 & 8 & 4 & 4 & 1 \\
\phi(\beta) & \\
4 & 3 & 1 & 4 & 8 & 5 & 6 & 1
\end{align*}
\]
Fubini Ranking to Unit Interval PF Bijection

**Definition**

Define $\phi : FR_n \mapsto UPF_n$ as follows: Given $\beta = (b_1, \ldots, b_n) \in FR_n$, $\phi(\beta) = (a_1, \ldots, a_n) \in UPF_n$, where

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\end{cases}$$

**Ex:**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\phi(\beta)$</th>
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</thead>
<tbody>
<tr>
<td>4 3 1 4 8 4 4 1</td>
<td>4 3 1 4 8 5 6 1</td>
</tr>
<tr>
<td>↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓</td>
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</tbody>
</table>
## Fubini Bijection Exercises

\[ \beta \in \text{FR}_5 \quad \text{and} \quad \phi(\beta) \in \text{UPF}_5 \]

<table>
<thead>
<tr>
<th>\beta \in \text{FR}_5</th>
<th>\phi(\beta) \in \text{UPF}_5</th>
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<tbody>
<tr>
<td>12345</td>
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<tr>
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### Fubini Bijection Exercises

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**Theorem (JES)**

*The number of unit interval parking functions of length $n$ is $F_{b_n}$.*

**Proof Sketch:** We proved $\phi$ is a bijection from $\text{FR}_n$ to $\text{UPF}_n$, so

$$|\text{UPF}_n| = |\text{FR}_n| = F_{b_n}.$$
Weakly Increasing/Decreasing Unit Interval Parking Functions
Weakly Increasing Unit Interval Parking Functions

| $n$ | $|\text{WIUPF}_n|$ |
|-----|--------------------|
| 1   | 1                  |
| 2   | 2                  |
| 3   | 4                  |
| 4   | 8                  |
| 5   | 16                 |
| 6   | 32                 |
| 7   | 64                 |

**Table:** The number of weakly increasing unit interval parking functions of length $n$
Weakly Increasing Unit Interval Parking Functions

| $n$ | $|\text{WIUPF}_n|$ |
|-----|-----------------|
| 1   | 1               |
| 2   | 2               |
| 3   | 4               |
| 4   | 8               |
| 5   | 16              |
| 6   | 32              |
| 7   | 64              |

Table: The number of weakly increasing unit interval parking functions of length $n$

$2^{n-1} \quad \text{Play}$
Weakly Increasing Unit Interval Parking Functions

**Theorem (JES)**

*The number of weakly increasing unit interval parking functions of length $n$ is $2^{n-1}$.*

Weakening increasing unit interval parking functions:

- $n = 1$: 1
- $n = 2$: 11, 12
- $n = 3$: 112, 113, 122, 123
- $n = 4$: 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234
- $n = 5$: 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345
Weakly Increasing Unit Interval Parking Functions

Theorem (JES)

The number of weakly increasing unit interval parking functions of length \( n \) is \( 2^{n-1} \).

Weaking increasing unit interval parking functions:

- \( n = 1 \): 1
- \( n = 2 \): 11, 12
- \( n = 3 \): 112, 113, 122, 123
- \( n = 4 \): 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234
- \( n = 5 \): 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345
Weakly Increasing Unit Interval Parking Functions

Theorem (JES)

The number of weakly increasing unit interval parking functions of length $n$ is $2^{n-1}$.

Weaking increasing unit interval parking functions:

- $n = 1$: 1
- $n = 2$: 11, 12
- $n = 3$: 112, 113, 122, 123
- $n = 4$: 1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234
- $n = 5$: 11234, 11244, 11334, 11344, 12234, 12244, 12334, 12344, 11235, 11245, 11335, 11345, 12235, 12245, 12335, 12345
Connections to Fubini Rankings

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{WIUPF}_n$</th>
<th>$\text{WIFR}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>11, 12</td>
<td>11, 12</td>
</tr>
<tr>
<td>3</td>
<td>112, 113, 122, 123</td>
<td>111, 113, 122, 123</td>
</tr>
<tr>
<td>4</td>
<td>1123, 1124, 1133, 1134, 1223, 1224, 1233, 1234</td>
<td>1111, 1114, 1222, 1133, 1134, 1224, 1233, 1234</td>
</tr>
</tbody>
</table>

Table: $\text{WIUPF}_n$ and $\text{WIFR}_n$
Weakly Decreasing Unit Interval Parking Functions

| $n$ | $|\text{WDUPF}_n|$ |
|-----|------------------|
| 1   | 1                |
| 2   | 2                |
| 3   | 3                |
| 4   | 5                |
| 5   | 8                |
| 6   | 13               |
| 7   | 21               |

**Table:** Weakly decreasing unit interval parking functions of length $n$
Weakly Decreasing Unit Interval Parking Functions

| $n$ | $|\text{WDUPF}_n|$ |
|-----|------------------|
| 1   | 1                |
| 2   | 2                |
| 3   | 3                |
| 4   | 5                |
| 5   | 8                |
| 6   | 13               |
| 7   | 21               |

**Table:** Weakly decreasing unit interval parking functions of length $n$

Fibonacci Play
Weakly Decreasing Unit Interval Parking Functions

Theorem (JES)

The number of weakly decreasing unit interval parking functions of length $n$ is $F_{n+1}$.
($F_n$ is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Weaking decreasing unit interval parking functions:

- $n = 1$: 1
- $n = 2$: 11, 21
- $n = 3$: 221, 311, 321
- $n = 4$: 3311, 3321, 4221, 4311, 4321
- $n = 5$: 44221, 44311, 44321, 53311, 53321, 54221, 54311, 54321
Weakly Decreasing Unit Interval Parking Functions

**Theorem (JES)**

The number of weakly decreasing unit interval parking functions of length $n$ is $F_{n+1}$.

