The A, C, shifted Berenstein-Kirillov groups and cacti

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Plan

- The cactus group $J_{\mathfrak{g}}$: $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{sp}_{2n}$.
- A_{n-1}, C_n-crystals of tableaux and restrictions.
  - Full and partial Schützenberger-Lusztig involutions.
- Internal cactus group action on a normal crystal.
- Berenstein-Kirillov groups: A and shifted.
  - C.

The cactus group $J_{\mathfrak{g}}$

- $\mathfrak{g}$ finite dimensional, complex, semisimple Lie algebra
  - I the Dynkin diagram, $\Delta = \{\alpha_i\}_{i \in I}$ the simple roots.
  - W the Weyl group, w_0 the long element.
  - $\theta : I \to I$ the Dynkin diagram automorphism of I, defined by

$$\alpha_{\theta(i)} = -w_0.\alpha_i, \ i \in I.$$

  - $\theta_J : J \to J$ the Dynkin diagram automorphism of a connected subdiagram $J \subseteq I$, defined by

$$\alpha_{\theta_J(j)} = -w_0^J.\alpha_j, \ j \in J,$$

    w_0^J the long element of the parabolic subgroup $W^J \subseteq W$.

The cactus group $J_{\mathfrak{g}}$

- [Halacheva 2016]. The *cactus group* $J_{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is the group defined by:
  - **Generators**: s_J, $J \subseteq I$ running over all connected subdiagrams of the Dynkin diagram I of $\mathfrak{g}$, and
  - **Relations**:

    1 $\mathfrak{g}$. $s_J^2 = 1$, for all $J \subseteq I$,

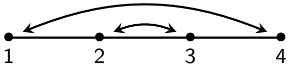
    2 $\mathfrak{g}$. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq I$ and $J \cap J' = \emptyset$,

    3 $\mathfrak{g}$. $s_J s_{J'} = s_{\theta_J(J')} s_J$, for all $J' \subseteq J \subseteq I$.

The cacti $J_{\mathfrak{gl}_n}$ and $J_{\mathfrak{sp}_{2n}}$

- Cartan type A_{n-1}: $I = [n-1]$, $\Delta = \{\alpha_i = \mathrm{e_i} - \mathrm{e_{i+1}}\}_{\mathrm{i} \in [\mathrm{n}-1]}$, $W = \mathfrak{S}_n$, $\theta(i) = n - i$,

$n = 5 \qquad A_4$



- Cartan type C_n: $I = [n]$, $\Delta = \{\alpha_i = \mathrm{e_i} - \mathrm{e_{i+1}}\}_{\mathrm{i} \in [\mathrm{n}-1]} \cup \{\alpha_\mathrm{n} = 2\mathrm{e_n}\}$, $\theta(i) = i$,
  $W = B_n =< r_1, \ldots, r_{n-1}, r_n : R1, R2, R3, R4 >$

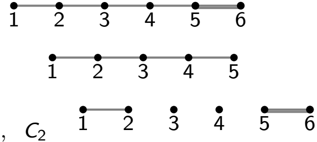
$$\begin{array}{lll} R1: & r_i^2 = 1, & 1 \leq i \leq n, \\ R2: & (r_i r_j)^2 = 1, & |i-j| > 1, \\ R3: & (r_i r_{i+1})^3 = 1, & 1 \leq i \leq n-2, \\ R4: & (r_{n-1} r_n)^4 = 1 & \end{array},$$

$w_0.\alpha_i = -\alpha_i, \quad \theta(i) = i.$

C_6

$J : A_5$

$J : A_2, \ A_1, \ A_1, \ \ C_2$



The cacti $J_{\mathfrak{gl}_n}$ and $J_{\mathfrak{sp}_{2n}}$

- The cactus group $J_{\mathfrak{gl}_n} = J_n$ is the group defined by
  - Generators: s_J, J connected subdiagram of the A_{n-1} Dynkin diagram,
  - Relations:
    1A. $s_J^2 = 1, J \subseteq [n-1]$,

    2A. $s_J s_{J'} = s_{J'} s_J, J \cap J' = \emptyset,\ J, J' \subseteq [n-1]$,

    3A. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]},\ [k,l] \subset [p,q] \subseteq [n-1]$.

