Out of the Parking Lot and into the Forest: Parking Functions, Bond Lattices, and Unimodal Forests

Davy Brooks, Sophie Rubenfeld, Bianca Teves

Summer @ ICERM

August 3, 2022

Outline

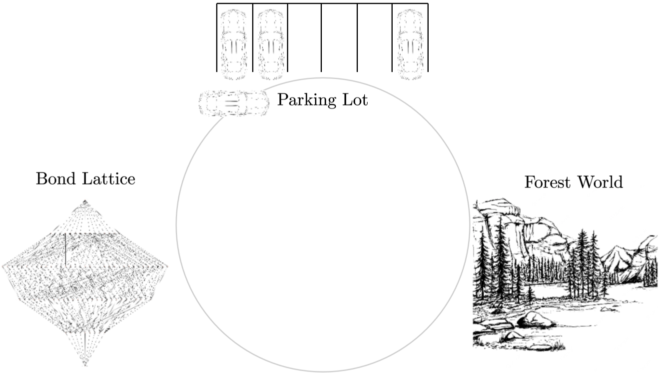


Figure 1: The Domains

Outline

i Triangulation graphs

ii. Partitions and merge graphs

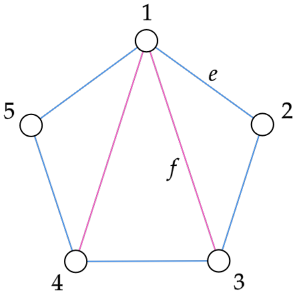
iii. The forest world

iv. Merge chains

v. The Ψ bijection

Graph Terminology

1. $G = (V, E)$
2. Inner edges
3. Outer edges
4. Cycle graph



Graphs with Inner Edges

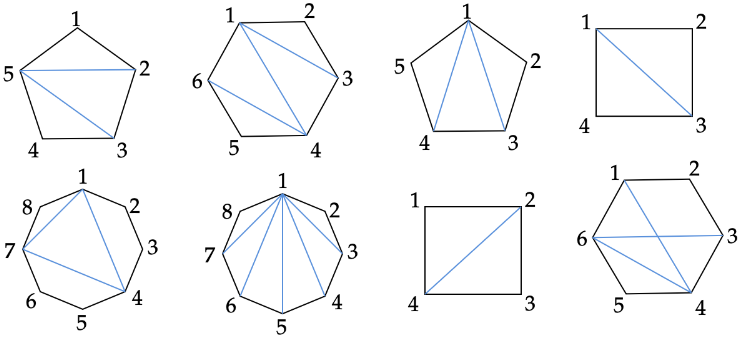


Figure 2: Various graphs that contain inner edges.

Triangulation Graphs

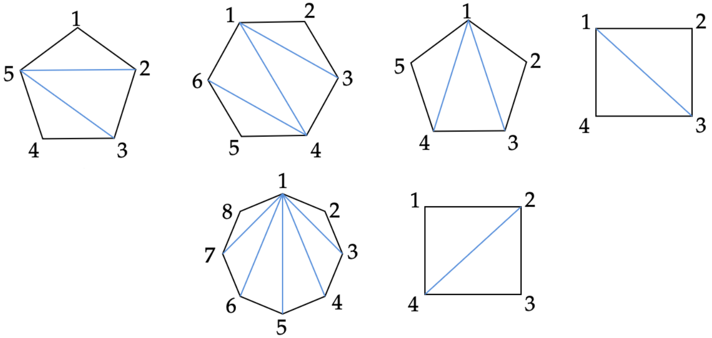


Figure 3: A few triangulation graphs

Spider Triangulation Graphs

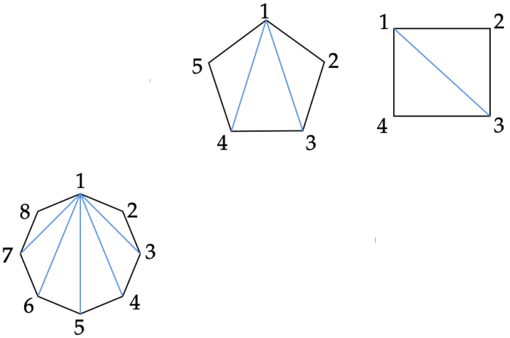
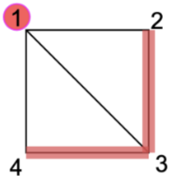


