Out of the Parking Lot and into the Forest: Parking Functions, Bond Lattices, and Unimodal Forests

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Summer @ ICERM

August 3, 2022

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└─ Map / Outline

Outline

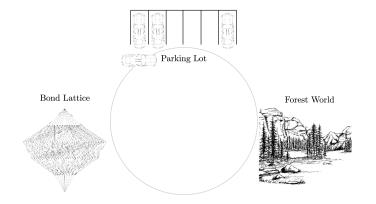


Figure 1: The Domains

Posets, Bond Lattices, and Their Bijections
Introduction & Research Questions
Map / Outline

Outline

- *i* Triangulation graphs
- ii. Partitions and merge graphs

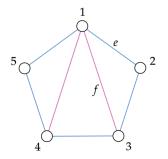
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- iii. The forest world
- iv. Merge chains
- v. The Ψ bijection

-Triangulation Graphs

Graph Terminology

- 1. G = (V, E)
- 2. Inner edges
- 3. Outer edges
- 4. Cycle graph



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Triangulation Graphs

Graphs with Inner Edges

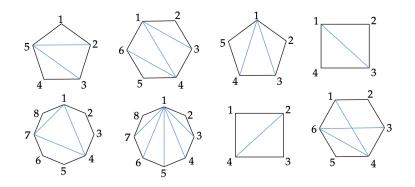


Figure 2: Various graphs that contain inner edges.

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Triangulation Graphs

Triangulation Graphs

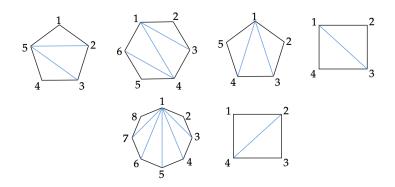


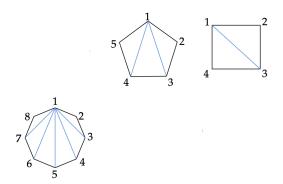
Figure 3: A few triangulation graphs

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Triangulation Graphs

Spider Triangulation Graphs



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Figure 4: A refinement of Figure 3 to only Spider Triangulations

Triangulation Graphs

(more) Graph Terminology



1. Induced subgraph

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- 2. Connected
- 3. Tree

Set Partitions and Merge Graphs

Bond Lattice

Recall: Noncrossing Set Partitions

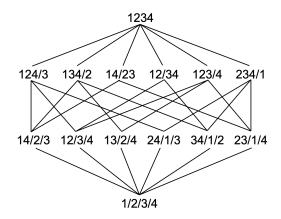


Figure 5: The noncrossing partition lattice NC₄

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Set Partitions and Merge Graphs

Bond Lattice

Introducing: the bond lattice

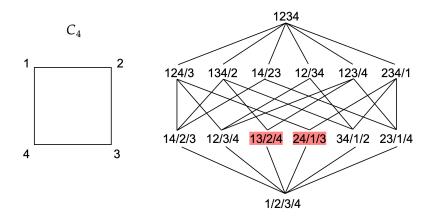


Figure 6: The noncrossing partition lattice NC_4

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Set Partitions and Merge Graphs

Bond Lattice

Introducing: the bond lattice

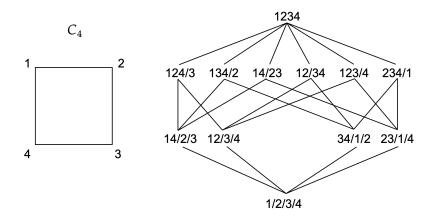


Figure 7: The bond lattice of C_4

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Set Partitions and Merge Graphs

Bond Lattice

Example: another bond lattice

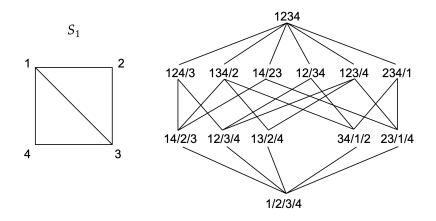


Figure 8: The bond lattice of a triangulation of a square

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Set Partitions and Merge Graphs

-Bond Lattice

Introducing: Merge Graphs

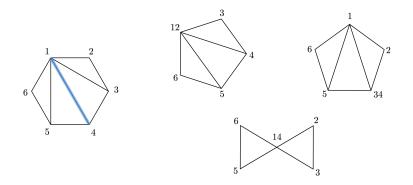


Figure 9: Merge Graph Possibilities

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Set Partitions and Merge Graphs