($F_n$ is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

**Weakly decreasing unit interval parking functions:**

- $n = 1$: 1
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($F_n$ is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Weakly decreasing unit interval parking functions:

- $n = 1$: 1
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### Connections to Fubini Rankings

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{WDUPF}_n$</th>
<th>$\text{WDFR}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>11, 21</td>
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<tr>
<td>3</td>
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<td>1111, 4111, 2221, 3311, 3321, 4221, 4311, 4321</td>
</tr>
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</table>

**Table:** $\text{WDUPF}_n$ and $\text{WDFR}_n$
$r$-Fubini Numbers
**Definition**

The $n$-tuple $\beta = (b_1, \ldots, b_n) \in [n]^n$ is an **$r$-Fubini ranking** of length $n$ if $\beta$ is a Fubini ranking starting with $r$ distinct values, we denote $\text{FR}^r_n$.

**Example:** $\beta = (1, 2, 2) \in \text{FR}^2_3$
r-Fubini Bijection

**Theorem**

*The number unit interval parking functions of length n starting with r distinct values is* $F_{b}^{r}_{n}$.

**Proof Sketch:** The bijection $\phi : FR_{n} \mapsto UPF_{n}$ maintains the maximal number of distinct starting values.

<table>
<thead>
<tr>
<th>$\beta \in FR_{4}$</th>
<th>$\phi(\beta) \in UPF_{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td>1234</td>
</tr>
<tr>
<td>123 3</td>
<td>123 3</td>
</tr>
<tr>
<td>14 11</td>
<td>14 12</td>
</tr>
<tr>
<td>12 22</td>
<td>12 23</td>
</tr>
</tbody>
</table>

So $|UPF_{n}^{r}| = |FR_{n}^{r}| = F_{b}^{r}_{n}$. 
Bijection to Unit Interval Parking Functions with Certain Tie Subsets

Theorem (JES)

Each of the following is equal to $F_{b,n}^r$:

1. $UPF_{n}^r$
2. The number of unit interval parking functions of length $n + r - 1$ with tie subset $\{2, 4, \ldots, 2r - 2\}$.
3. The number of unit interval parking functions of length $n + r$ with tie subset $\{1, 3, \ldots, 2r - 1\}$.
Add Tie Algorithm

**Question:** Given a $k \in [n]$, $\alpha \in \text{UPF}_n^k$, can we find a unique $\alpha' \in \text{UPF}_{n+1}^k$ with an added tie at index $k$?
Add Tie Algorithm

Definition (JES)

Define \( \lambda(\alpha, k) : \text{UPF}_{n}^{k}, [n] \mapsto \text{UPF}_{n+1}^{k} \) as \( \lambda(\alpha, k) = (a'_1, \ldots, a'_n, a'_{n+1}) \), where

\[
a'_i = \begin{cases} 
  a_i & \text{if } i < k \text{ and } a_i < a_k, \text{ or } i = k, \text{ or } i = k + 1 \\
  a_i + 1 & \text{if } i < k \text{ and } a_i \geq a_k \\
  a_{i-1} & \text{if } i > k + 1 \text{ and } a_{i-1} < a_k \\
  a_{i-1} + 1 & \text{if } i > k + 1 \text{ and } a_{i-1} \geq a_k
\end{cases}
\]

For any \( \alpha \in \text{UPF}_{n}^{k} \), \( \lambda(\alpha, k) \in \text{UPF}_{n+1}^{k} \).
Add Tie Algorithm

**Definition (JES)**

Define $\lambda(\alpha, k) : \text{UPF}_n^k, [n] \mapsto \text{UPF}_{n+1}^k$ as $\lambda(\alpha, k) = (a'_1, \ldots, a'_n, a'_{n+1})$, where

$$a'_i = \begin{cases} 
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  a_{i-1} + 1 & \text{if } i > k + 1 \text{ and } a_{i-1} \geq a_k 
\end{cases}.$$

For any $\alpha \in \text{UPF}_n^k$, $\lambda(\alpha, k) \in \text{UPF}_{n+1}^k$.

**Example:** $\alpha = (5, 1, 3, 3, 2) \in \text{UPF}_5^3 \mapsto \lambda(\alpha, 3) = (6, 1, 3, 3, 4, 2) \in \text{UPF}_6^3$
Bijections Summary

\[ \text{FR}_n^r \leftrightarrow \text{UPF}_n^r \]

\[ \text{UPF}_{n+r-1}^r \text{ with tie subset } \{2,4,\ldots,2r-2\} \]

\[ \text{UPF}_{n+r}^r \text{ with tie subset } \{1,3,\ldots,2r-1\} \]
Future Directions

- Finding a bijection between descent set and inverse descent set
- Using inclusion/exclusion principle to get ascent/descent/tie set formulas from subset formulas
- Enumerating parking function with a given peak or valley set
- Analyzing these statistics for $\ell$-interval parking functions
Thanks for listening!
References I


Sequence Music
Sound example

NARAYANA Play
FUBINI Play
POWER OF 2 Play
FIBONACCI Play