Figure 4: A refinement of Figure 3 to only Spider Triangulations

(more) Graph Terminology



1. Induced subgraph
2. Connected
3. Tree

Recall: Noncrossing Set Partitions

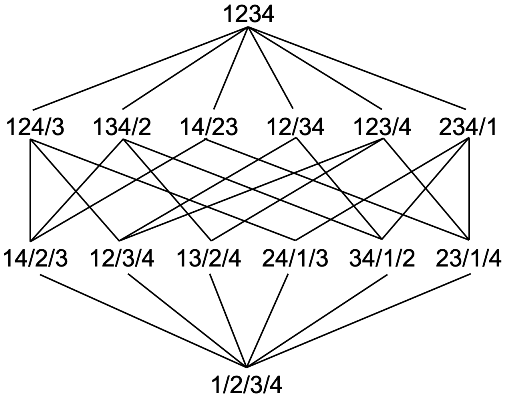


Figure 5: The noncrossing partition lattice NC_4

Introducing: the bond lattice

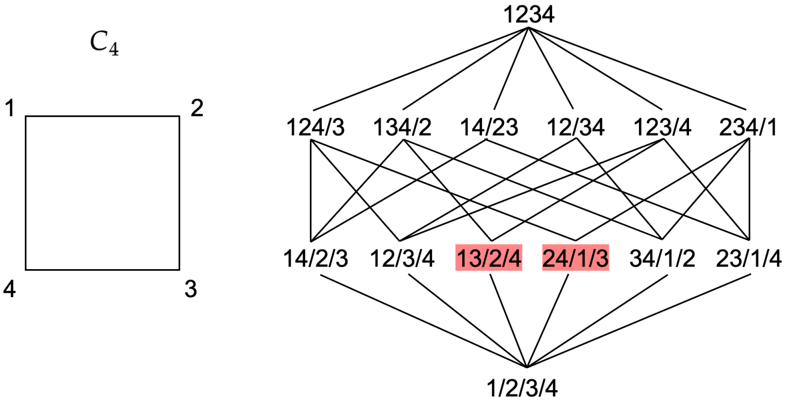


Figure 6: The noncrossing partition lattice NC_4

Introducing: the bond lattice

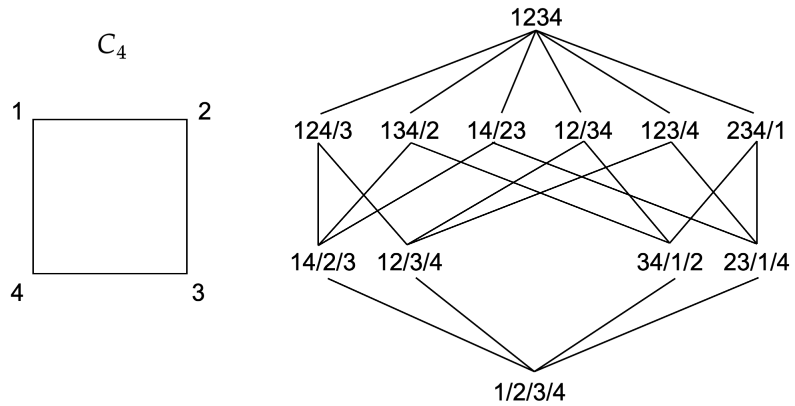


Figure 7: The bond lattice of C_4

Example: another bond lattice

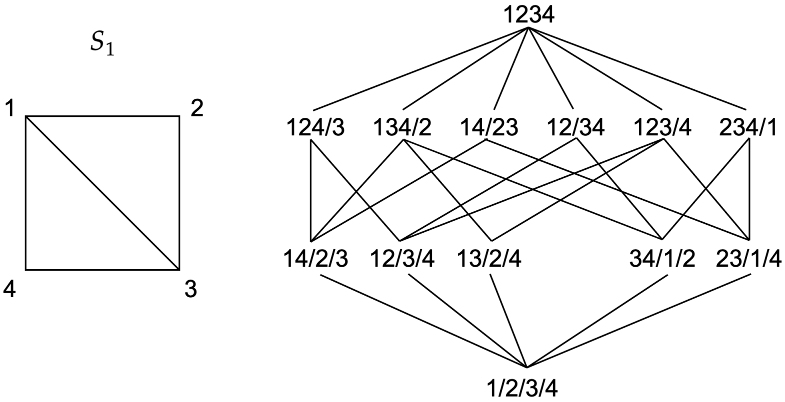


Figure 8: The bond lattice of a triangulation of a square

Introducing: Merge Graphs

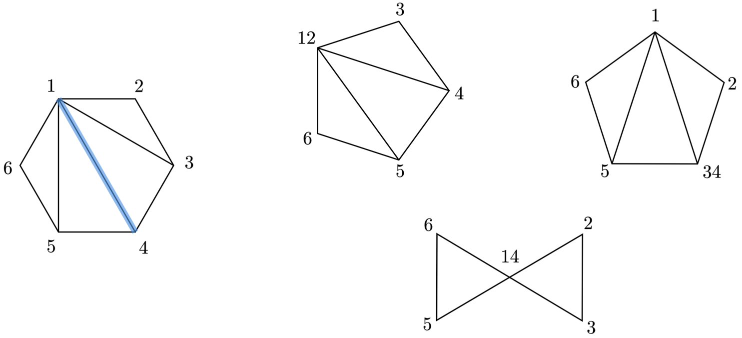


Figure 9: Merge Graph Possibilities

The Bond Lattice of the Spider Hexagon

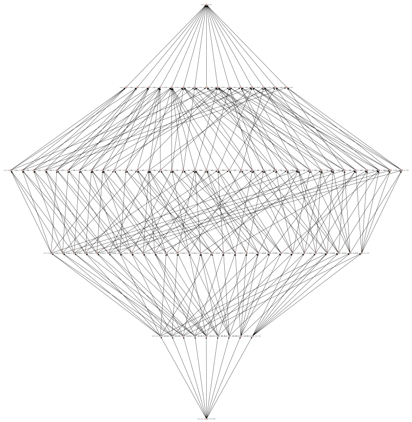


Figure 10: The Bond Lattice of the Hexagon 1

Merge Chain for Parking Function (3, 1, 5, 4)

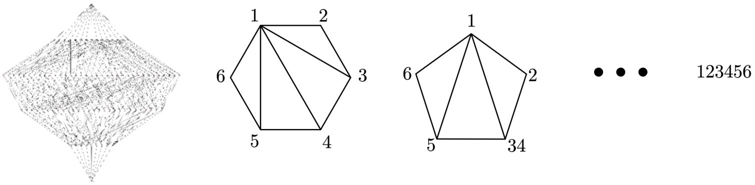
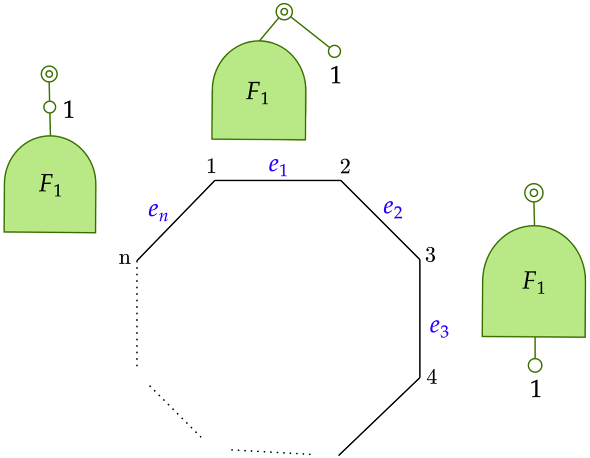
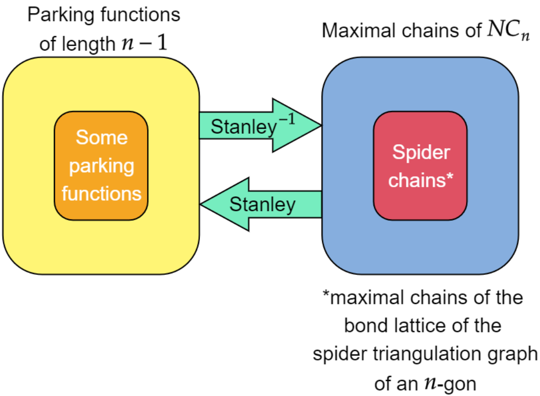


