On Flattened Parking Functions

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Table of Contents

Definitions

Given a *word* $w = w_1 w_2 \dots w_n$

- The index i is an **ascent** if $w_i \leq w_{i+1}$
- The index i is a **descent** if $w_i > w_{i+1}$
- A **run** is a maximal string of consecutive ascents

Example

Let $n = 6$

1 2 5 3 6 4

Definitions

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- The index i is an **ascent** if $w_i \leq w_{i+1}$
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Let $n = 6$

$$\underline{1\ 2\ 5}\ \underline{3\ 6}\ \underline{4}$$

Background

- Zhuang provided enumerations of permutations having certain restrictions on run lengths (2021)
- Nabawanda, Rakotondrajao, Bamunoba (2020) and Beyene, Mantaci (2022) studied distribution of runs in flattened partitions
- We extend these ideas to parking functions (i.e. flatness, enumerations, run distribution)

Flattened Parking Functions

Flattened Parking Functions $\subset$ Parking Functions

Definition

A parking function π is said to be a **flattened parking function** if its leading values of each runs are arranged in weakly increasing order.

Example

Parking Functions	Leading Terms	Flattened?
1 2 5 1 4 3		

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$\underline{1}$ 2 5 $\underline{1}$ 4 $\underline{3}$	113	✓
$\underline{1}$ 5 $\underline{4}$ 5 $\underline{2}$ 3	142	$\times$

Flattened Parking Functions

n \ k	Total	1 Run	2 Runs	3 Runs	4 Runs
1	1	1	0	0	0
2	2	2	0	0	0
3	8	5	3	0	0
4	46	14	32	0	0
5	336	42	245	49	0
6	2937	132	1656	1149	0
7	29629	429	10563	17008	1629
8	336732	1430	65472	204815	65015

Table: Total flattened parking functions of length n with k runs

Flattened Parking Functions

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Table: Total flattened parking functions of length n with k runs

Flattened $\mathcal{S}$-Insertion Parking Functions

Example

Take a permutation of length n

$$1\ 2\ 4\ 3$$

and insert every element of the multiset $\mathcal{S} = \{2, 2, 3\}$

$$1\ 2\ 2\ 4\ 2\ 3\ 3$$

Then, check if it's flattened

$$\underline{1\ 2\ 2\ 4}\ \underline{2\ 3\ 3}.$$

We have a flattened parking function of length 7 with the multiset $\{2, 2, 3\}$ inserted.

Flattened $\mathcal{S}$-Insertion Parking Functions

Definition

Let $\mathcal{S}$ be a multiset containing elements from $[n+1]$. A $\mathcal{S}$-**insertion parking functions** is formed by taking a permutation $\pi \in \mathfrak{S}_n$ and inserting each element $s \in \mathcal{S}$ into π. We will denote this set of parking functions as $\mathcal{PF}_n(\mathcal{S})$.

Lemma

The elements in the set $\mathcal{PF}_n(\mathcal{S})$ are parking functions of length $n + |\mathcal{S}|$.

Proof.

Arrange the permutation π in weakly increasing order, so $\pi_i = i$. Since, for all $s \in \mathcal{S}$, we have $s \leq n+1$, inserting s into π and rearranging in weakly increasing order will satisfy $a_i \leq i$. Thus, resulting in a parking function of length $n + |\mathcal{S}|$. □

Flattened $\mathcal{S}$-Insertion Parking Functions

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Let $\mathcal{S}$ be a multiset containing elements from $[n+1]$. A $\mathcal{S}$-**insertion parking functions** is formed by taking a permutation $\pi \in \mathfrak{S}_n$ and inserting each element $s \in \mathcal{S}$ into π. We will denote this set of parking functions as $\mathcal{PF}_n(\mathcal{S})$

- We denote the flattened subset of these as $\text{flat}(\mathcal{PF}_n(\mathcal{S}))$.
- We denote the flattened subset of these with k runs as $\text{flat}_k(\mathcal{PF}_n(\mathcal{S}))$.
- The cardinality of $\text{flat}(\mathcal{PF}_n(\mathcal{S}))$ is denoted $f(\mathcal{S}; n)$.
- The cardinality of $\text{flat}_k(\mathcal{PF}_n(\mathcal{S}))$ is denoted $f(\mathcal{S}; n, k)$.

