

Higher Dimensional Chip-Firing

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Chip-Firing

Discrete Dynamical System

Combinatorial Game

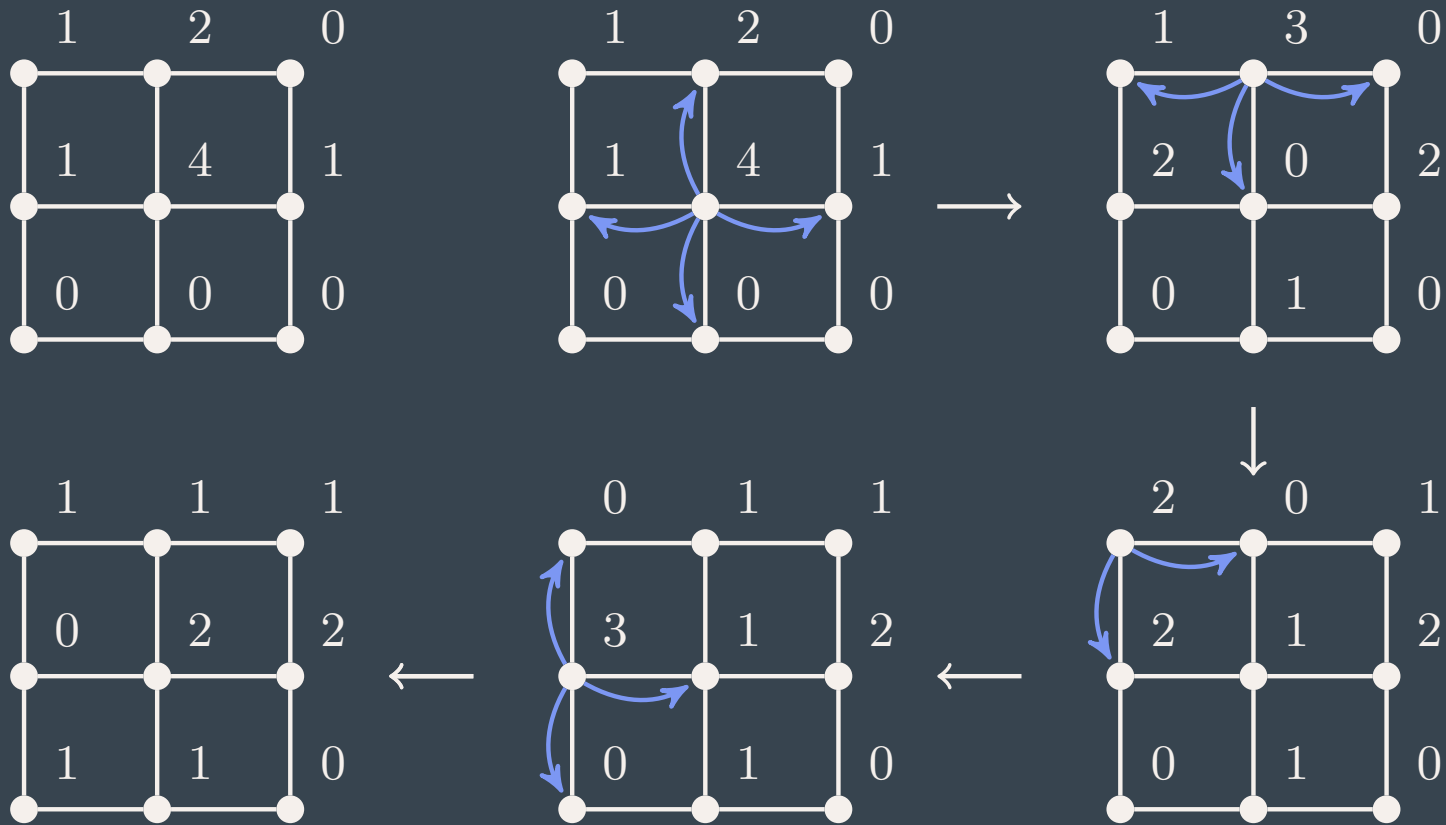
Physics Model

A commodity (chips, dollars, sand) is moved (fired, exchanged, toppled) amongst sites of a network.

Dynamics are given by a local thresholding rule.

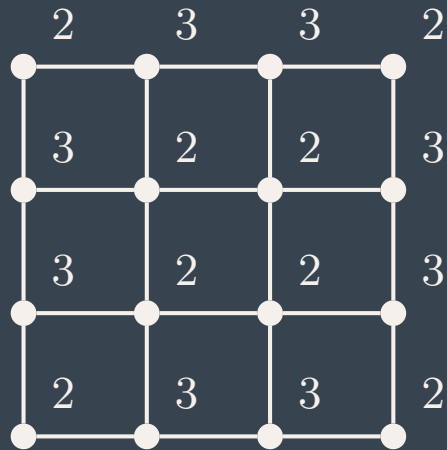
If you have enough to share with all of your neighbors, then share!

Chip-Firing – Basic Dynamics

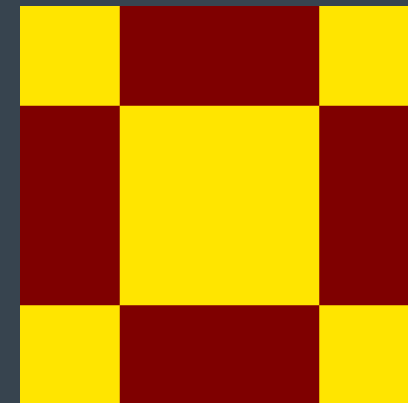


Let's look at some larger examples. How can we visualize them?

Color	Number of chips
	0
	1
	2
	3



2	3	3	2
3	2	2	3
3	2	2	3
2	3	3	2



Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.