Figure 11: 1/2/3/4/5/6
$\to$ 1/2/34/5/6 $\to$ 134/2/5/6 $\to$ 134/2/56 $\to$ 1234/56 $\to$ 123456

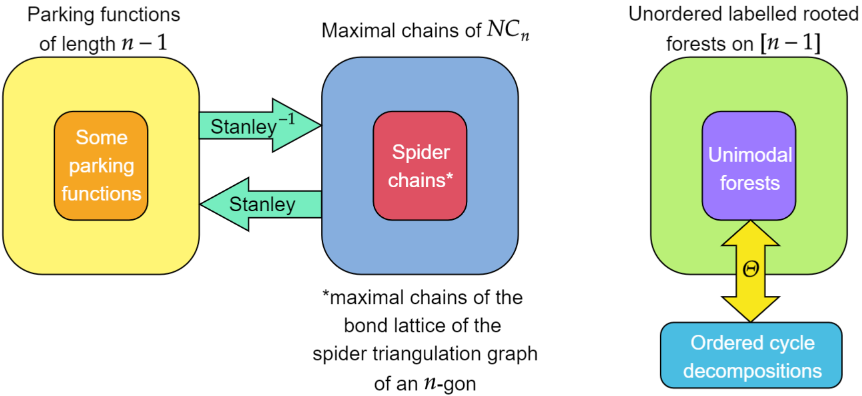
Ψ Preview: Linking Merge Chains and Forests



Recall: Stanley's Bijection

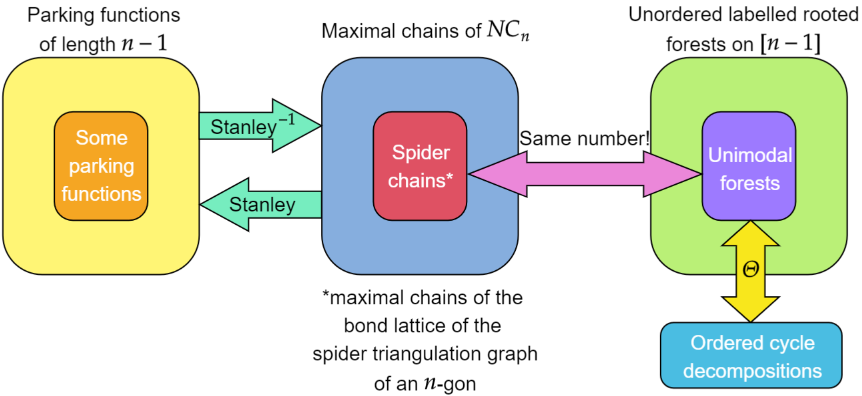


Introducing: forests



A result from Anders and Archer.

Introducing: forests



A result from the 2020 REU group.

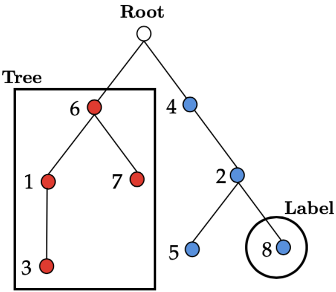
Unordered Labelled Rooted Forests

Unordered: Left to right tree order is not fixed

Labelled: Every node has a label

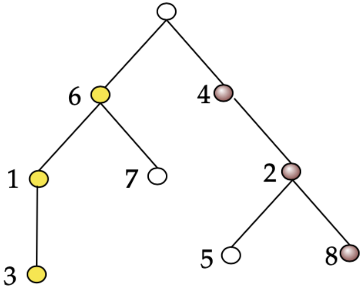
Rooted: One unlabelled node connects all trees

Forest: One or more trees



Unimodal Unordered Labelled Rooted Forests

Unimodal: Avoiding 312 and 213 patterns.



A non-unimodal forest (left), and a unimodal forest (right).

Building The Set of Unimodal Rooted Forests on $[n]$, Recursively

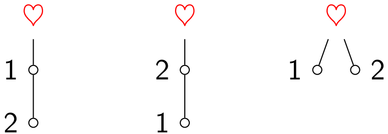


Figure 13: $F_2(312, 213)$, the set of 14 unimodal rooted forests on $[2]$.