Bijection 1

Theorem

Let $\mathcal{S}$ be a multiset with elements in $[n]$ and $\mathcal{S}' = \{s-1 \mid 1 < s \in \mathcal{S}\} \cup \{1\}$. Then, the set $\mathrm{flat}(\mathcal{PF}_n(\mathcal{S}))$ is in bijection with $\mathrm{flat}(\mathcal{PF}_{n-1}(\mathcal{S}'))$.

Example

Let $n = 4$ and $\mathcal{S} = \{2, 2, 3\}$.
Then, $\mathrm{flat}\left(\mathcal{PF}_4\left(\{2,2,3\}\right)\right) \Longleftrightarrow \mathrm{flat}\left(\mathcal{PF}_3\left(\{1,1,1,2\}\right)\right)$.
Consider $\Longrightarrow$

$$1\ 2\ 2\ 4\ 2\ 3\ 3 \mapsto 1\ 1\ 1\ 3\ 1\ 2\ 2.$$

To go from $\mathrm{flat}\left(\mathcal{PF}_4\left(\{2,2,3\}\right)\right)$ to $\mathrm{flat}\left(\mathcal{PF}_3\left(\{1,1,1,2\}\right)\right)$, subtract 1 from every term other than the first.

Bijection 1

Theorem

Let $\mathcal{S}$ be a multiset with elements in $[n]$ and $\mathcal{S}' = \{s-1 \mid 1 < s \in \mathcal{S}\} \cup \{1\}$. *Then, the set* $\text{flat}(\mathcal{PF}_n(\mathcal{S}))$ *is in bijection with* $\text{flat}(\mathcal{PF}_{n-1}(\mathcal{S}'))$.

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Consider $\Longleftarrow$

$$1\ 1\ 1\ 3\ 1\ 2\ 2 \mapsto 1\ 2\ 2\ 4\ 2\ 3\ 3.$$

To go from $\text{flat}\left(\mathcal{PF}_3\left(\{1,1,1,2\}\right)\right)$ to $\text{flat}\left(\mathcal{PF}_4\left(\{2,2,3\}\right)\right)$, add 1 to every term other than the first.

Bijection 1

Theorem

Let $\mathcal{S}$ be a multiset with elements in $[n]$ and $\mathcal{S}' = \{s - 1 \mid 1 < s \in \mathcal{S}\} \cup \{1\}$. Then, the set $\text{flat}(\mathcal{PF}_n(\mathcal{S}))$ *is in bijection with* $\text{flat}(\mathcal{PF}_{n-1}(\mathcal{S}'))$.

In general,

- ($\Longrightarrow$) Subtract 1 from the value at every index other than the first.
- ($\Longleftarrow$) Add 1 to the value at every index other than the first.

Bijective Results

Theorem (Bijection 1)

Let $\mathcal{S}$ *be a multiset with elements in* $[n]$ *and* $\mathcal{S}' = \{s-1 \mid 1 < s \in \mathcal{S}\} \cup \{1\}$*. Then, the set* $\mathrm{flat}(\mathcal{PF}_n(\mathcal{S}))$ *is in bijection with* $\mathrm{flat}(\mathcal{PF}_{n-1}(\mathcal{S}'))$*.*

Theorem (Bijection 2)

Let $\mathcal{S}$ *be a multiset with elements in* $[n]$ *where, for all* $s \in \mathcal{S}$*, we have* $s < n-1$*. Then, the set* $\mathrm{flat}(\mathcal{PF}_n(\mathcal{S} \cup \{n\}))$ *is in bijection with* $\mathrm{flat}(\mathcal{PF}_n(\mathcal{S} \cup \{n-1\}))$*.*