Bond Lattice

The Bond Lattice of the Spider Hexagon

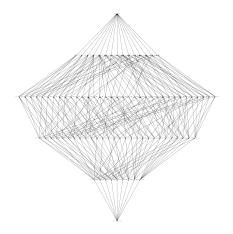


Figure 10: The Bond Lattice of the Hexagon 1

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Set Partitions and Merge Graphs

Bond Lattice

Merge Chain for Parking Function (3, 1, 5, 4)

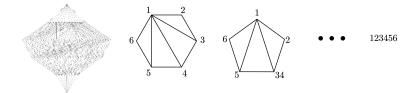


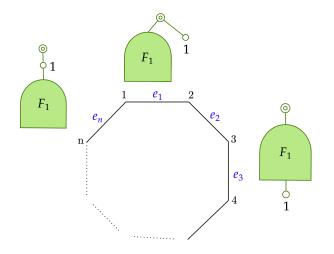
Figure 11: 1/2/3/4/5/6 $\rightarrow 1/2/34/5/6 \rightarrow 134/2/5/6 \rightarrow 134/2/56 \rightarrow 1234/56 \rightarrow 123456$

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Set Partitions and Merge Graphs

-Bond Lattice

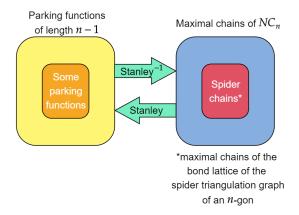
Ψ Preview: Linking Merge Chains and Forests



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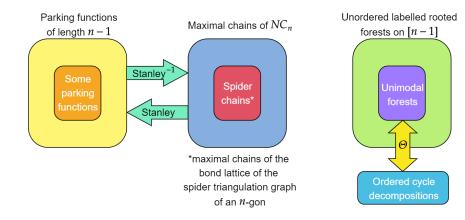
Posets, Bond Lattices, and Their Bijections — Counting - Bond Lattices of Triangulation Graphs and Parking Functions

Recall: Stanley's Bijection



Posets, Bond Lattices, and Their Bijections — Counting - Bond Lattices of Triangulation Graphs and Parking Functions

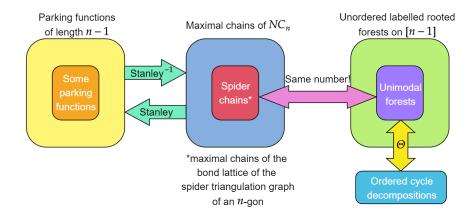
Introducing: forests



A result from Anders and Archer.

Posets, Bond Lattices, and Their Bijections — Counting - Bond Lattices of Triangulation Graphs and Parking Functions

Introducing: forests



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A result from the 2020 REU group.

The Forest World

Unimodal Forests

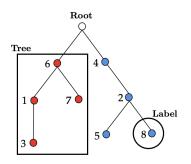
Unordered Labelled Rooted Forests

Unordered: Left to right tree order is not fixed

Labelled: Every node has a label

Rooted: One unlabelled node connects all trees

Forest: One or more trees



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The Forest World

Unimodal Forests

Unimodal Unordered Labelled Rooted Forests

Unimodal: Avoiding 312 and 213 patterns.

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A non-unimodal forest (left), and a unimodal forest (right).

The Forest World

Forest recursion

Building The Set of Unimodal Rooted Forests on [n], Recursively

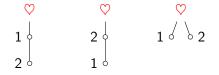


Figure 13: $F_2(312, 213)$, the set of 14 unimodal rooted forests on [2].

The Forest World

Forest recursion

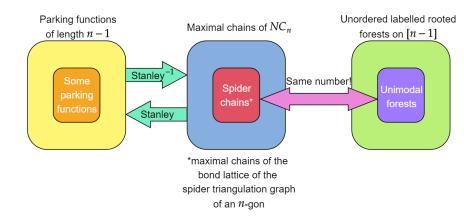
Building The Set of Unimodal Forests on [n], Recursively

Type Ia. Node 1 is a leaf child of the root Type Ic. Node 1 is an only child of the root Type lb. Node 1 is a leaf child of a non-root node Type II. Node 1 is the root of a subforest.

Figure 14: $F_3(312, 213)$, the set of unimodal rooted forests on [3].

Chain Recursion

Now: back to maximal chains



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Chain Recursion

Two types of edge contraction

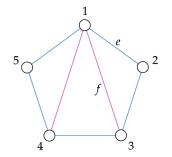
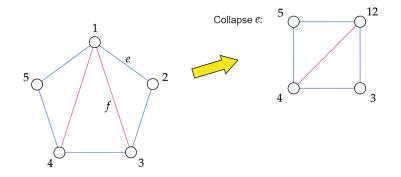


Figure 15: The spider triangulation graph of a pentagon.