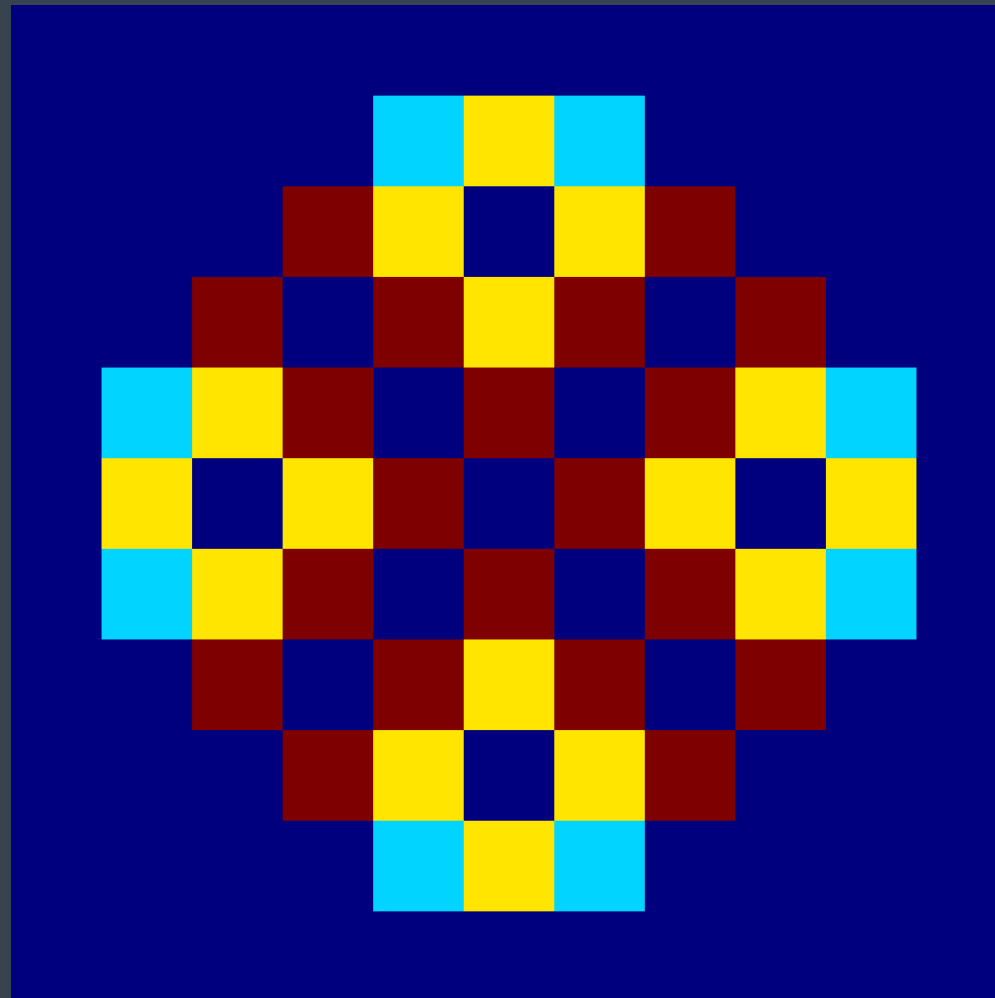
$$N = 10$$



Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.

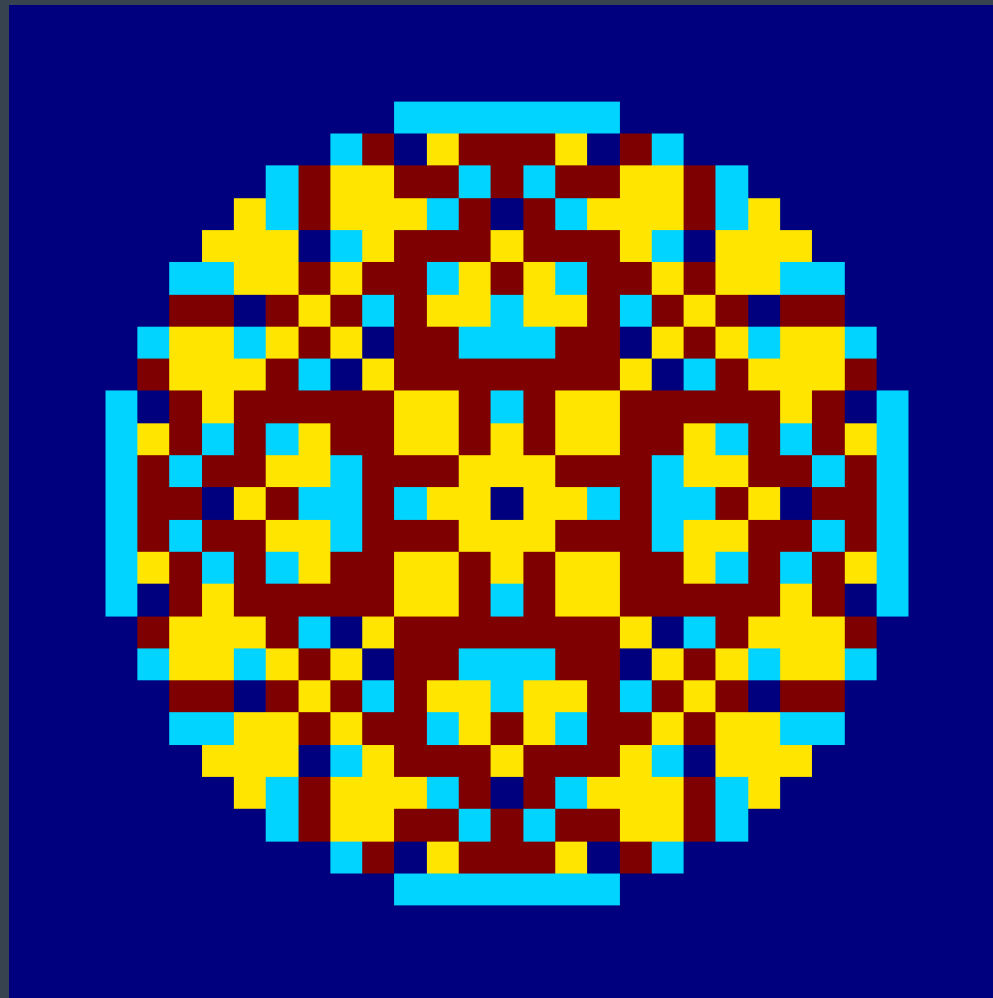
$$N = 100$$



Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.