Theorem (Bijection 3)

Let $2 \leq \ell \leq n$*. The set* $\mathrm{flat}_2\left(\mathcal{PF}_n\left(\{\ell - 1\}\right)\right)$ *is in bijection with* $\mathrm{flat}_2\left(\mathcal{PF}_n\left(\{\ell\}\right)\right)$*.*

Flattened $\{1\}$-Insertion Parking Functions

n \ k	Total	1 Run	2 Runs	3 Runs	4 Runs
1	1	1	0	0	0
2	2	1	1	0	0
3	5	1	4	0	0
4	15	1	11	3	0
5	52	1	26	25	0
6	203	1	57	130	15
7	877	1	120	546	210

Table: $\text{flat}_k\left(\mathcal{PF}_n\left(\{1\}\right)\right)$

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Eulerian Numbers

n \ k	Total	1 Run	2 Runs	3 Runs	4 Runs
1	1	1	0	0	0
2	2	1	1	0	0
3	5	1	4	0	0
4	15	1	11	3	0
5	52	1	26	25	0
6	203	1	57	130	15
7	877	1	120	546	210

Definition

The **Eulerian numbers** $A(n, m)$ are the number of permutations of $[n]$ with exactly m descents.

Eulerian Numbers

Theorem

The set $\mathrm{flat}_2(\mathcal{PF}_n(\{1\}))$ *is enumerated by* $A(n, 1)$.

Example

Let $n = 5$.

2 3 4 1 5

Eulerian Numbers

Theorem

The set $\text{flat}_2(\mathcal{PF}_n(\{1\}))$ *is enumerated by* $A(n, 1)$*.*

Example

Let $n = 5$.

$$\underline{2\ 3\ 4}\ \underline{1\ 5}$$

- Notice that

  Permutations with 2 runs $\Leftrightarrow$ Permutations with 1 descent

Eulerian Numbers

Theorem

The set $\mathrm{flat}_2(\mathcal{PF}_n(\{1\}))$ *is enumerated by* $A(n, 1)$.

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Let $n = 5$.

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- Question: where can a 1 be inserted in a permutation without changing the number of runs and so that the new object is flat?

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  Permutations with 2 runs $\Leftrightarrow$ Permutations with 1 descent

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$$|\text{flat}_2(\mathcal{PF}_n(\{1\}))| = A(n,1)$$

Connection to Set Partitions

Definition

The numbers $T(n,k)$ are the number of ways to partition $[n]$ such that there are exactly k set partitions with at least two elements.

Theorem

For $n, k \geq 1$, we have

$$|\text{flat}_{k+1}\mathcal{PF}_n(\{1\})| = T(n,k).$$

Connection to Set Partitions

Example

Let $n = 3$ and $k = 1$.

- We want to partition $\{1,2,3\}$ into subsets where exactly one subset has greater than 2 elements

$$123,\ 1/23,\ 2/13,\ 3/12 \Rightarrow T(3,1) = 4$$

- $\text{flat}_2\left(\mathcal{PF}_3\left(\{1\}\right)\right)$

$$1213,\ 1231,\ 1132, \text{ and } 1312 \ \Rightarrow\ |\text{flat}_2(\mathcal{PF}_3\left(\{1\}\right))| = 4$$

- To map from $T(3,1)$ to $\text{flat}_2\left(\mathcal{PF}_3\left(\{1\}\right)\right)$:

$$1/23 \mapsto 1/32 \mapsto 132 \mapsto 1132$$

Connection to Set Partitions

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- To map from $T(3,1)$ to $\text{flat}_2\left(\mathcal{PF}_3\left(\{1\}\right)\right)$:

$$1/23 \mapsto 1/32 \mapsto 132 \mapsto 1132$$
$$13/2 \mapsto 31/2 \mapsto 312 \mapsto 1312$$