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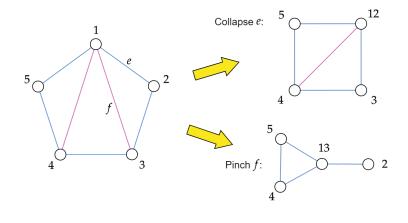
Chain Recursion

Two types of edge contraction



Chain Recursion

Two types of edge contraction



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Chain Recursion

A bigger pinch

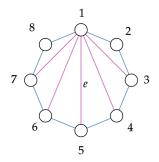


Figure 16: The spider triangulation graph of an octagon

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Chain Recursion

A bigger pinch

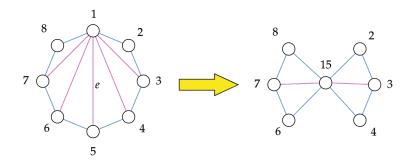


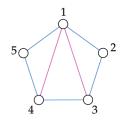
Figure 17: The spider triangulation graph of an octagon, after a pinch

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Chain Recursion

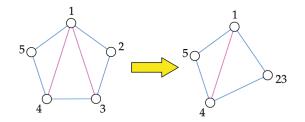
Type I: collapse first chain example

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Chain Recursion

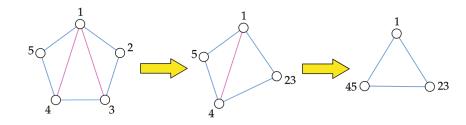
Type I: collapse first chain example



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Chain Recursion

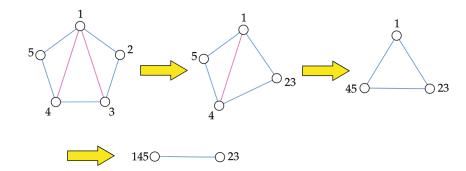
Type I: collapse first chain example



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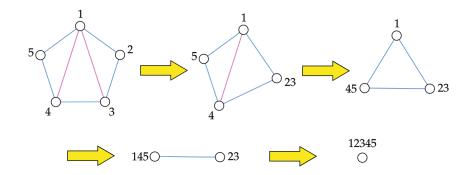
Chain Recursion

Type I: collapse first chain example



Chain Recursion

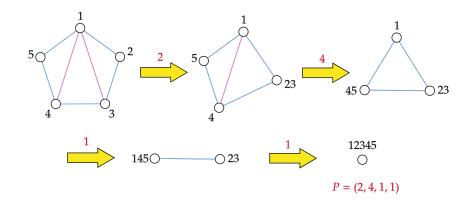
Type I: collapse first chain example



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Chain Recursion

Type I: collapse first chain example

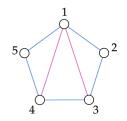


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Chain Recursion

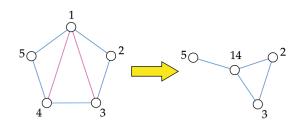
Type II: pinch first chain example

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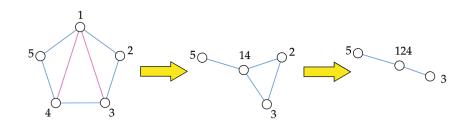
Type II: pinch first chain example



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Chain Recursion

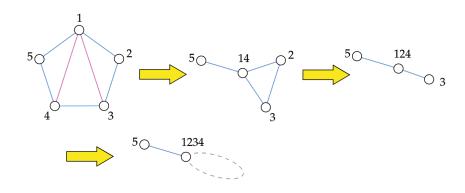
Type II: pinch first chain example



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Chain Recursion

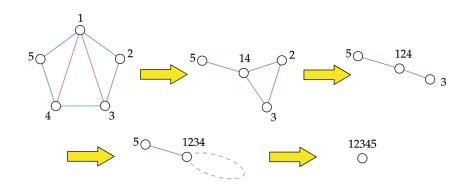
Type II: pinch first chain example



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Chain Recursion

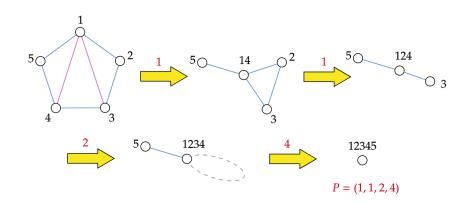
Type II: pinch first chain example



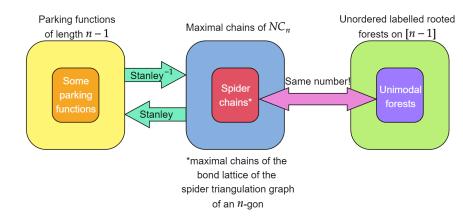
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Chain Recursion

Type II: pinch first chain example



Bijection time...

