Out of the Parking Lot and into the Forest: Parking Functions, Bond Lattices, and Unimodal Forests

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August 3, 2022
Figure 1: The Domains
Outline

i. Triangulation graphs

ii. Partitions and merge graphs

iii. The forest world

iv. Merge chains

v. The $\Psi$ bijection
Graph Terminology

1. $G = (V, E)$
2. Inner edges
3. Outer edges
4. Cycle graph
Graphs with Inner Edges

Figure 2: Various graphs that contain inner edges.
Figure 3: A few triangulation graphs
Spider Triangulation Graphs

Figure 4: A refinement of Figure 3 to only Spider Triangulations
(more) Graph Terminology

1. Induced subgraph
2. Connected
3. Tree
Recall: Noncrossing Set Partitions

Figure 5: The noncrossing partition lattice $NC_4$
Introducing: the bond lattice

Figure 6: The noncrossing partition lattice $NC_4$
Introducing: the bond lattice

Figure 7: The bond lattice of $C_4$
Example: another bond lattice

Figure 8: The bond lattice of a triangulation of a square
Introducing: Merge Graphs

Figure 9: Merge Graph Possibilities
The Bond Lattice of the Spider Hexagon

Figure 10: The Bond Lattice of the Hexagon 1
Merge Chain for Parking Function $(3, 1, 5, 4)$

Figure 11: $1/2/3/4/5/6$  
$\rightarrow 1/2/34/5/6 \rightarrow 134/2/5/6 \rightarrow 134/2/56 \rightarrow 1234/56 \rightarrow 123456$
Preview: Linking Merge Chains and Forests
Recall: Stanley’s Bijection

Parking functions of length \( n - 1 \) \( \xrightarrow{\text{Stanley}} \) Maximal chains of \( NC_n \)

Some parking functions \( \xrightarrow{\text{Stanley}^{-1}} \) Spider chains*

*maximal chains of the bond lattice of the spider triangulation graph of an \( n \)-gon
Introducing: forests

A result from Anders and Archer.
Introducing: forests

- Parking functions of length $n - 1$
- Some parking functions
- Maximal chains of $NC_n$
- Spider chains*
- Same number!
- Unordered labelled rooted forests on $[n - 1]$
- Unimodal forests
- *maximal chains of the bond lattice of the spider triangulation graph of an $n$-gon
- Ordered cycle decompositions

A result from the 2020 REU group.
Unordered Labelled Rooted Forests

**Unordered:** Left to right tree order is not fixed

**Labelled:** Every node has a label

**Rooted:** One unlabelled node connects all trees

**Forest:** One or more trees
**Unimodal** Unordered Labelled Rooted Forests

**Unimodal:** Avoiding 312 and 213 patterns.

A non-unimodal forest (left), and a unimodal forest (right).
Building The Set of Unimodal Rooted Forests on \([n]\), Recursively

**Figure 13**: \(F_2(312, 213)\), the set of 14 unimodal rooted forests on \([2]\).
Building The Set of Unimodal Forests on \([n]\), Recursively

**Type Ia.** Node 1 is a leaf child of the root

**Type Ib.** Node 1 is a leaf child of a non-root node

**Type Ic.** Node 1 is an only child of the root

**Type II.** Node 1 is the root of a subforest.

*Figure 14:* \(F_3(312, 213)\), the set of unimodal rooted forests on \([3]\).*
Now: back to maximal chains

Parking functions of length $n - 1$ ⟷ Maximal chains of $NC_n$ ⟷ Unordered labelled rooted forests on $[n - 1]$

Some parking functions ⟷ Spider chains* ⟷ Unimodal forests

*maximal chains of the bond lattice of the spider triangulation graph of an $n$-gon

Stanley \(^{-1}\) ⟷ Same number!
Two types of edge contraction

Figure 15: The spider triangulation graph of a pentagon.
Two types of edge contraction

 Collapse $e$: 

![Diagram showing two types of edge contraction]
Two types of edge contraction

Collapse $e$: 

Pinch $f$: 

$\begin{align*}
\text{Posets, Bond Lattices, and Their Bijections} \\
\text{Chain Recursion}
\end{align*}$
A bigger pinch

Figure 16: The spider triangulation graph of an octagon
Figure 17: The spider triangulation graph of an octagon, after a pinch
Type I: collapse first chain example
Type I: collapse first chain example
Type I: collapse first chain example
Type I: collapse first chain example
Type I: collapse first chain example

145 → 12345
Type I: collapse first chain example

$P = (2, 4, 1, 1)$
Type II: pinch first chain example

![Graph Diagram]
Type II: pinch first chain example
Type II: pinch first chain example
Type II: pinch first chain example
Type II: pinch first chain example
Type II: pinch first chain example

$P = (1, 1, 2, 4)$
Bijection time...

Parking functions of length $n - 1$

Some parking functions

Maximal chains of $NC_n$

Spider chains*

Unordered labelled rooted forests on $[n-1]$

Same number!