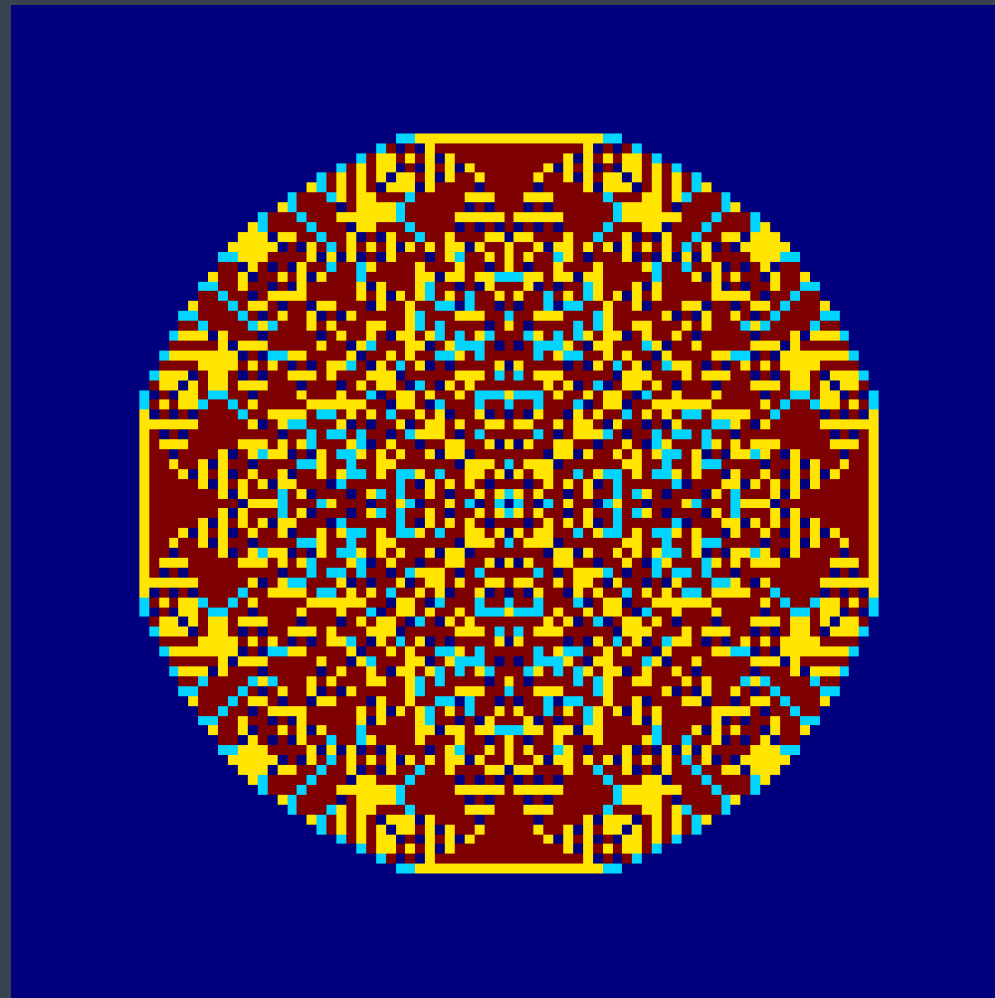
$N = 1,000$



Pattern Formation

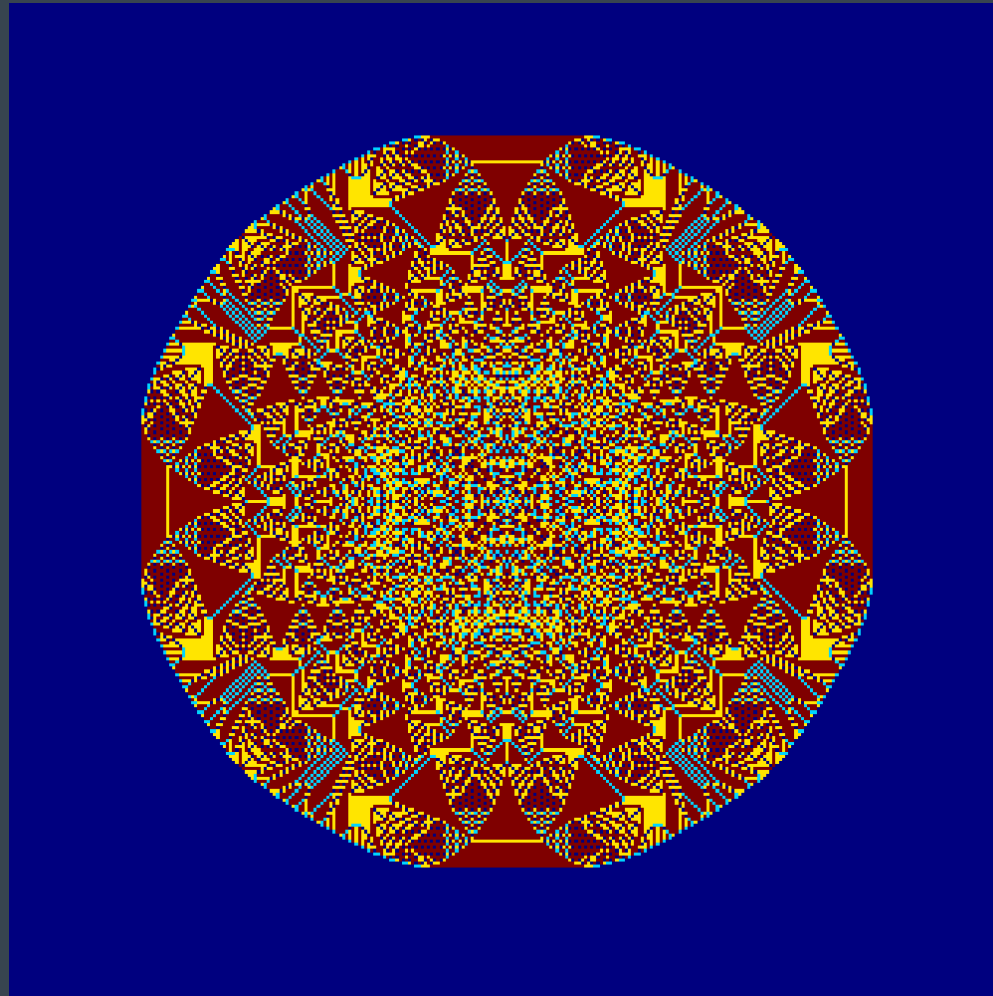
Suppose we drop N chips at the origin of the two-dimensional grid.