- The cactus group $J_{\mathfrak{sp}_{2n}}$ is the group defined by
  - Generators: s_J, J connected subdiagrams of the C_n Dynkin diagram,
  - Relations:
    1C. $s_J^2 = 1,\ J \subseteq [n]$,

    2C. $s_J s_{J'} = s_{J'} s_J,\ J \cap J' = \emptyset,\ J, J' \subseteq [n]$,

    3C. 1. $s_{[p,n]} s_{[q,l]} = s_{[q,l]} s_{[p,n]},\ [q,l] \subset [p,n] \subseteq [n]$,

    2. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]},\ [k,l] \subset [p,q] \subseteq [n-1]$.

- Alternative $n-1$ generators for J_n, and $2n-1$ generators for $J_{\mathfrak{sp}_{2n}}$

$$s_{[1,j]},\ 1 \leq j \leq n-1, \qquad s_{[j,n]},\ 1 \leq j \leq n.$$

Normal crystals

A $\mathfrak{g}$-crystal B is a finite set B together with the maps

$$e_i, f_i : \mathrm{B} \to \mathrm{B} \sqcup \{0\},\ i \in I,\ \mathrm{wt} : \mathrm{B} \to \Lambda,$$

where for all $b, c \in \mathrm{B}$ and $i \in I$,

- $f_i(b) = c$ if and only if $b = e_i(c)$,
- if $e_i(b) \in B$, then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$,
  if $f_i(b) \in B$, then $\mathrm{wt}(f_i(b)) = wt(b) - \alpha_i$,
- $\varphi_i(b) = max\{k : f_i^k(b) \neq 0\}$, $\varepsilon_i(b) = \max\{k : e_i^k(b) \neq 0\}$,
- $\varphi_i(b) - \varepsilon_i(b) =< wt(b), \alpha_i^\vee >$.

- Let B be a normal crystal (crystal corresponding to the finite-dimensional quantum enveloping algebra $U_q(\mathfrak{g})$-representation).
  - For $J \subseteq I$, B_J is the *restriction of the normal crystal* B to the connected subdiagram J of I.
  - The crystal graph of B_J has the same vertices as B but the arrows are only those labelled in J, that is, we forget the crystal maps e_i, f_i, φ_i, and ε_i, for $i \notin J$.

Crystals of A_{n-1} and C_n tableaux and restrictions

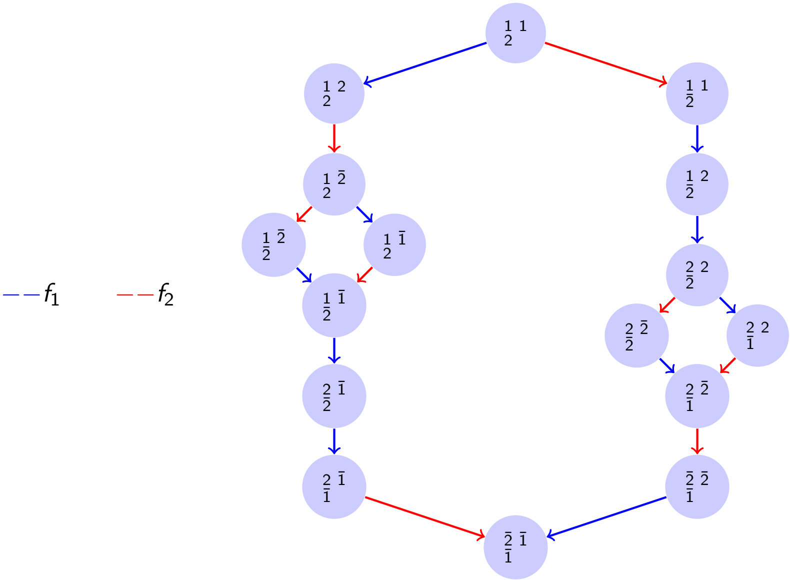
- SSYT(λ, n) crystal of semistandard Young tableaux of straight shape λ in the alphabet $[n]$.
- KN(λ, n) crystal of C_n Kashiwara-Nakashima (De Concini) tableaux of straight shape λ in the alphabet $[\pm n] = \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$