$\mathbb{1}_r$-Insertion Parking Functions

Let $\mathbb{1}_r$ be the multiset consisting of r ones. Then, we have the following:

n	$\text{flat}(\mathcal{PF}_n(\mathbb{1}_1))$
1	1
2	2
3	5
4	15
5	52
6	203

n	$\text{flat}(\mathcal{PF}_n(\mathbb{1}_2))$
1	1
2	3
3	10
4	37
5	151
6	674

n	$\text{flat}(\mathcal{PF}_n(\mathbb{1}_3))$
1	1
2	4
3	17
4	77
5	372
6	1915

r-Bell Numbers

Definition

The r-**Bell numbers**, denoted $B(n, r)$, count the number of set partitions of $[n + r]$ where the first r elements are in different blocks.

Theorem

$B(n, r)$ is in bijection with $\mathrm{flat}(\mathcal{PF}_{n+1}(\mathbb{1}_r))$

r-Bell Numbers

Theorem

$B(n, r)$ is in bijection with $\mathrm{flat}(\mathcal{PF}_{n+1}(\mathbb{1}_r))$

Example

We construct a bijection from $B(2, 3)$ to $\mathrm{flat}(\mathcal{PF}_3(\mathbb{1}_3))$:

$$1\,/\,\underline{24}\,/\,\underline{35} \quad\mapsto\quad 1\,\underline{42}\,\underline{53} \quad\mapsto\quad 12131 \quad\mapsto\quad \underline{1}\,12131$$

r-Bell Numbers

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$$\underline{145} \,/\, 2 \,/\, 3 \quad \mapsto$$

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$$\underline{145} \,/\, 2 \,/\, 3 \quad \mapsto \quad \underline{451}\,2\,3 \quad \mapsto$$

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$$\begin{array}{ccccccc} 1 \,/\, \underline{24} \,/\, \underline{35} & \mapsto & 1\,\underline{42}\,\underline{53} & \mapsto & 12131 & \mapsto & \underline{1}\,12131 \\ \underline{145} \,/\, 2 \,/\, 3 & \mapsto & \underline{451}\,2\,3 & \mapsto & 23111 & \mapsto & \end{array}$$

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$$\underline{145}\,/\,2\,/\,3 \quad\mapsto\quad \underline{451}\,2\,3 \quad\mapsto\quad 23111 \quad\mapsto\quad \underline{1}\,23111$$

Recursions Recursions Recursions!

We have three different ways of recursively counting $\mathrm{flat}_k(\mathcal{PF}_{n+1}(\mathbb{1}_r))$.

- Method 1: Count by where $n+1$ shows up
- Method 2: Count by construction of the first run
- Method 3: Count by whether the ones are in the same run or different runs

Recursions Recursions Recursions!

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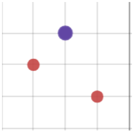
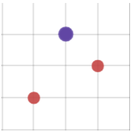
Recursion (#1)

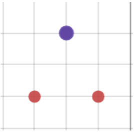
Method 1: Count by where $n+1$ shows up



$2(n+1)1$

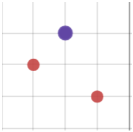


$1(n+1)2$



$1(n+1)1$

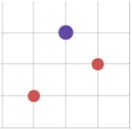
Recursion (#1)



$2(n+1)1$

- $2(n+1)1$ or $(n+1)$ at the end means removing $n+1$ does not reduce the number of runs
  - Start with $\text{flat}_k(\mathcal{PF}_n(\mathbb{1}_r))$
  - $n+1$ gets inserted at the end of any of the k runs.
- $k \cdot |\text{flat}_k(\mathcal{PF}_n(\mathbb{1}_r))|$

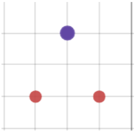
Recursion (#1)