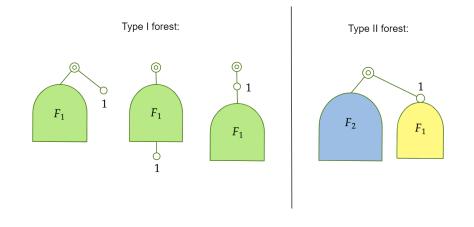


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Ψ bijection

Definition

Forest shapes for each type of chain



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-Ψ bijection

-Ψ-Bijection - Parking Function to Forest

Ψ -Bijection - Parking Function to Forest Example

Consider parking function (3, 1, 1, 2).

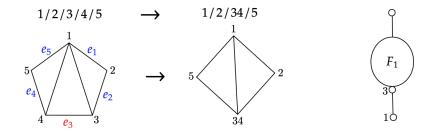
 $1/2/3/4/5 \rightarrow 1/2/34/5 \rightarrow \textbf{15}/2/34 \rightarrow \textbf{125}/34 \rightarrow \textbf{12345}$

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 $-\Psi$ bijection

-Ψ-Bijection - Parking Function to Forest

Ψ -Bijection - Parking Function to Forest Example

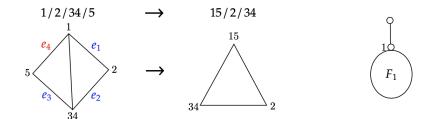


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 $-\Psi$ bijection

-Ψ-Bijection - Parking Function to Forest

Ψ -Bijection - Parking Function to Forest Example

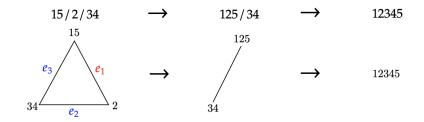


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 $-\Psi$ bijection

-Ψ-Bijection - Parking Function to Forest

Ψ -Bijection - Parking Function to Forest Example

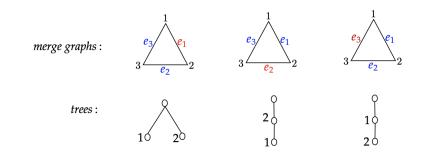


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LΨ bijection

Ψ-Bijection - Parking Function to Forest

Ψ -Bijection - Base Cases

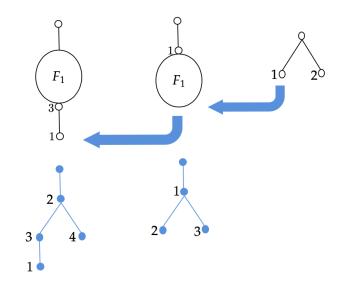


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 $-\Psi$ bijection

-Ψ-Bijection - Parking Function to Forest

Ψ -Bijection- Parking Function to Forest Example



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-Ψ bijection

Ψ-Bijection - Forest to Parking Function

Ψ -Bijection - The Forest for Parking Function (3, -, -, -)

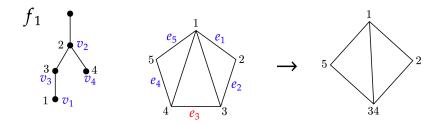


Figure 18: Car 1 wants spot 3.

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-Ψ bijection

Ψ-Bijection - Forest to Parking Function

Ψ -Bijection - The Forest for Parking Function (3, 1, -, -)

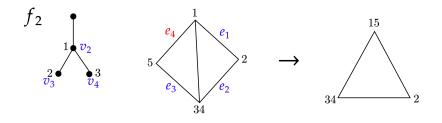


Figure 19: Car 2 wants spot 1.

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-Ψ bijection

Ψ-Bijection - Forest to Parking Function

Ψ -Bijection - The Forest for Parking Function (3, 1, 1, -)

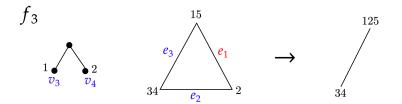


Figure 20: Car 3 wants spot 1.