$$n = 4 \qquad Q = \begin{matrix} 1 \\ 2 \\ \overline{4} \\ \overline{2} \end{matrix} \qquad \begin{matrix} 1 \\ \emptyset \end{matrix} \quad \begin{matrix} 2 \\ \overline{2} \end{matrix} \quad \begin{matrix} \emptyset \\ \emptyset \end{matrix} \quad \begin{matrix} \emptyset \\ \overline{4} \end{matrix} \quad Q \text{ not KN column}, \qquad T = \begin{matrix} 2 \\ \overline{4} \\ \overline{2} \end{matrix} \qquad \begin{matrix} \emptyset \\ \emptyset \end{matrix} \quad \begin{matrix} 2 \\ \overline{2} \end{matrix} \quad \begin{matrix} \emptyset \\ \emptyset \end{matrix} \quad \begin{matrix} \emptyset \\ \overline{4} \end{matrix} \quad OK!$$

$$(\ell T, rT) = \begin{matrix} 1 \\ \overline{4} \\ \overline{2} \end{matrix} \quad \begin{matrix} 2 \\ \overline{4} \\ \overline{1} \end{matrix}$$

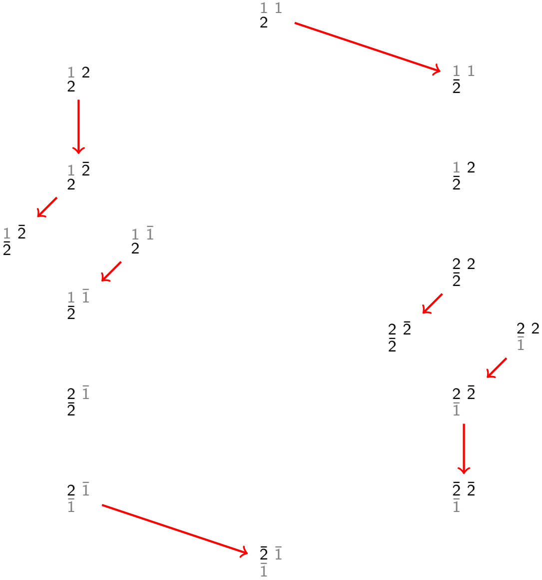
$$P = \begin{matrix} 2 \\ 4 \\ \overline{2} \end{matrix} \quad \begin{matrix} 2 \\ \overline{3} \\ \overline{1} \end{matrix} \quad \begin{matrix} \overline{1} \\ \\ \end{matrix} \qquad (\ell P, rP) = \begin{matrix} 1 \\ \overline{4} \\ \overline{2} \end{matrix} \quad \begin{matrix} 2| \\ \overline{4}| \\ \overline{1}| \end{matrix} \quad \begin{matrix} 2 \\ \overline{3} \\ \overline{1} \end{matrix} \quad \begin{matrix} 2| \\ \overline{3}| \\ \overline{1}| \end{matrix} \quad \begin{matrix} \overline{1} \\ \\ \end{matrix} \quad \begin{matrix} \overline{1} \\ \\ \end{matrix} \qquad \text{KN tableau}$$

Type C_2 crystal $\mathrm{KN}((2,1),2)$



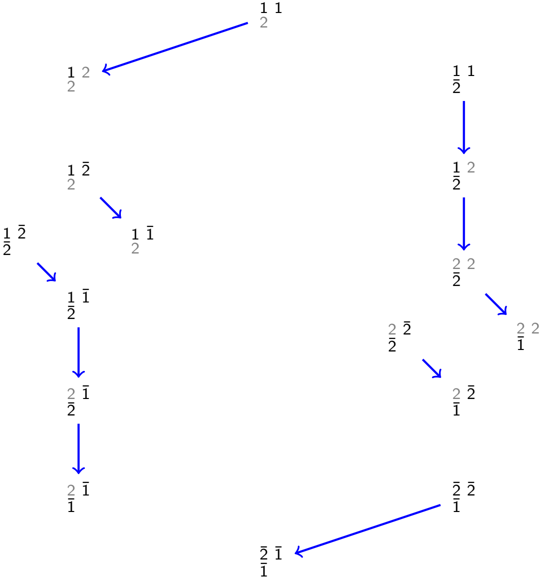
$\mathrm{KN}_{\{2\}}((2,1),2)$ crystal

$J = \{2\}$, C_1 crystal, alphabet $\{1 < 2 < \bar{2} < \bar{1}\}$ $f_2 : 2 \mapsto \bar{2}$