$N = 10,000$



Pattern Formation

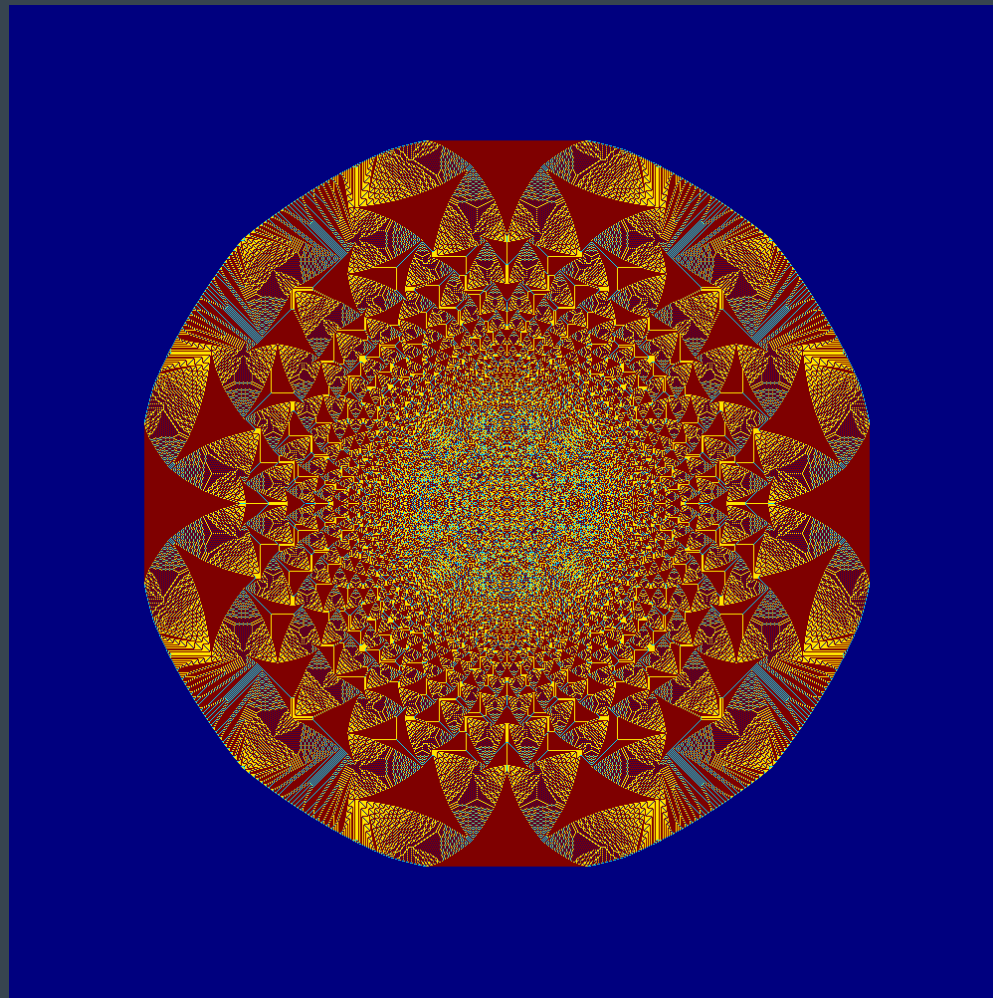
Suppose we drop N chips at the origin of the two-dimensional grid.
 $N = 100,000$



Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.