Building The Set of Unimodal Forests on $[n]$, Recursively

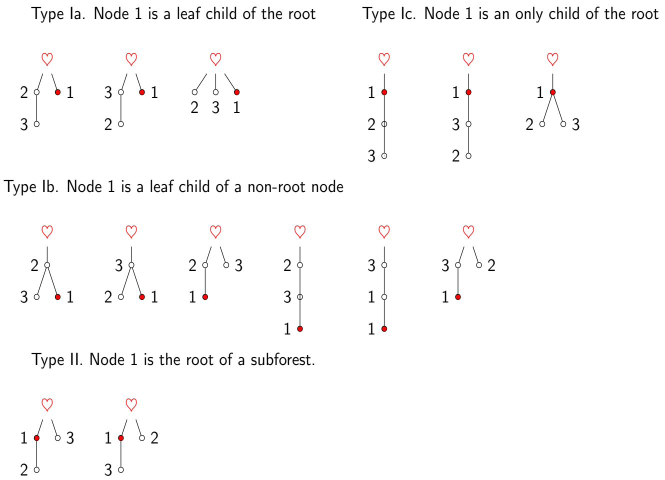
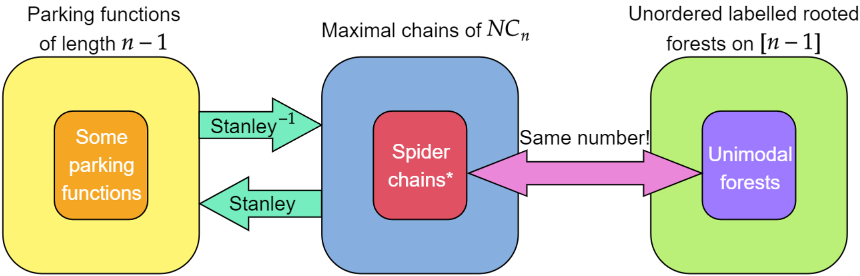


Figure 14: $F_3(312, 213)$, the set of unimodal rooted forests on [3].

Now: back to maximal chains



*maximal chains of the bond lattice of the spider triangulation graph of an n-gon

Two types of edge contraction

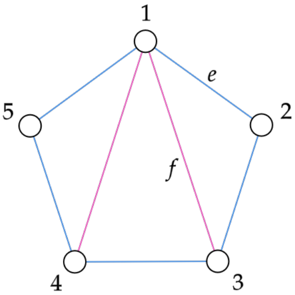
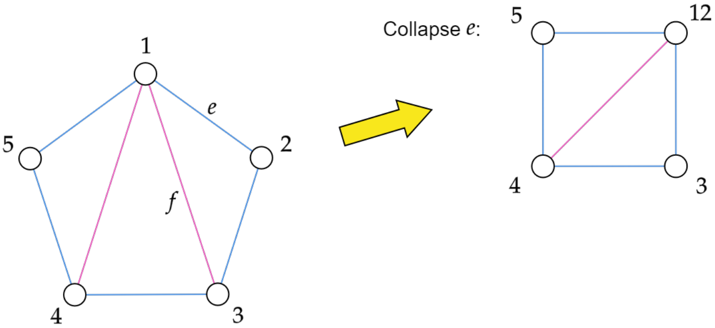
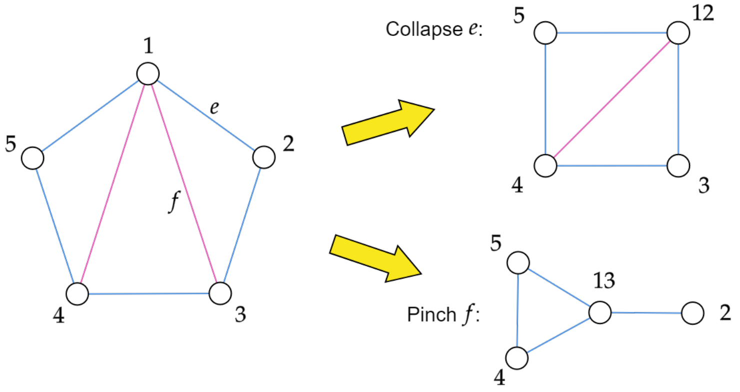


Figure 15: The spider triangulation graph of a pentagon.

Two types of edge contraction



Two types of edge contraction



A bigger pinch

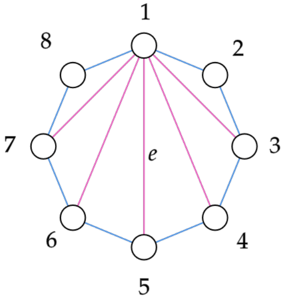


Figure 16: The spider triangulation graph of an octagon

A bigger pinch

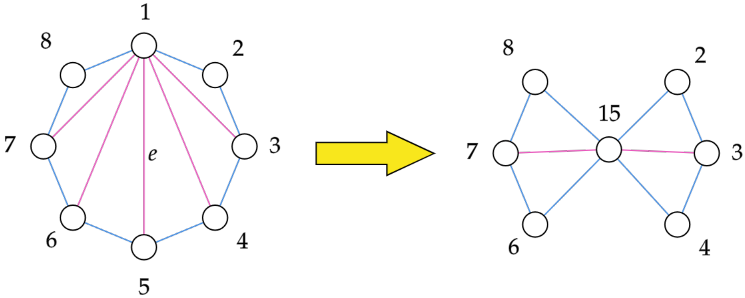
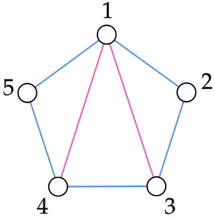
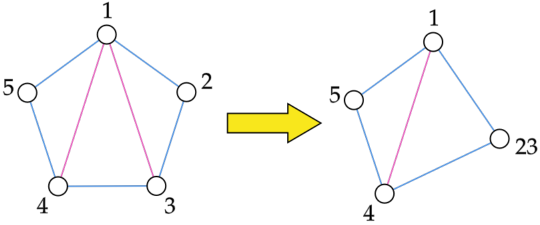