$\mathrm{KN}_{\{1\}}((2,1),2)$ crystal

$J = \{1\}$, A_1 crystal, alphabet $\{1 < 2 < \overline{2} < \overline{1}\}$ $f_1 : 1 \mapsto 2,\ \overline{2} \mapsto \overline{1}$



Full and partial Schützenberger-Lusztig involutions

- Let $\mathrm{B}(\lambda)$ a normal crystal with highest weight λ, and
  - u_λ^{high}, u_λ^{low} highest and lowest weight elements.
- The Schützenberger-Lusztig involution ξ is the unique set involution $\xi : \mathrm{B}(\lambda) \to \mathrm{B}(\lambda)$ such that, for all $b \in \mathrm{B}(\lambda)$, and $i \in I$,
  - $e_i \xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i \xi(b) = \xi e_{\theta(i)}(b)$
  - $\mathrm{wt}(\xi(b)) = w_0.\mathrm{wt}(\mathrm{b})$, w_0 the long element of the Weyl group W.

- Let $b \in \mathrm{B}(\lambda)$ and $b = f_{j_r} \cdots f_{j_1}(u_\lambda^{\text{high}})$.
  - in type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\text{low}})$, $\mathrm{wt}(\xi(b)) = \text{reverse}\,\mathrm{wt}(\mathrm{b})$
  - in type C_n, $\quad \xi(b) = e_{j_r} \cdots e_{j_1}(u_\lambda^{\text{low}})$, $\mathrm{wt}(\xi(b)) = -\mathrm{wt}(\mathrm{b})$

Full evacuation and reversal: C_n direct algorithms

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & \bar{1} \\ \hline 3 & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array} \in \mathrm{KN}((3,2,1),3) \qquad Q = \begin{array}{|c|c|}\hline & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} \\ \cline{1-1}\end{array} \in \mathrm{KN}((3,2,1)/(1),3)$$

- [Santos, 2021] $\mathrm{evac}^C(T) = \xi(T)$

$$T^\pi = \begin{array}{|c|c|c|}\hline * & * & 3 \\ \hline * & 2 & \bar{3} \\ \hline 1 & \bar{2} & \bar{1} \\ \hline\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|c|}\hline * & 2 & 3 \\ \hline * & \bar{3} & \bar{1} \\ \hline 1 & \bar{2} & * \\ \hline\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|c|}\hline * & 2 & 3 \\ \hline 1 & 3 & \bar{1} \\ \hline \bar{2} & * & * \\ \hline\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline * & \bar{3} & \bar{1} \\ \hline \bar{2} & * & * \\ \hline\end{array}$$

$$\overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|c|}\hline 1 & 2 & \bar{2} \\ \hline 3 & * & \bar{1} \\ \hline \bar{3} & * & * \\ \hline\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \bar{3} & \bar{1} & * \\ \hline \bar{2} & * & * \\ \hline\end{array} \qquad \mathrm{evac}^C(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \bar{3} & \bar{1} \\ \cline{1-2} \bar{2} \\ \cline{1-1}\end{array}$$

- $\mathrm{reversal}(Q) = \xi(Q)$

$$\begin{array}{|c|c|}\hline a & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & a \\ \hline 3 & \bar{1} \\ \hline \bar{3} \\ \cline{1-1}\end{array} \overset{\mathrm{rect}}{\longrightarrow} \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 3 & a \\ \hline \bar{3} \\ \cline{1-1}\end{array} \overset{\mathrm{evac}}{\longrightarrow} \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \bar{3} & a \\ \hline \bar{2} \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|}\hline 2 & 3 \\ \hline a & \bar{3} \\ \hline \bar{2} \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\longrightarrow} \begin{array}{|c|c|}\hline a & 3 \\ \hline 2 & \bar{3} \\ \hline \bar{2} \\ \cline{1-1}\end{array} = \mathrm{reversal}(Q)$$

Problems to explore

Problem

A symplectic version for the Benkart-Sottille-Stroomer tableau-switching on KN tableaux.

Problem

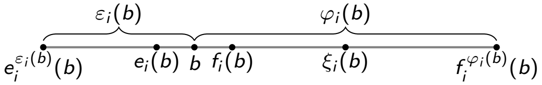
A symplectic rectification Fomin's growth diagram-like for KN tableaux.