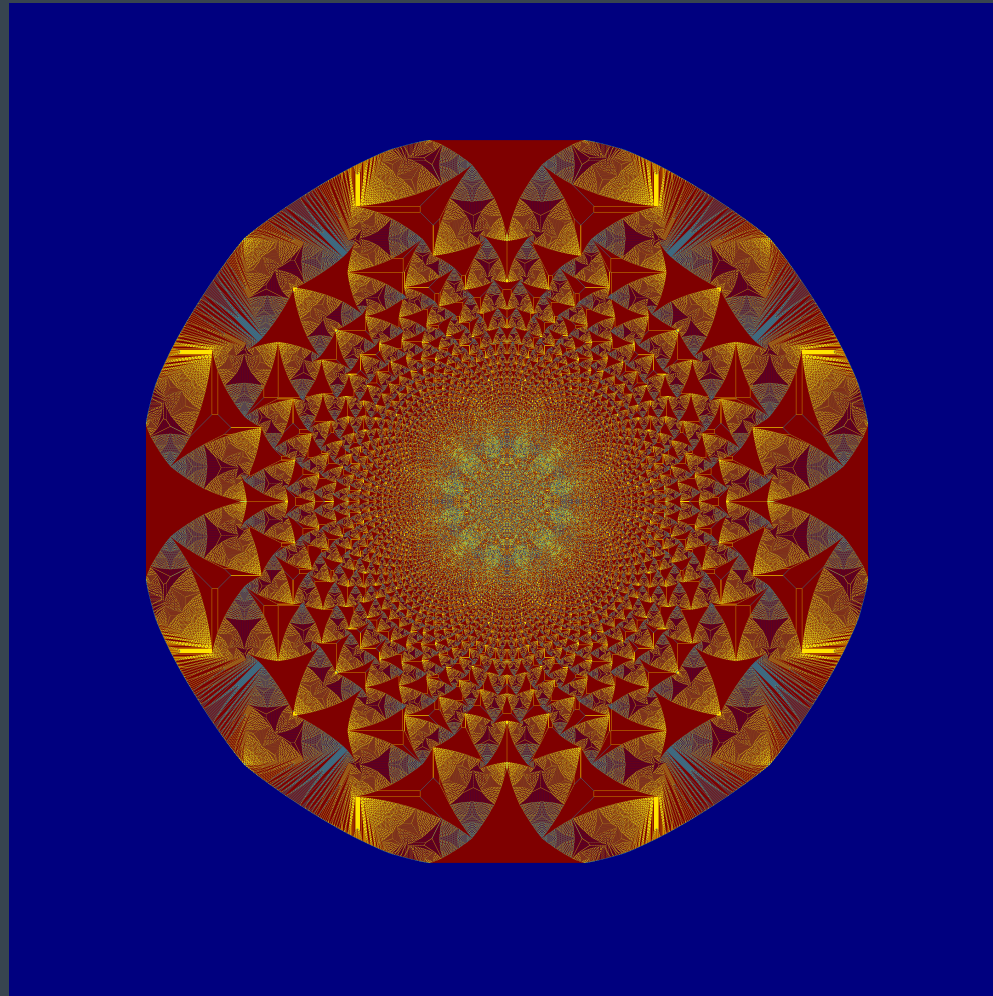
$N = 1,000,000$

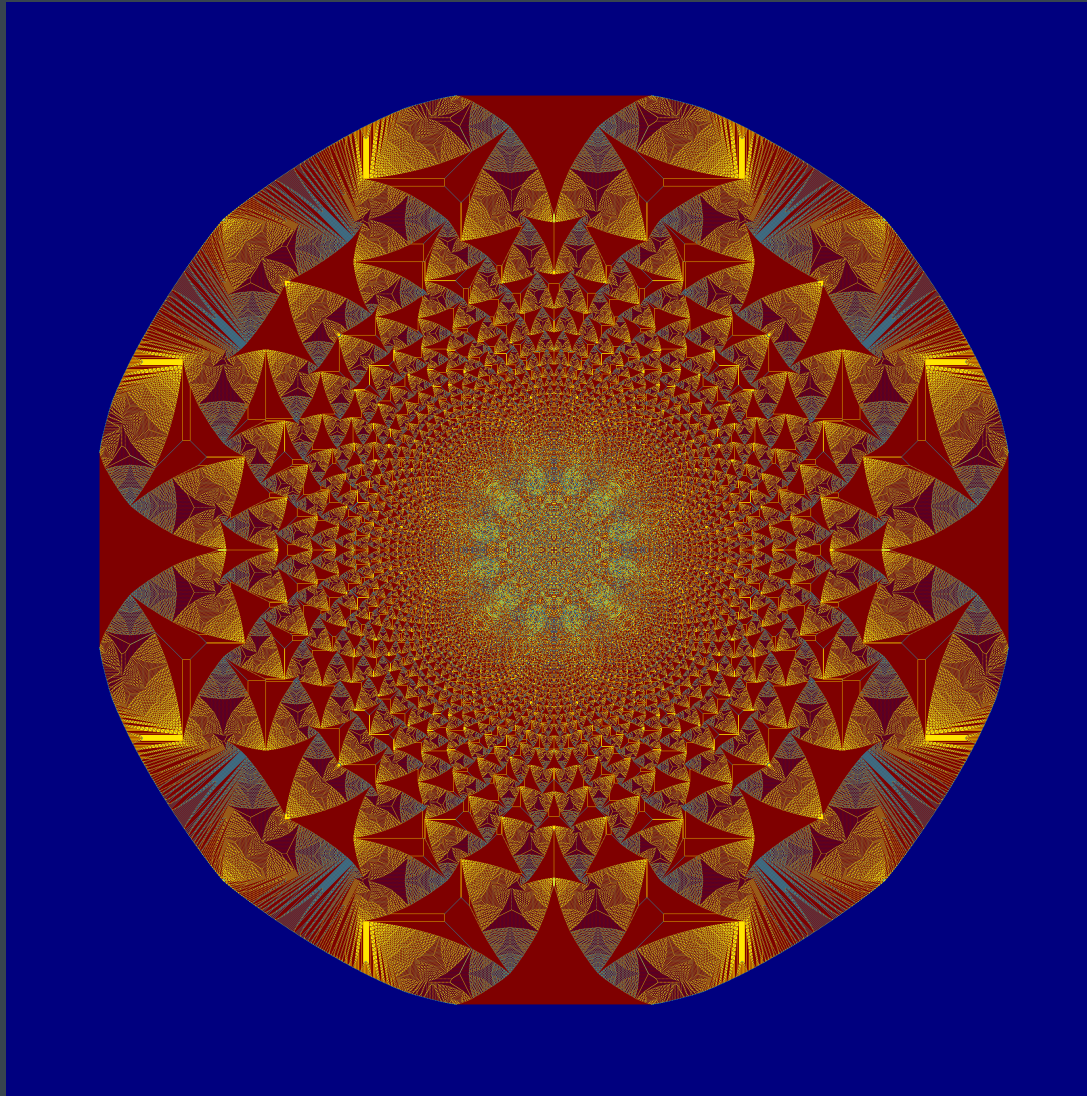


Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.

$N = 10,000,000$

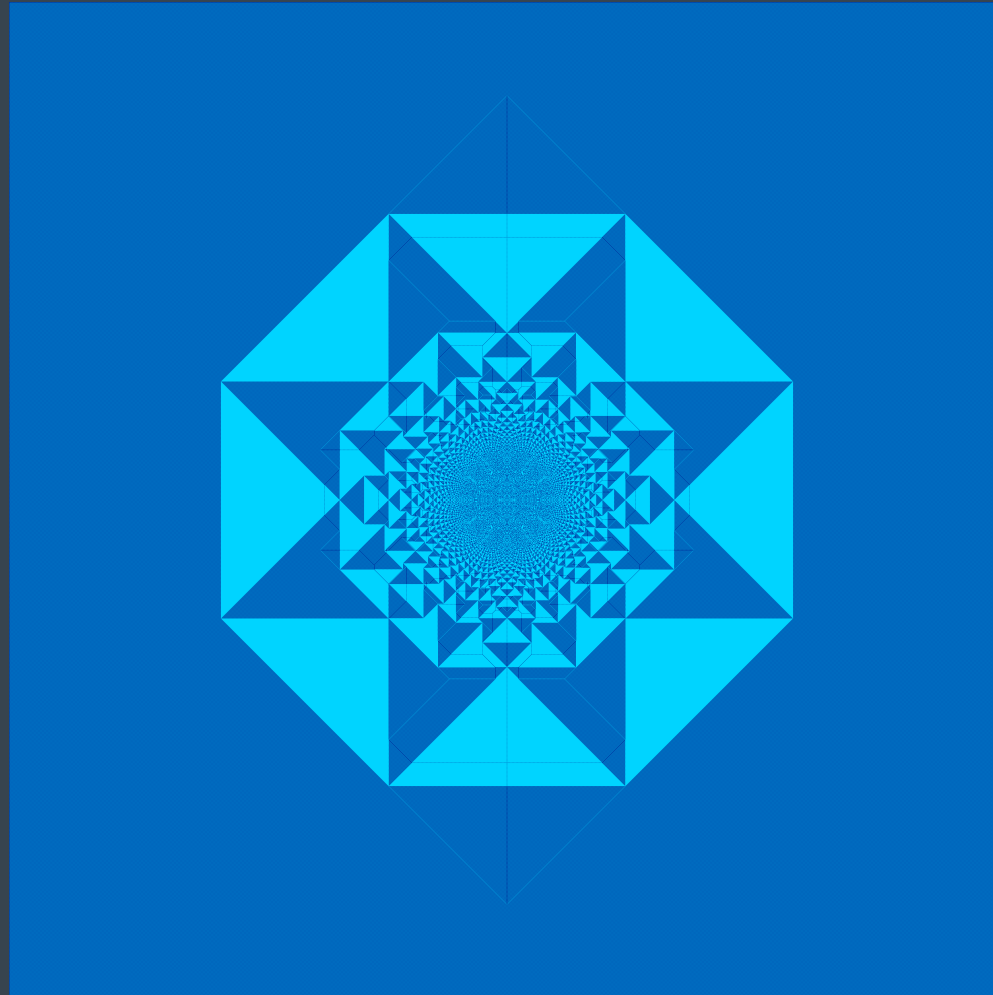




(Bak, Tang, Wiesenfeld '88, Dhar '06, Creutz '04, Pstojic '03, Caracciolo, Paoletti, Sportiello '08, Paoletti '14, Levine, Pegden, Smart '13, '16, '17)