$1(n+1)2$

- $1(n+1)2$ means removing $n+1$ reduces the number of runs. We establish the i, π method
  - Take a word $\pi \in \mathrm{flat}_{k-1}(\mathcal{PF}_{n-1}(\mathbb{1}_r))$
  - Pick an element i, $1 \leq i \leq n-1$. Take every element in π that is greater than i and add 1.
  - Now take the sequence $(n+1)(i+1)$ and insert it after the rightmost element of $\{1, 2, 3, \ldots, i\}$.
- $(n-1) \cdot |\mathrm{flat}_{k-1}(\mathcal{PF}_{n-1}(\mathbb{1}_r))|$

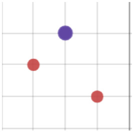
Recursion (#1)



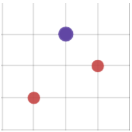
$1(n+1)1$

- $1(n+1)1$ means removing $n+1$ reduces the number of runs.
  - Start with the word $\pi \in \text{flat}_{k-1}(\mathcal{PF}_n(\mathbb{1}_{r-1}))$
  - Take the sequence $(n+1)1$. Insert this after any 1 in π.
- $r \cdot |\text{flat}_{k-1}(\mathcal{PF}_n(\mathbb{1}_{r-1}))|$

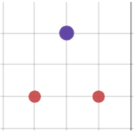
Recursion (#1)



$2(n+1)1$



$1(n+1)2$



$1(n+1)1$

Theorem (EHMTV)

$$
\begin{aligned}
|\mathrm{flat}_k(\mathcal{PF}_{n+1}(\mathbb{1}_r))| = k \cdot |\mathrm{flat}_k(\mathcal{PF}_n(\mathbb{1}_r))| \\
+ (n-1) \cdot |\mathrm{flat}_{k-1}(\mathcal{PF}_{n-1}(\mathbb{1}_r))| \\
+ r \cdot |\mathrm{flat}_{k-1}(\mathcal{PF}_n(\mathbb{1}_{r-1}))|
\end{aligned}
$$

Run Distribution

What happens if we let the first s terms be in different runs?

Example

Let $s = 3$.

$$1423657 \times$$
$$1425736 \checkmark$$

These had already been counted for permutations! (Nabawanda, Rakotondrajao, Bamunoba 2020)

Theorem (NRB)

$$f^{(s+1)}_{n+s+1,k} = \sum_{i_1,i_2,\ldots,i_s \geq 1} \binom{n}{i_1, i_2, \ldots, i_s} f_{n+1-\sum_{j=1}^{s-1} i_j, k-s}$$

Run Distribution - Generalization?

What happens if we let the first s terms be in different runs?

Example

Let $s = 3$.

- All ones in one run:

  111456273

- All ones in different runs:

  141516273

- Something in between?

  114156273

Run Distribution - Generalization! (#1)

All 1's in one run:

Example

$$\text{flat}_2(\mathcal{PF}_4(\mathbb{1}_3)), \text{ with } (s = 2)$$

- Take the list of flattened **permutations** on n with the first s terms in different runs:

  1423, 1342, 1324

- Place all ones in the first run:

  1111423, 1111342, 1111324

This gives us the same formula as for flattened permutations:

$$f^{(s+1)}(\mathbb{1}_r; n+s+1, k) = \sum_{i_1, i_2, \ldots, i_s \geq 1} \binom{n}{i_1, i_2, \ldots, i_s} f_{n+1-\sum_{j=1}^{s-1} i_j, k-s}$$

Run Distribution - Generalization (#2)

All 1's in different runs:

Example

$$\mathrm{flat}_4(\mathcal{PF}_5(\mathbb{1}_3)) \text{ with } (s = 2)$$

- Treat each of the ones as a separate integer, i.e. $1_a, 1_b, 1_c$. Set up an ordering of the ones, i.e. $1_a < 1_b < 1_c$.
- Find the number of flattened permutations on $n + r$ with $r + s$ integers in different runs.