Figure 17: The spider triangulation graph of an octagon, after a pinch

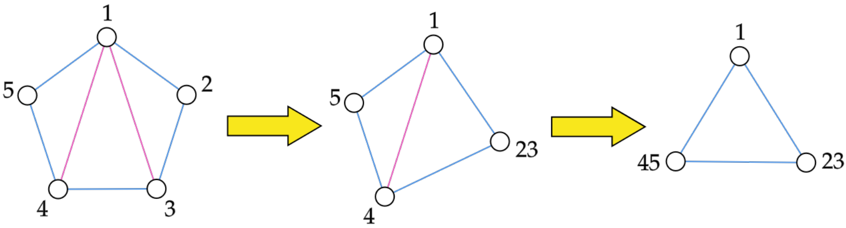
Type I: collapse first chain example



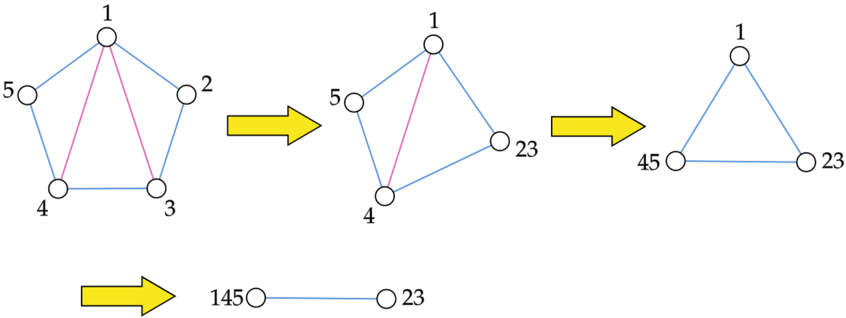
Type I: collapse first chain example



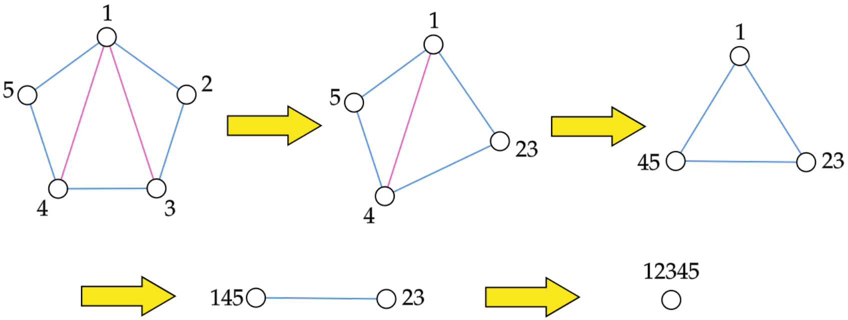
Type I: collapse first chain example



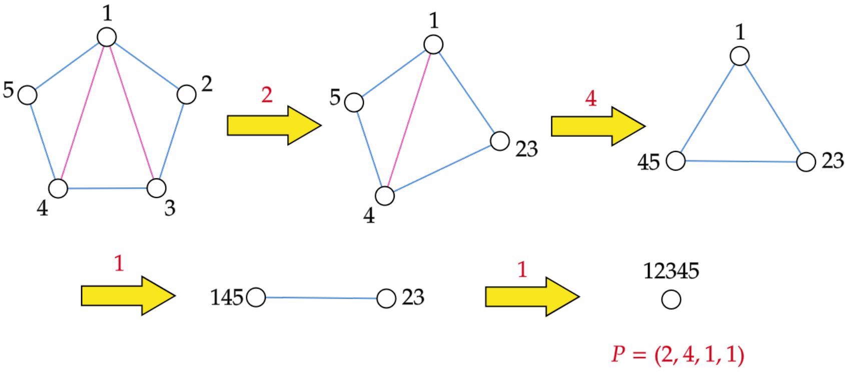
Type I: collapse first chain example



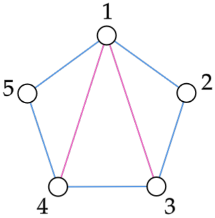
Type I: collapse first chain example



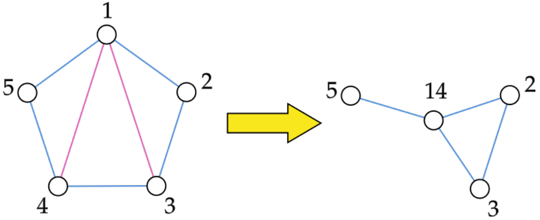
Type I: collapse first chain example



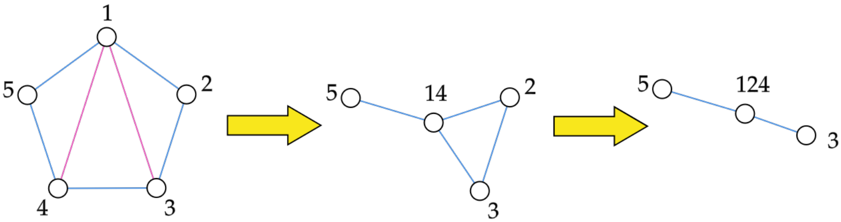
Type II: pinch first chain example