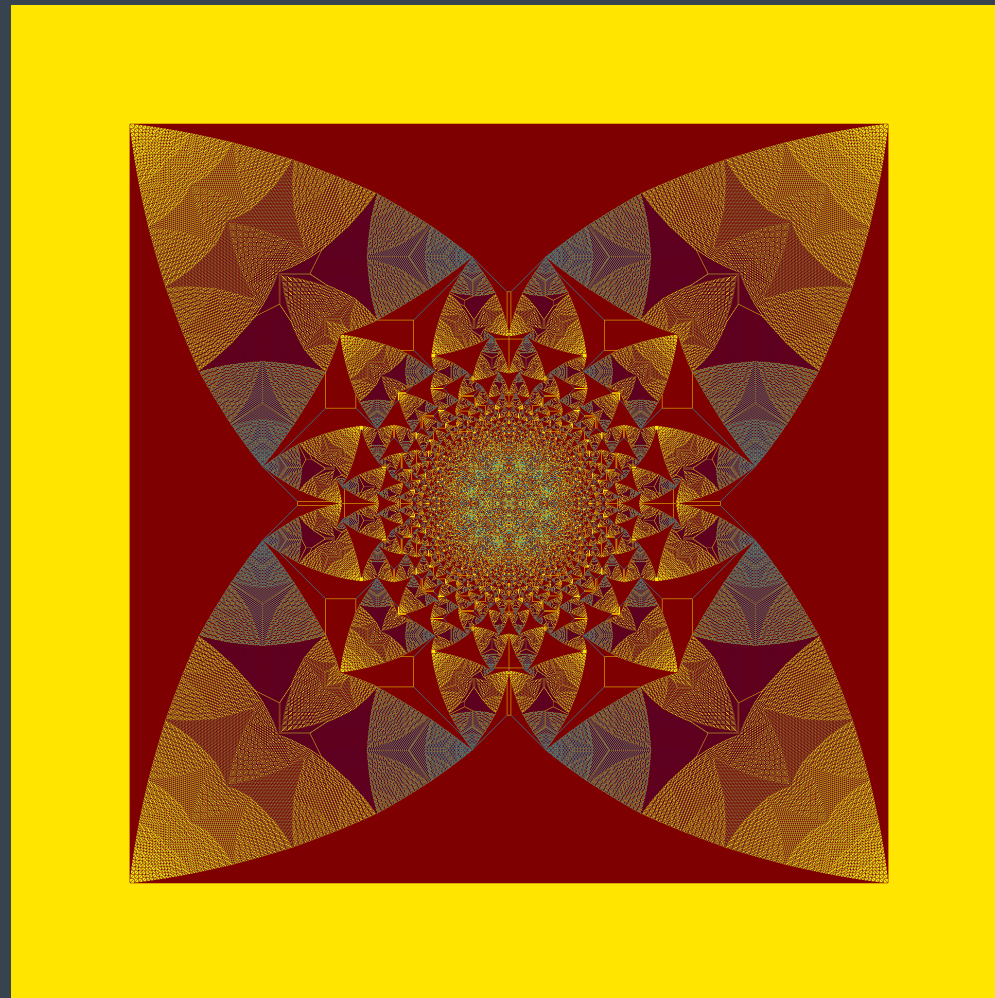
Pattern Formation

Suppose we drop N chips at the origin of the F-Lattice.
With checkerboard background 0 / 1.



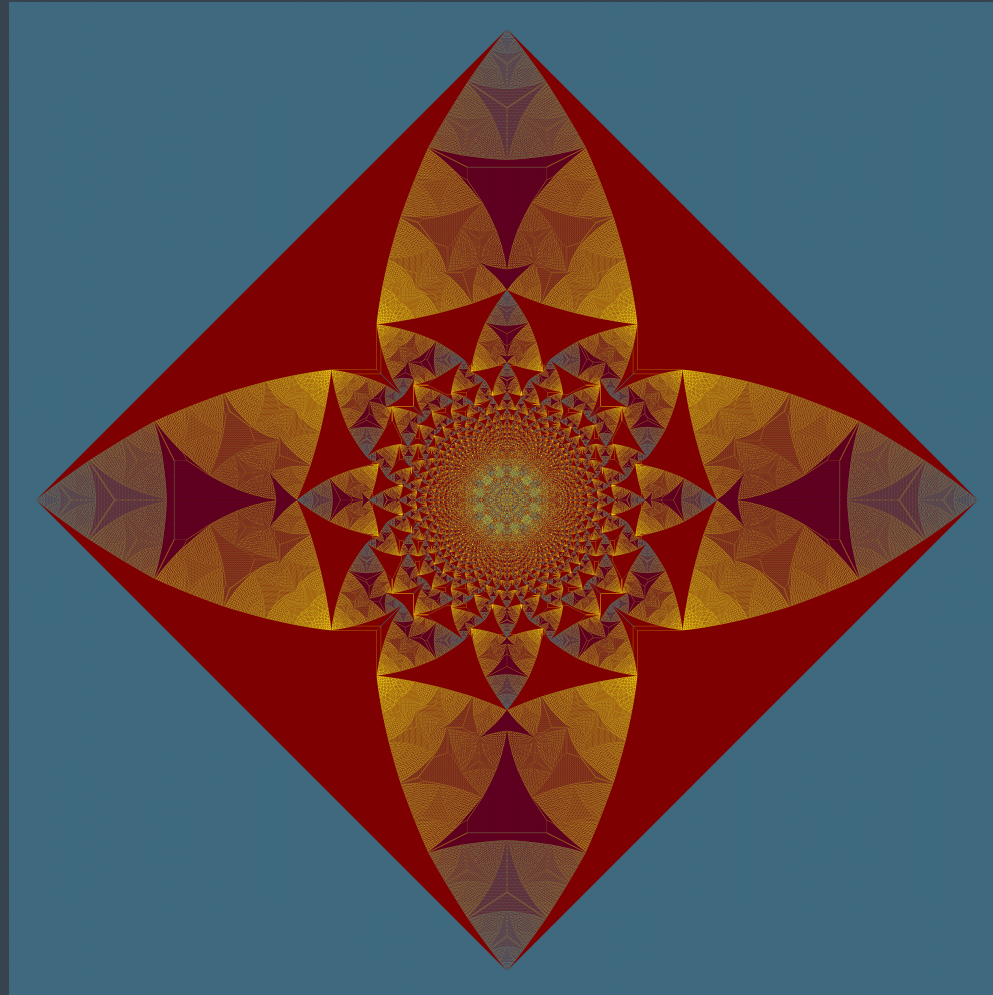
Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.
With background height 2.



Pattern Formation

Suppose we drop N chips at the origin of the two-dimensional grid.
With checkerboard background 1 / 3.



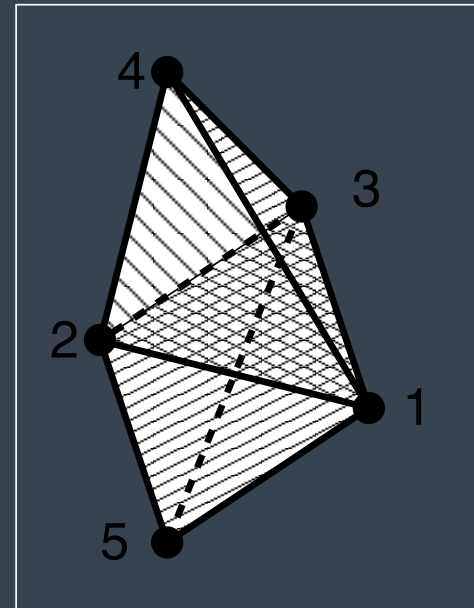
Chip-Firing in Higher Dimensions

- Algebraic (Duval, K., Martin '11, '14)