Run Distribution - Generalization (#2)

All 1's in different runs:

Example

$$\text{flat}_4(\mathcal{PF}_5(\mathbb{1}_3)) \text{ with } (s = 2)$$

- Treat each of the ones as a separate integer, i.e. $1_a, 1_b, 1_c$. Set up an ordering of the ones, i.e. $1_a < 1_b < 1_c$.
- Find the number of flattened permutations on $n + r$ with $r + s$ integers in different runs.

This gives us a modified formula from flattened partitions:

$$f^{(s+r+1)}(\mathbb{1}_r; n+s+1, k) = \sum_{i_1,\ldots,i_{s+r} \geq 1} \binom{n}{i_1, \ldots, i_{s+r}} f_{n+1-\sum_{j=1}^{s+r} i_j, k-s-r}$$

Run Distribution - Generalization (#3)

All 1's in any arrangement:

Example

$$\text{flat}_3(\mathcal{PF}_4(\mathbb{1}_3)) \text{ with } (s = 2)$$

- Set your number of "boxes" that ones are in to $1 \leq x \leq r + 1$. This means the ones are in x different runs
- Start by assuming that there are x ones in different runs,

  13142
- Multiply by the number of ways to distribute the ones across j boxes: 111/1, 11/11, 1/111

  1113142, 1131142, 1311142
- Integer compositions: $\binom{r}{x-1}$

Run Distribution - Generalization (#3)

All 1's in any arrangement:
This gives us a very modified formula from flattened permutations:

$$f^{(s+1)}(\mathbb{1}_r; n+s+1, k) =$$
$$\sum_{x=1}^{r+1} \left(\binom{r}{x-1} \sum_{i_1,\ldots,i_{s+x} \geq 1} \binom{n}{i_1,\ldots,i_{s+x}} f_{n+1-\sum_{j=1}^{s+x} i_j, k-s-x} \right)$$

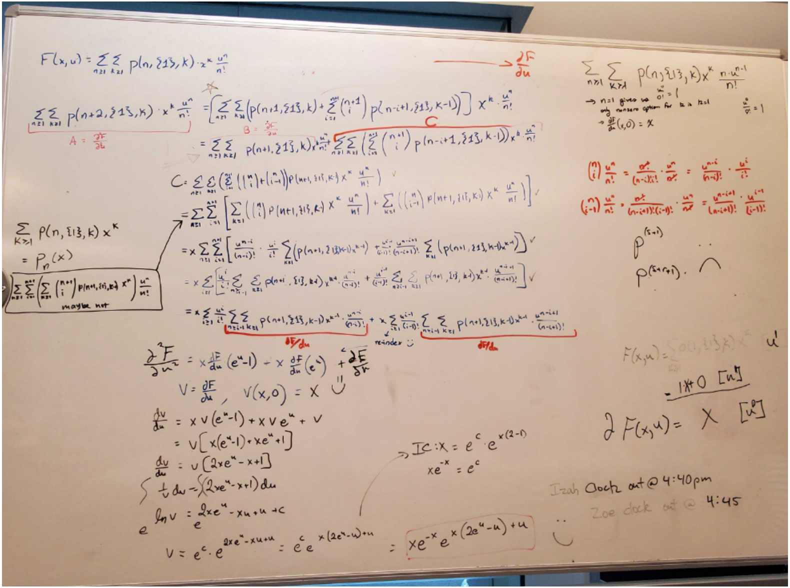
Generating FUN-ctions

- A **generating function** is a way of encoding information from a recursion
- We used exponential generating functions to encode our sequence as the coefficients of a power series

Example

Take our favorite recursion $f_{n+2} = f_{n+1} + f_n$

- $G(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ using what we know about Maclaurin series
- $\frac{d}{dx} G(x) = \sum_{n\geq 1} f_n \cdot \frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 0} f_{n+1} \cdot \frac{x^n}{n!}$