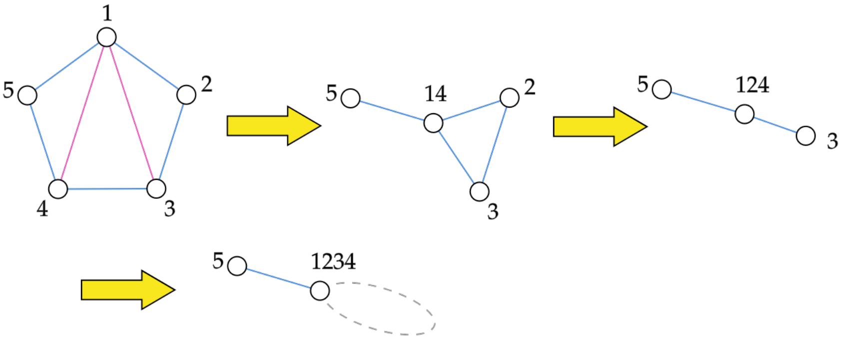
Type II: pinch first chain example



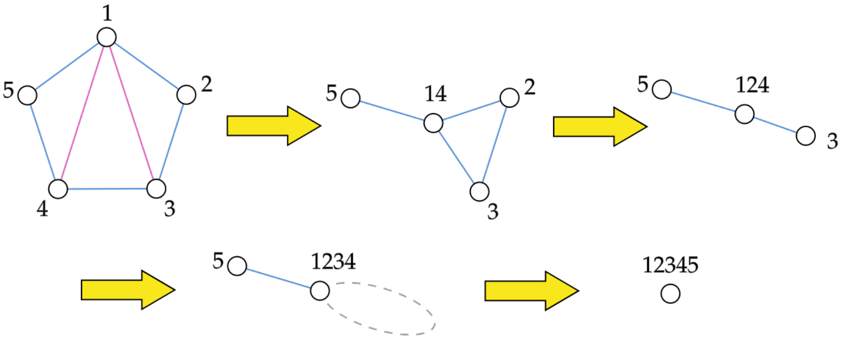
Type II: pinch first chain example



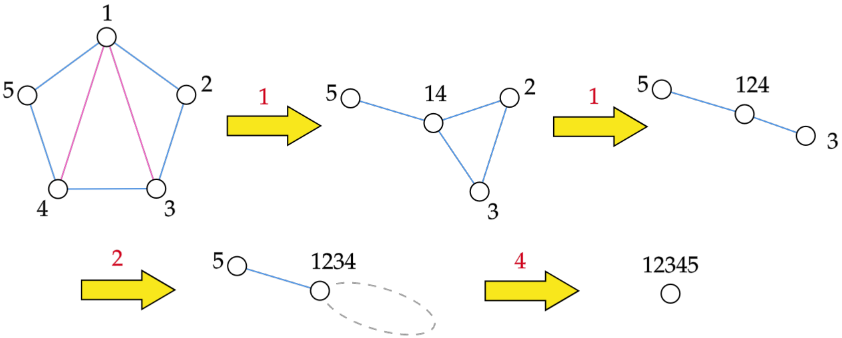
Type II: pinch first chain example



Type II: pinch first chain example

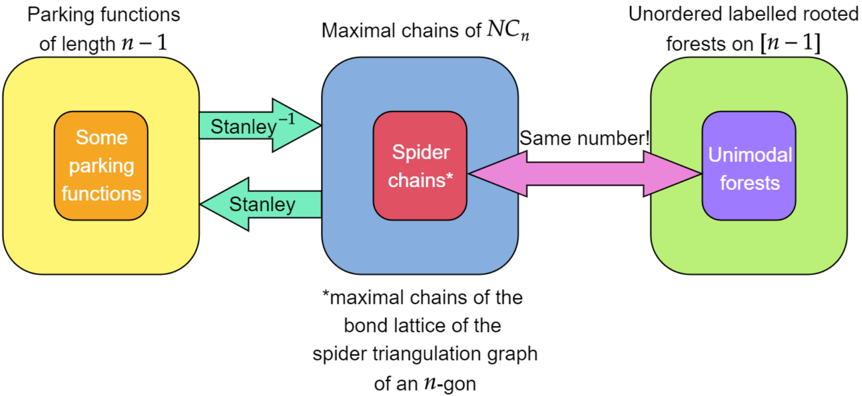


Type II: pinch first chain example

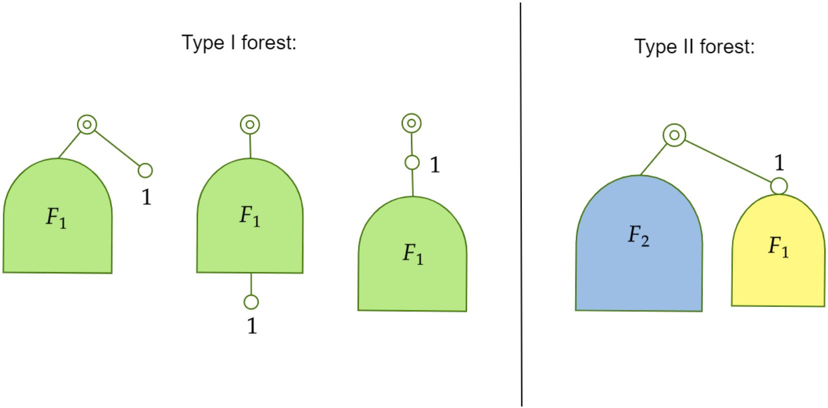


$P = (1, 1, 2, 4)$

Bijection time...



Forest shapes for each type of chain

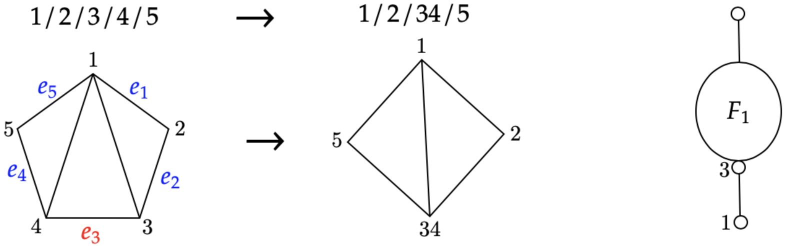


Ψ-Bijection - Parking Function to Forest Example

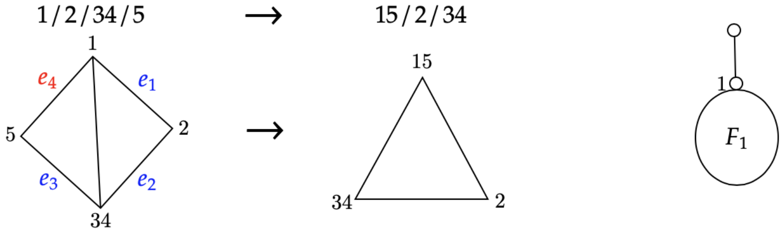
Consider parking function $(3, 1, 1, 2)$.