Combinatorial Laplacian
(Hodge Laplacian)

Higher dimensional spanning trees
(simplicial matroids)

Sandpile group $\mathcal{S}(G)$
(family of group invariants)



Flow on edges

Reroute across incident faces

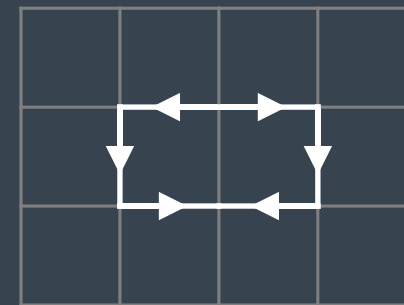
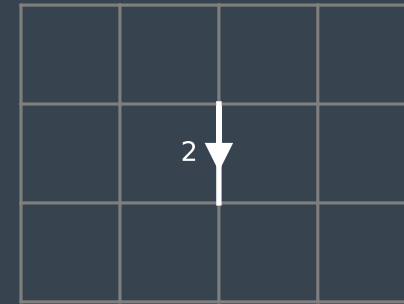
Chip-Firing in Higher Dimensions

- Dynamic (Felzenszwalb, K. '19)

Does the process stop?

Order of firings?

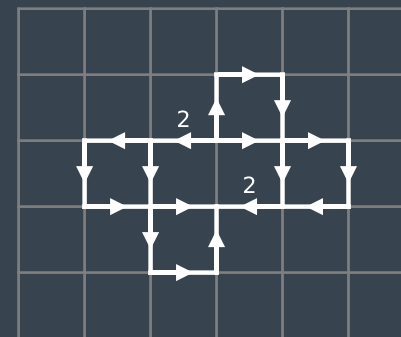
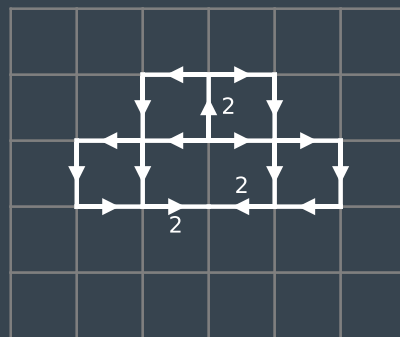
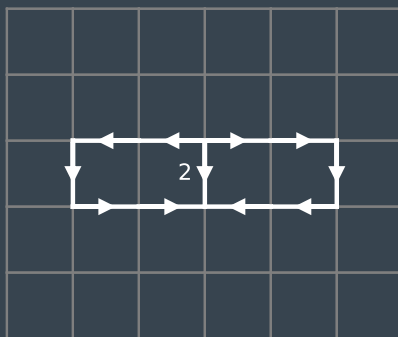
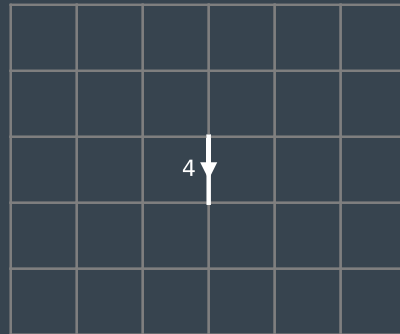
Pattern Formation?



- Labeled chip-firing
(Hopkins, McConville, Propp '17)
- Root system chip-firing
(Galashin, Hopkins, McConville, Postnikov '18)

Chip-Firing in Higher Dimensions

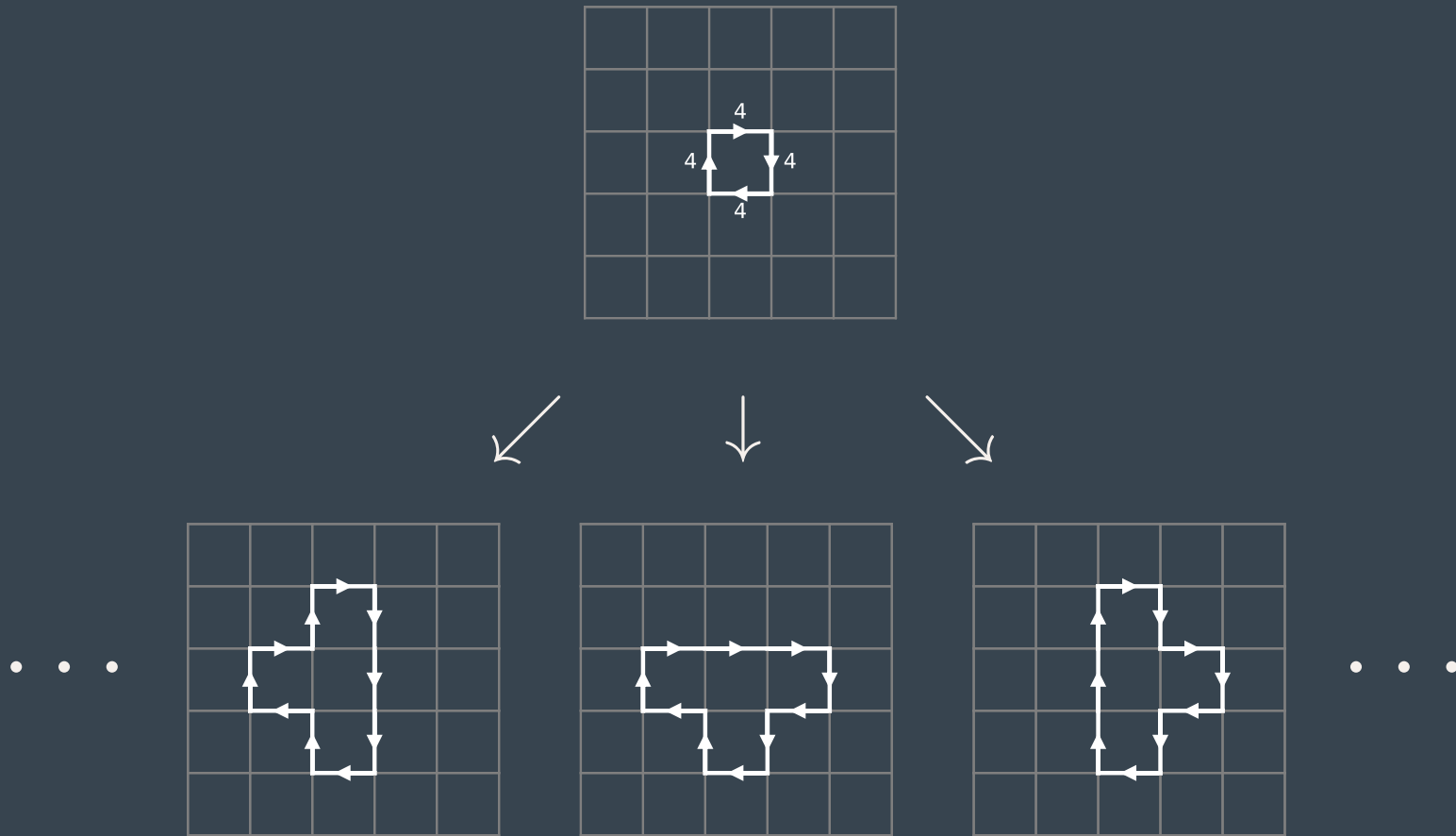
- A non-terminating example:



...

