Generalized Splines

ICERM · Algebraic Combinatorics · 2021

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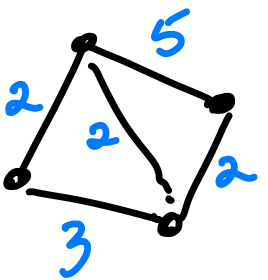
Ring R (commutative, id.)
$\mathbb{Z}$, $\mathbb{C}[x_1, \ldots, x_n]$



Graph $G = (V, E)$

Edge labels

$\alpha : E \to$ ideals in R

or just elements of R

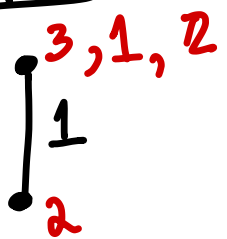
(principal ideal in R)

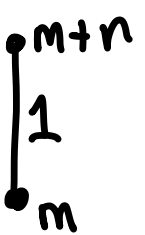
A <u>spline</u> on edge-labeled graph (G, α) is a vertex-labeling p so that adjacent vertex labels differ by elt of ideal

$$p_u - p_v \in \alpha(uv)$$

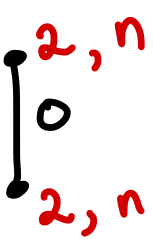
-OR- $p_u - p_v$ is multiple of edge-label

<u>Examples</u>: What are the splines on these?





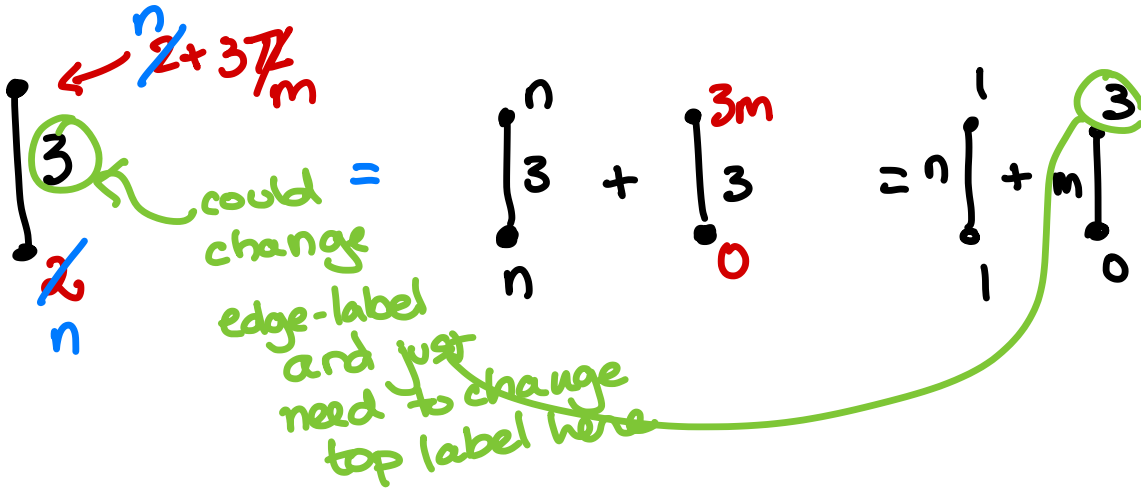
splines on this graph are R^2 and have R-module + ring structure



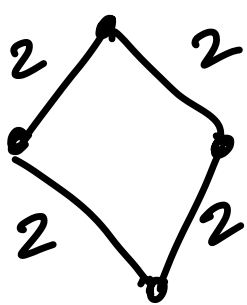
splines are usually $\subsetneq R^2$ ← $|V|$

$R = \mathbb{Z}$

don't need $\mathbb{Z}$



- It's natural to look for "upper Δ" basis



Basic Questions:

1) Find a basis (or dimension, or algorithm for a basis)

2) Characterize the multiplication structure (structure constants, or "Schubert calculus")

3) Representations of appropriate groups of graph automorphisms

Splines in analysis/applied math

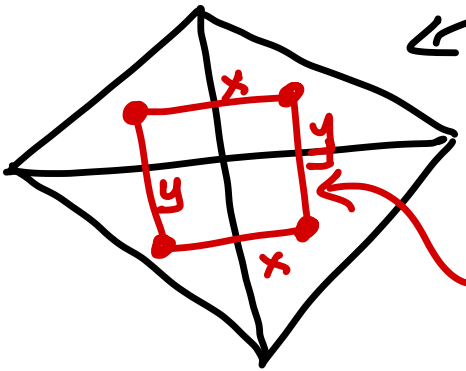
Piecewise polynomials on a $\left\{ \begin{array}{l} \text{polytope} \\ \text{triangulation} \\ \text{simplicial complex} \\ \ldots \end{array} \right.$

that are

- r-differentiable
- degree at most d

<u>Billera</u>: If Δ is a strongly connected simplicial complex and the link of each vertex is strongly connected then as $\mathbb{C}$ vector spaces

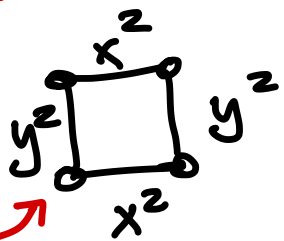
$$S^{r}_{\infty}(\Delta) \cong \text{splines on the dual graph to } \Delta \text{, edges labeled with equation of the codimension-one intersection raised to } r+1$$



Triangulation (in the plane)

Dual graph with edges labeled by equations of lines

$S^{1}_{\infty}(\Delta) \cong$ splines on



exponent is $r+1$

<u>Anderson-Matherne-Tymoczko</u> : with Billera's assumptions,

$S^r_d(\Delta) \cong$ splines on dual graph with edge labels to $r+1^{st}$ power

"over quotient ring" $\mathbb{C}[x_1,\dots,x_n] / \text{polys of deg } d+1$

Lower bound conjecture (Schumaker, Strang, ...)