$1/2/3/4/5 \rightarrow 1/2/\mathbf{34}/5 \rightarrow \mathbf{15}/2/34 \rightarrow \mathbf{125}/34 \rightarrow 12345$

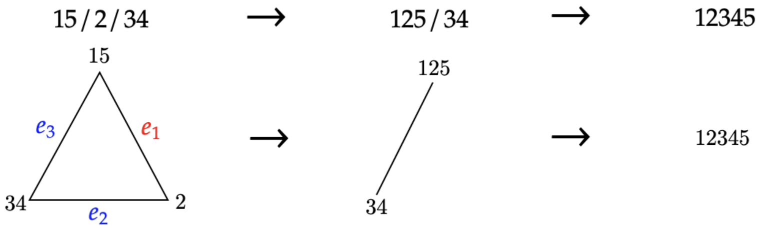
Ψ-Bijection - Parking Function to Forest Example



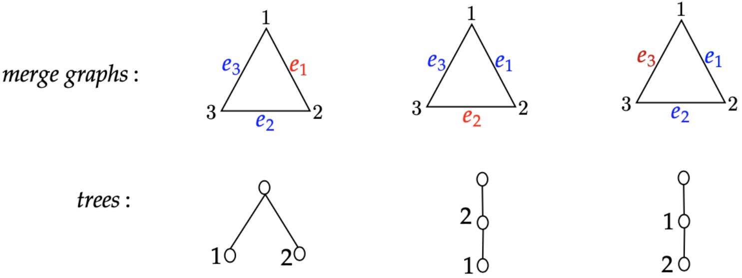
Ψ-Bijection - Parking Function to Forest Example



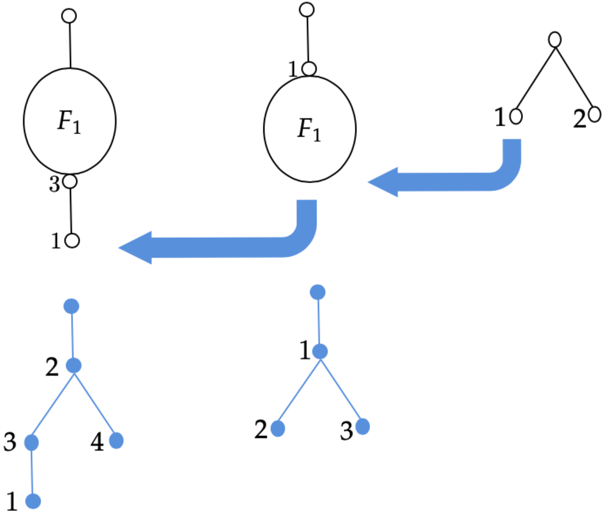
Ψ-Bijection - Parking Function to Forest Example



Ψ-Bijection - Base Cases



Ψ-Bijection- Parking Function to Forest Example



Ψ-Bijection - The Forest for Parking Function (3, -, -, -)

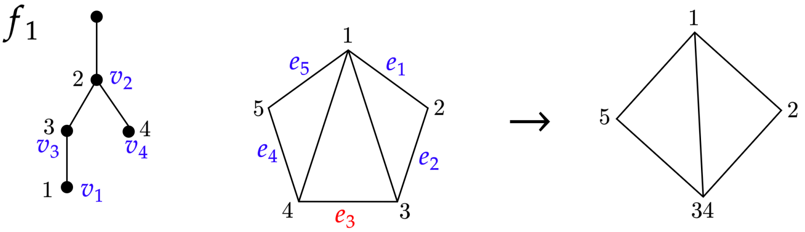


Figure 18: Car 1 wants spot 3.

Ψ-Bijection - The Forest for Parking Function (3, 1, -, -)

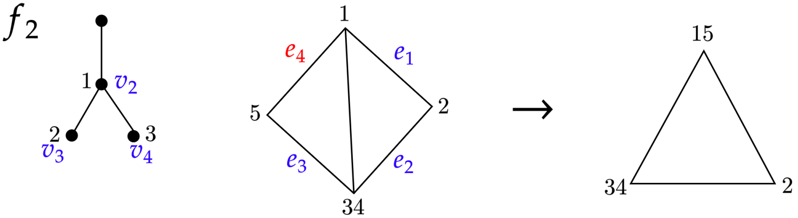


Figure 19: Car 2 wants spot 1.

Ψ-Bijection - The Forest for Parking Function (3, 1, 1, -)

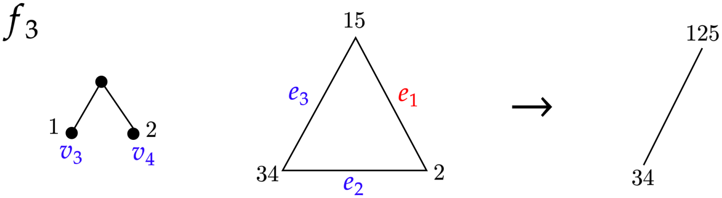


Figure 20: Car 3 wants spot 1.

Ψ-Bijection - The Forest for Parking Function (3, 1, 1, 2)

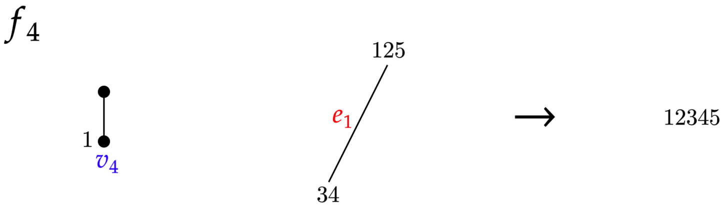
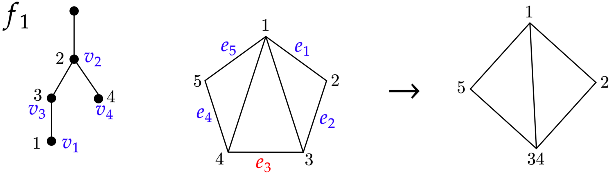