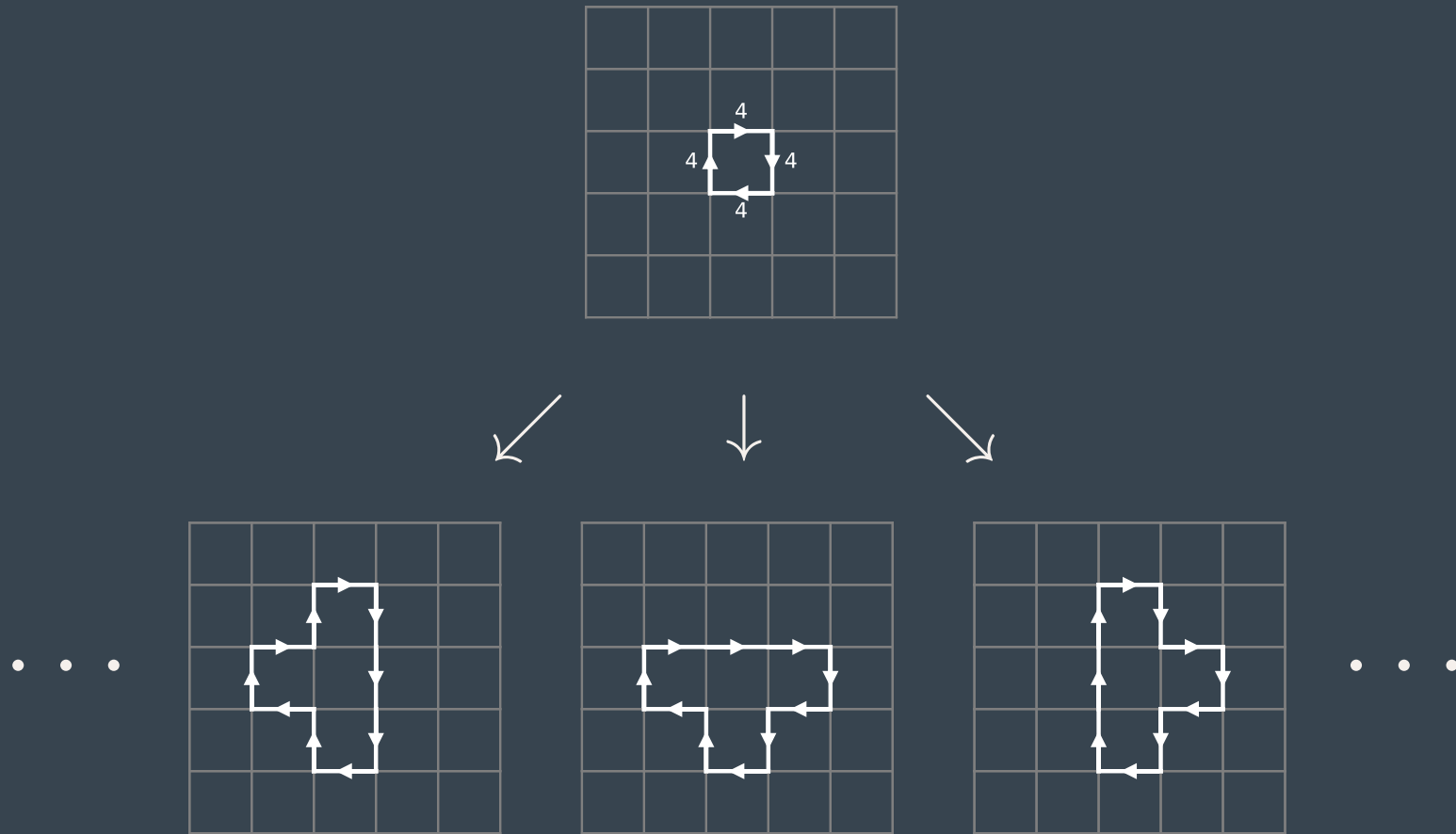
Chip-Firing in Higher Dimensions

- Conservative flows (circulations) terminate:



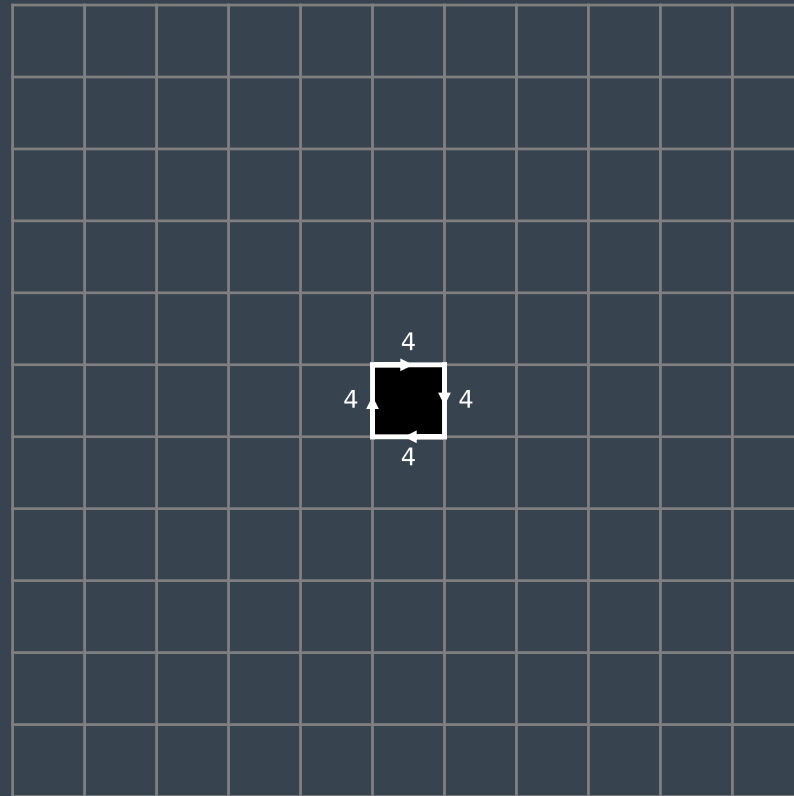
Chip-Firing in Higher Dimensions

- Order matters!



Chip-Firing in Higher Dimensions

- Remove a face from the grid:
(Topological Constraint)



Chip-Firing in Higher Dimensions