F(x,u) = Σ_{n≥1} Σ_{k≥1} p(n,{1},k) · x^k u^n/n!
∂F/∂u
A = ∂F/∂u
B = ∂F/∂u
C
maybe not
Σ_{k≥1} p(n,{1},k) x^k
= P_n(x)
∂F/du
∂F/du
reindex
V = ∂F/du , V(x,0) = X
IC: X = e^c · e^{x(2-1)}
xe^{-x} = e^c
V = e^c · e^{2xe^u - xu + u} = e^c e^{x(2e^u - u) + u}
= xe^{-x} e^{x(2e^u - u) + u}
-> n=1 gives us u^0/0! = 1
only nonzero option for k is k=1
-> ∂F/du (x,0) = x
∂F(x,u) = X [u^0]
Izah Clock out @ 4:40pm
Zoe clock out @ 4:45

Generating FUN-ctions

Theorem

The exponential generating function $F(x, y, z)$ *of the run distribution over* $\text{flat}(\mathcal{PF}_n(\mathbb{1}_r))$ *has the closed differential form*

$$\frac{\partial^2 F(x, y, z)}{\partial y^2} = x\frac{\partial F}{\partial y}\left((exp(y) - 1) + exp(y)\right)\left(1 - \frac{z}{r}\right)^{-1}.$$

Theorem

The exponential generating function $F^{[s+1]}(x, u)$ *for the numbers* $f^{(s)}_{n+s,k}$ *has closed differential form*

$$\frac{\partial^{s+1} F^{[s+1]}(x, u)}{\partial u^{s+1}} = (x(exp(u) - 1)^s \frac{\partial F(x, u)}{\partial u}.$$

Future Directions

- Recursive and/or closed formula for total number of flattened parking functions
- Recursive and/or closed formula for general $\mathcal{S}$-insertion parking functions
- Where do the flattened parking functions live in the poset of set partitions?
- Pattern avoidance: the only pattern of length 3 that every flattened partition avoids is 321. What about flattened parking functions?

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References

- Armstrong, Drew; Loehr, Nicholas A.; Warrington, Gregory S. Rational parking functions and Catalan numbers. (English summary) Ann. Comb. 20 (2016), no. 1, 21–58. 05E10 (05A30 05E05 05E18)
- Beyene, Fufa; Mantaci, Roberto. Merging-Free Partitions and Run-Sorted Permutations. *Preprint*, arxiv.org/abs/2101.07081 (2021)
- Broder, Andrei Z. The r-Stirling numbers. Discrete Math. 49 (1984), no. 3, 241–259. (Reviewer: F. T. Howard) 05A10 (11B73)
- Mansour, T.; Shattuck, M., Wagner, S. Counting subwords in flattened partitions of sets, Discrete Math. 338 (2015), 1989-2005.
- Mező, István The r-Bell numbers. J. Integer Seq. 14 (2011), no. 1, Article 11.1.1, 14 pp. (Reviewers: Steven Joel Miller and Alec Greaves-Tunnell) 05A18 (05A15)
- Nabawanda, Olivia; Rakotondrajao, Fanja; Bamunoba, Alex Samuel. Run distribution over flattened partitions. J. Integer Seq. 23 (2020), no. 9, Art. 20.9.6, 14 pp. (Reviewer: Sergey Kitaev) 05A18 (05A10 05A15)
- SageMath Inc. *CoCalc Collaborative Computation Online*, 2022. https://cocalc.com/
- W. A. Stein et al., *Sage Mathematics Software (Version 9.4)*, The Sage Development Team, 2022. www.sagemath.org.
- Steinhardt, J. Permutations with ascending and descending blocks, Electron. J. Combin. 17 (2010), #R14.
- Zhuang, Y. Counting permutations by runs, J. Combin. Theory Ser. A 142 (2016), 147-176.

Questions?

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