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—Ψ bijection

Ψ-Bijection - Forest to Parking Function

Ψ -Bijection - The Forest for Parking Function (3, 1, 1, 2)

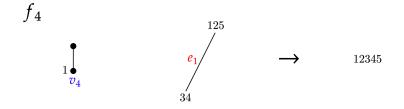


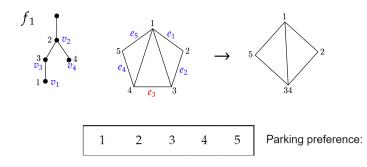
Figure 21: Car 4 wants spot 2

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Posets, Bond Lattices, and Their Bijections $\Box \Psi$ bijection

Chart

Ψ -Bijection - Forest to Parking Function Example

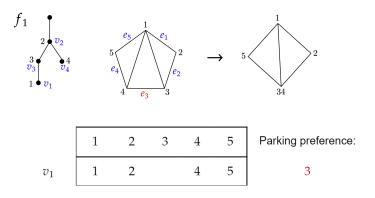


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Posets, Bond Lattices, and Their Bijections $\sqcup \Psi$ bijection

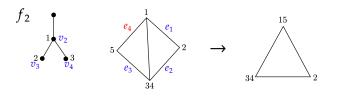
Chart

Ψ -Bijection - Forest to Parking Function Example



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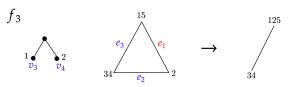
Ψ -Bijection - Forest to Parking Function Example



	1	2	3	4	5	Parking preference:
v_1	1	2		4	5	3
v_2	1	2		4		1

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Ψ -Bijection - Forest to Parking Function Example



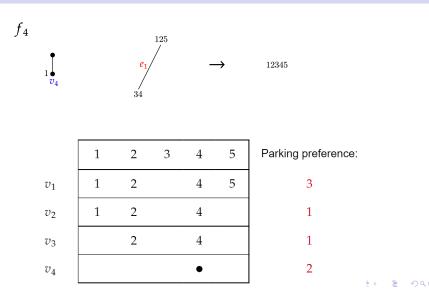
	1	2	3	4	5	Parking preference:
v_1	1	2		4	5	3
v_2	1	2		4		1
v_3		2		4		1

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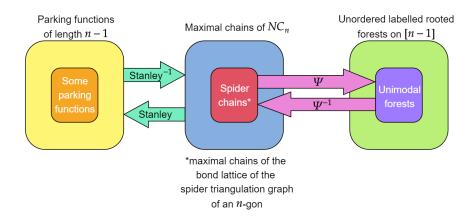
Posets, Bond Lattices, and Their Bijections $-\Psi$ bijection

Chart

Ψ -Bijection - Forest to Parking Function Example



Our resulting bijection



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Posets, Bond Lattices, and Their Bijections └─ Next Steps

Next Steps...

- 1. Expressed parking functions
- 2. Isomorphic bond lattices
- 3. Applying ψ to non-unimodal forests

4. Statistics preserved by the map

Posets, Bond Lattices, and Their Bijections └─ Next Steps

Thank you for your time! \heartsuit



Extra Information

Alternative Bond Lattice Definition

The bond lattice BL(G) is the poset of closed subsets of edges of G, ordered by inclusion.

ie: G(V, E) is a graph on vertices V and edges E. Let S be the power set of E, check if $s \in S$ contains all possible edges between components that are connected. If so, $s \in S$ creates an induced subgraph of G, which is a closed subset of edges.

Extra Information

Partitions as Transpositions

Consider the maximal chain in NC_4 :

$$1/2/3/4 \rightarrow 12/3/4 \rightarrow 12/34 \rightarrow 1234$$

Which is equivalent to:

$$(1) \to (12) \to (12)(34) \to (12)(34)(24)$$

Which provides the parking function:

(1, 3, 2)

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Extra Information

Forest to parking function chart

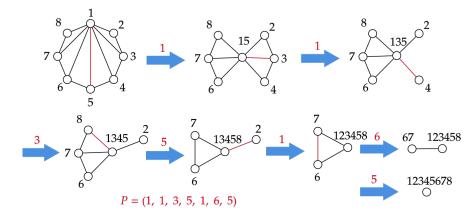


Figure 22: A merge chain of an octagon

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Extra Information

Forest to parking function chart

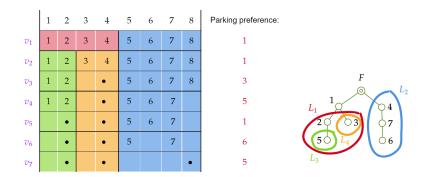


Figure 23: The corresponding forest and chart, with lobes color coded

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