Partial Schützenberger involutions

- For $J \subseteq I$, the partial Schützenberger-Lusztig involution ξ_J is the Schützenberger involution defined on the normal crystal $\mathrm{B}_J(\lambda)$.

1. ***J* has a sole node: the Weyl group action on the crystal**
   - [Kashiwara 94] The ξ_i, for $i \in I$, define an action of the Weyl group W on $\mathrm{B}(\lambda)$,

$$r_i.b = \xi_i(b), \quad r_i.\mathrm{wt} b = \mathrm{wt}\, \xi_i(b),\ b \in B,\ \ u_\lambda^{\mathrm{low}} = w_0.u_\lambda^{\mathrm{high}}$$



2. ***J* has more than one node**

   b_J^{high}, b_J^{low} for the connected component of $\mathrm{KN}_J(\lambda, n)$ containing b, and $b = f_{j_r} \cdots f_{j_1}(b_J^{\mathrm{high}})$, for $j_r, \ldots, j_1 \in J$.
   - $J = [p, n]$, $\mathrm{KN}_J(\lambda, n)$ type C_{n-p+1} crystal,

$$\xi_J(b) = e_{j_r} \cdots e_{j_1}(b_J^{\mathrm{low}}).$$

   - $J = [1, p]$, $p < n$, $\mathrm{KN}_J(\lambda, n)$ type A_p, crystal,

$$\xi_J(b) = e_{p-j_r+1} \cdots e_{p-j_1+1}(b_J^{\mathrm{low}}).$$

Problem

Problem

- *For* $1 < p \leq n$*, a direct procedure to calculate* $\xi_{[p,n]}$ *on the type* C_{n-p+1} *crystal* $KN_{[p,n]}(\lambda, n)$.

$$n = 3,\ p = 2 \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & \bar{1} \\ \hline 3 & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}$$

- *For* $1 \leq p \leq n-1$*, a direct procedure to calculate* $\xi_{[1,p]}$ *on the type* A_p *crystal* $KN_{[1,p]}(\lambda, n)$.

$$n = 3,\ p = 1 \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & \bar{1} \\ \hline 3 & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}$$

Internal cactus group action on a normal crystal

Theorem

[Halacheva, 2016] *The following is a group homomorphism*

$$\begin{array}{ccc} J_{\mathfrak{g}} & \rightarrow & \mathfrak{S}_B \\ s_J & \mapsto & \xi_J, \end{array}$$

$wt(\xi_J(b)) = w_0^J.wt(b)$, $b \in B$.

- For the $\mathfrak{gl}_n$-crystal $\mathrm{SSYT}(\lambda, n)$, the map

$$s_{[1,j]} \mapsto \xi_{[1,j]} = \mathrm{evac}_{j+1}, \ 1 \leq j \leq n-1,$$

defines an action of the cactus group J_n on the set $\mathrm{SSYT}(\lambda, n)$.

- For the $\mathfrak{sp}_{2n}$-crystal $\mathrm{KN}(\lambda, n)$, the map

$$\begin{array}{cccc} s_{[1,j]} & \mapsto & \xi^A_{[1,j]}, & 1 \leq j \leq n-1, \\ s_{[j,n]} & \mapsto & \xi^C_{[j,n]}, & 1 \leq j \leq n, \end{array}$$

defines an action of $J_{\mathfrak{sp}_{2n}}$ on the set $\mathrm{KN}(\lambda, n)$.

The A and shifted Berenstein-Kirillov groups

The *Berenstein–Kirillov group* $\mathcal{BK}$ (*Gelfand-Tsetlin group*) [Berenstein, Kirillov, 1995], is the free group generated by the Bender-Knuth involutions t_i, for $i > 0$, modulo the relations they satisfy on straight shaped semistandard Young tableaux.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&2&2&3&3&3&3\\\hline 2&2&3&3&3&3\\\cline{1-6}\end{array} \xrightarrow{t_2} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&2&2&2&2&3&3\\\hline 2&3&3&3&3&3\\\cline{1-6}\end{array} = t_2(T) \neq \xi_2(T)$$

List of known relations satisfied by the t_i's in $\mathcal{BK}$:

1. $t_i^2 = 1, \qquad t_i t_j = t_j t_i,$ for $|i - j| > 1$ [BK 95],
2. $(t_1 q_{[1,i]})^4 = 1,$ for $i > 2$ [BK95],
3. $(t_1 t_2)^6 = 1$ [BK 95],
4. $(t_i q_{[j,k-1]})^2 = 1,$ for $i + 1 < j < k$ [Chmutov, Glick, Pylyavskii 2016],

   where

   - $q_{[1,i]} := t_1(t_2 t_1) \cdots (t_i t_{i-1} \cdots t_1)$, for $i \geq 1$,
   - $q_{[j,k-1]} := q_{[1,k-1]} q_{[1,k-j]} q_{[1,k-1]}$, for $j < k$.

Remark

$q_{[1,i]} = \xi_{[1,i]}$, $i \geq 1$, and $q_{[j,k-1]} = \xi_{[j,k-1]}$, $j < k$.

The $\mathcal{BK}_n$, $\mathcal{SBK}_n$ groups and the cactus J_n

Proposition

[Berenstein, Kirillov, 1995] *The elements* $q_{[1,1]}, \ldots, q_{[1,n-1]}$ *are generators of* $\mathcal{BK}_n$*,*

$$t_1 = q_{[1,1]}, \qquad t_i = q_{[1,i-1]} q_{[1,i]} q_{[1,i-1]} q_{[1,i-2]}, \text{ for } i \geq 2, \;\; q_0 := 1.$$

- The following are group epimorphisms from J_n to $\mathcal{BK}_n$.
  1. $s_{[i,j]} \mapsto q_{[i,j]}$ [Chmutov, Glick, Pylyavskii 2016].
  2. $s_{[1,j]} \mapsto q_{[1,j]}$ [Halacheva 2016, 2020].

  The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n.

- [Rodrigues, 2020] There is a natural action of cactus group J_n on a shifted tableau crystal-like $\mathrm{ShST}(\lambda, n)$ [Gillespie-Levinson-Purbhoo, 2019], given by the group homomorphism:

$$J_n \longrightarrow \mathfrak{S}_{\mathrm{ShST}(\lambda,n)}$$
$$s_{[1,i]} \longmapsto \xi_{[1,i]}.$$

- [Rodrigues, 2021] The following map is an epimorphism from J_n to $\mathcal{BK}_n$

$$s_{[i,j]} \longmapsto \mathsf{q}_{[i,j]}.$$

$\mathcal{SBK}_n$ is isomorphic to a quotient of J_n.

The type C Berenstein-Kirillov group $\mathcal{BK}^C$

- Neither symplectic Bender-Knuth involutions t_i^C nor symplectic tableau switching are known for KN tableaux.
- Motivation:For $n \geq 1$, the partial Schützenberger-Lusztig involutions $\xi_1^A = q_{[1,1]}, \ldots, \xi_{[1,n-1]}^A = q_{[1,n-1]}$ are generators for the Berenstein-Kirillov group $\mathcal{BK}_n$.

Definition

[Proposal] The *symplectic Berenstein–Kirillov group* $\mathcal{BK}_n^C$, $n \geq 1$, is the free group generated by the $2n-1$ partial Schützenberger involutions

$$q_{[1,i]}^A =: \xi_{[1,i]}^A,\ 1 \leq i < n, \ \text{ and } \ q_{[i,n]}^C =: \xi_{[i,n]}^C,\ 1 \leq i \leq n,$$

on straight shaped KN tableaux on the alphabet $[\pm n]$ modulo the relations they satisfy on those tableaux.

Theorem

The following is a group epimorphism from $J_{\mathfrak{sp}_{2n}}$ to $\mathcal{BK}_n^C$:

$$s_{[1,j]} \mapsto q_{[1,j]}^A,\ 1 \leq j < n, \qquad s_{[j,n]} \mapsto q_{[j,n]}^C,\ 1 \leq j \leq n$$

$\mathcal{BK}_n^C$ is isomorphic to a quotient of $J_{\mathfrak{sp}_{2n}}$.

Problems

Problem

Make explicit the symplectic Bender-Knuth involutions t_i^C on KN tableaux.

Problem

Find explicit relations for $\mathcal{BK}_n^C$ using the generators t_i^C.

Problem

An approach to $\mathcal{BK}_n^C$ via King tableaux [King 1975].