*maximal chains of the bond lattice of the spider triangulation graph of an $n$-gon

Stanley

Stanley$^{-1}$

$\psi$ bijection

Definition
Forest shapes for each type of chain

Type I forest:

- $F_1$
- $F_1$
- $F_1$

Type II forest:

- $F_2$
- $F_1$
Consider parking function $(3, 1, 1, 2)$.

$1/2/3/4/5 \rightarrow 1/2/34/5 \rightarrow 15/2/34 \rightarrow 125/34 \rightarrow 12345$
ψ-Bijection - Parking Function to Forest Example
\[ \Psi - \text{Bijection - Parking Function to Forest Example} \]
\( \Psi \)-Bijection - Parking Function to Forest Example

\[
\begin{align*}
15/2/34 & \rightarrow 125/34 & \rightarrow 12345 \\
15 & \rightarrow 125 & \rightarrow 12345 \\
34 & \rightarrow 34 & \rightarrow 12345 \\
34 & \rightarrow 2 & \rightarrow 12345 \\
e_3 & \rightarrow e_1 & \rightarrow e_2
\end{align*}
\]
Ψ-Bijection - Base Cases

merge graphs:

trees:
\(\Psi\)-Bijection - Parking Function to Forest Example
\( \Psi \)-Bijection - The Forest for Parking Function \((3, -, -, -)\)

Figure 18: Car 1 wants spot 3.
ψ-Bijection - The Forest for Parking Function (3, 1, -, -)

Figure 19: Car 2 wants spot 1.
Figure 20: Car 3 wants spot 1.
Ψ-Bijection - The Forest for Parking Function (3, 1, 1, 2)

Figure 21: Car 4 wants spot 2
Ψ-Bijection - Forest to Parking Function Example

\[
f_1
\]

Parking preference:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
\( \Psi \)-Bijection - Forest to Parking Function Example

\[ f_1 \]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 \\
\end{array}
\]

Parking preference:

\[ v_1 \]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 & 3 \\
\end{array}
\]
\( \Psi \)-Bijection - Forest to Parking Function Example

\[
\begin{array}{c}
\text{Parking preference:} \\
\begin{array}{c|ccccc}
& 1 & 2 & 3 & 4 & 5 \\
\hline
\nu_1 & 1 & 2 & 4 & 5 & \quad 3 \\
\nu_2 & 1 & 2 & \quad 4 & \quad 1 \\
\end{array}
\end{array}
\]
\( \Psi \)-Bijection - Forest to Parking Function Example

\[ f_3 \]

\[ v_3 \quad v_4 \]

\[ 1 \quad 2 \]

\[ 34 \quad 15 \quad e_2 \quad e_1 \quad e_3 \]

\[ 34 \quad 125 \]

Parking preference:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>( v_2 )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_3 )</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Ψ-Bijection - Forest to Parking Function Example

\[ f_4 \]

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
v_1 & 1 & 2 & 4 & 5 & 3 \\
v_2 & 1 & 2 & 4 & 1 & 1 \\
v_3 & 2 & 4 & 1 & 1 & 2 \\
v_4 & & & & \bullet & 2 \\
\end{array} \]

Parking preference:

- \( v_1 \): 3
- \( v_2 \): 1
- \( v_3 \): 1
- \( v_4 \): 2
Our resulting bijection

Parking functions of length $n - 1$

Some parking functions

Maximal chains of $NC_n$

Spider chains*

Unimodal forests

Unordered labelled rooted forests on $[n - 1]$

$\Psi$ bijection

Chart

*maximal chains of the bond lattice of the spider triangulation graph of an $n$-gon

Stanley

Stanley$^{-1}$

$\Psi$

$\Psi^{-1}$
Next Steps...

1. Expressed parking functions
2. Isomorphic bond lattices
3. Applying $\psi$ to non-unimodal forests
4. Statistics preserved by the map
Thank you for your time! 💕
Alternative Bond Lattice Definition

The bond lattice $BL(G)$ is the poset of closed subsets of edges of $G$, ordered by inclusion.

ie: $G(V, E)$ is a graph on vertices $V$ and edges $E$. Let $S$ be the power set of $E$, check if $s \in S$ contains all possible edges between components that are connected. If so, $s \in S$ creates an induced subgraph of $G$, which is a closed subset of edges.
Partitions as Transpositions

Consider the maximal chain in $NC_4$:

$$1/2/3/4 \rightarrow 12/3/4 \rightarrow 12/34 \rightarrow 1234$$

Which is equivalent to:

$$(1) \rightarrow (12) \rightarrow (12)(34) \rightarrow (12)(34)(24)$$

Which provides the parking function:

$$(1, 3, 2)$$
Figure 22: A merge chain of an octagon

$P = (1, 1, 3, 5, 1, 6, 5)$
### Forest to parking function chart

![Forest and chart with lobes color coded](image)

**Figure 23:** The corresponding forest and chart, with lobes color coded