Figure 21: Car 4 wants spot 2

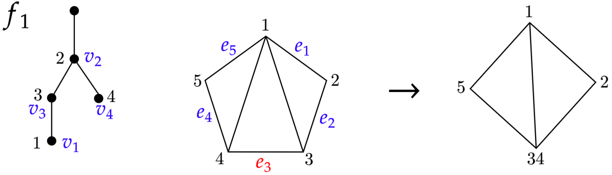
Ψ-Bijection - Forest to Parking Function Example



1	2	3	4	5

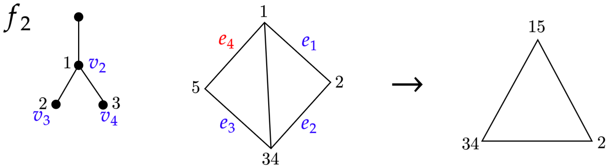
Parking preference:

Ψ-Bijection - Forest to Parking Function Example



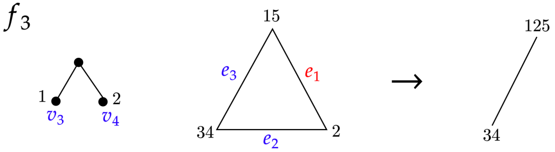
		1	2	3	4	5	Parking preference:
v_1		1	2		4	5	3

Ψ-Bijection - Forest to Parking Function Example



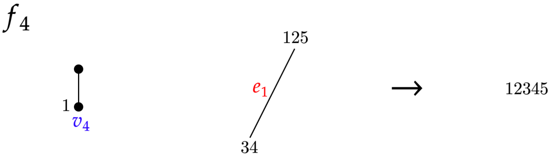
	1	2	3	4	5	Parking preference:
v_1	1	2		4	5	3
v_2	1	2		4		1

Ψ-Bijection - Forest to Parking Function Example



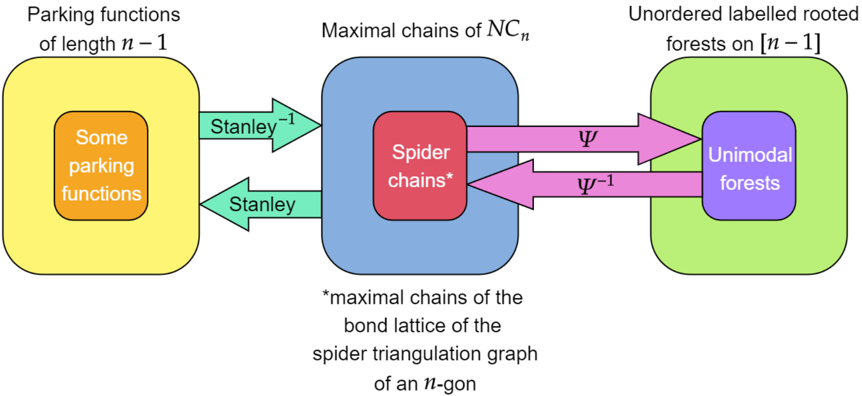
	1	2	3	4	5	Parking preference:
v_1	1	2		4	5	3
v_2	1	2		4		1
v_3		2		4		1

Ψ-Bijection - Forest to Parking Function Example



	1	2	3	4	5	Parking preference:
v_1	1	2		4	5	3
v_2	1	2		4		1
v_3		2		4		1
v_4				●		2

Our resulting bijection



Next Steps...

1. Expressed parking functions
2. Isomorphic bond lattices
3. Applying ψ to non-unimodal forests
4. Statistics preserved by the map

Thank you for your time! ♡

Alternative Bond Lattice Definition

The bond lattice $BL(G)$ is the poset of closed subsets of edges of G, ordered by inclusion.

ie: $G(V, E)$ is a graph on vertices V and edges E. Let S be the power set of E, check if $s \in S$ contains all possible edges between components that are connected. If so, $s \in S$ creates an induced subgraph of G, which is a closed subset of edges.

Partitions as Transpositions

Consider the maximal chain in NC_4:

$$1/2/3/4 \to 12/3/4 \to 12/34 \to 1234$$

Which is equivalent to:

$$(1) \to (12) \to (12)(34) \to (12)(34)(24)$$

Which provides the parking function:

$$(1, 3, 2)$$

Forest to parking function chart

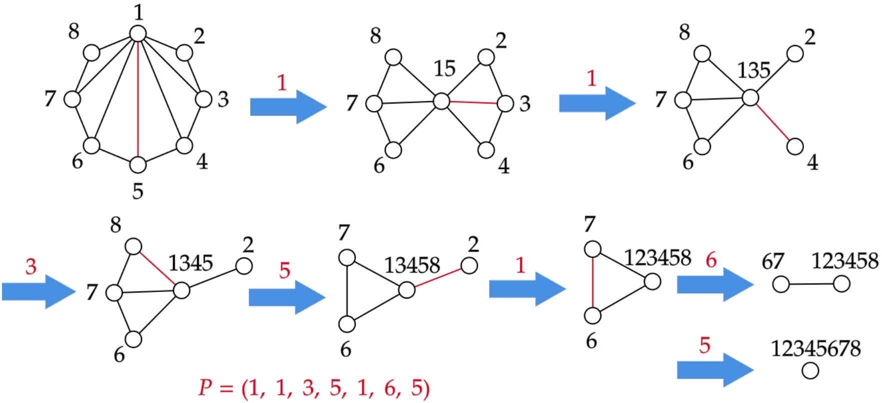


Figure 22: A merge chain of an octagon

Forest to parking function chart

	1	2	3	4	5	6	7	8	Parking preference:
v_1	1	2	3	4	5	6	7	8	1
v_2	1	2	3	4	5	6	7	8	1
v_3	1	2		•	5	6	7	8	3
v_4	1	2		•	5	6	7		5
v_5		•		•	5	6	7		1
v_6		•		•	5		7		6
v_7		•		•				•	5

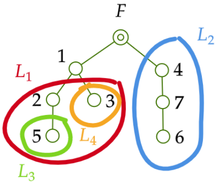


Figure 23: The corresponding forest and chart, with lobes color coded