When Δ is a triangulation in the plane

- $\dim S^r_d(\Delta)$ known for $r=1$, $d \geq 4$
- there's a beautiful combinatorial formula conjectured for $\dim S^1_3(\Delta)$
- $\dim S^1_2(\Delta)$ is utterly mysterious

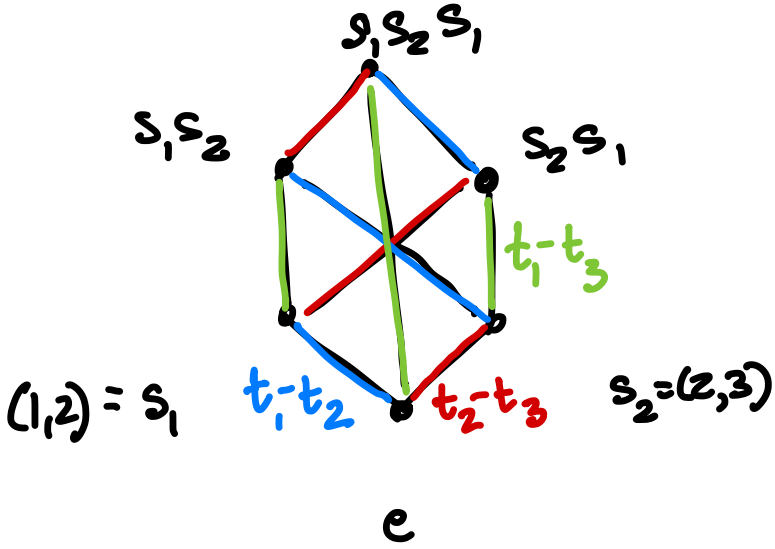
Splines in Geometry / Topology

Goresky-Kottwitz-MacPherson:

When X is a "nice" algebraic variety with a "nice" torus action $T \circlearrowright X$ then 0- and 1-dim'l torus orbits give a graph, labeled by torus weights.

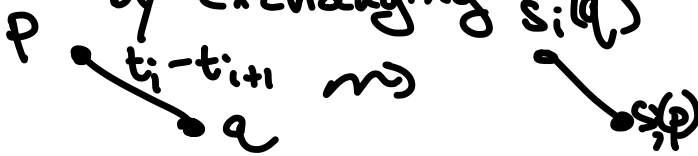
$H_T^*(X) \cong$ splines on this labeled graph

Question: Find combinatorial formulas for structure constants of these rings when X is ... $\begin{cases} \text{orthogonal Grassmannian} \\ \text{flag variety} \\ \dots \end{cases}$

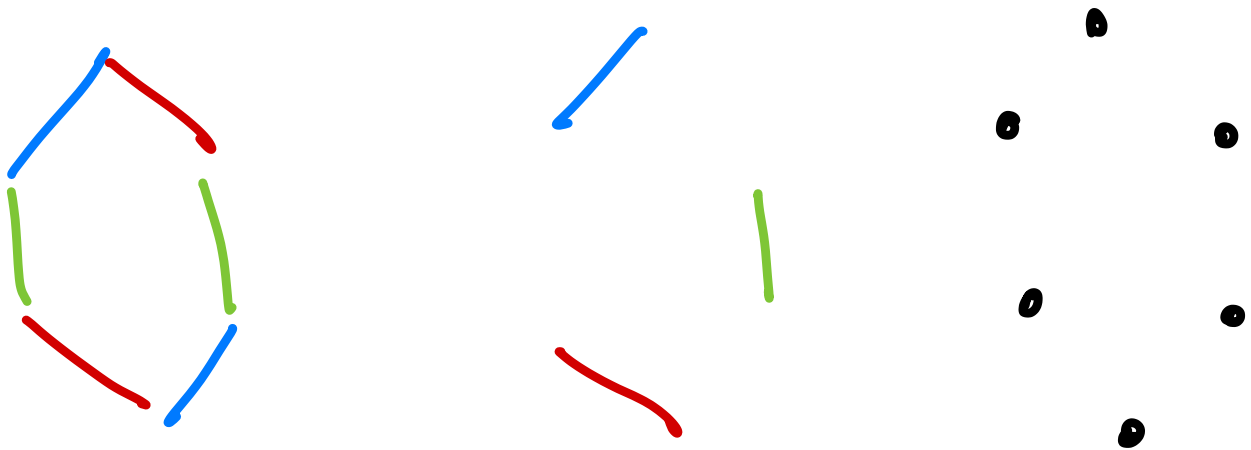


GKM result:
Splines on this graph are $H_T^*(GL_3/B)$

s_i acts on a spline by exchanging



$$S_n \circlearrowright H_T^*(GL_n/B)$$



Splines on these graphs also inherit the same S_n representation

<u>Question</u> (related to Shareshian-Wachs conjecture, and also Stanley-Stembridge) find a nice closed combinatorial formula to describe these repns.

<u>Question</u> (Billey) What can we say about representations on splines more generally?