Enumerative Combinatorics with Filling of Polyominoes

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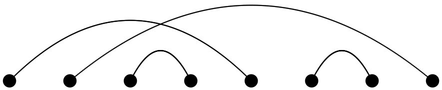
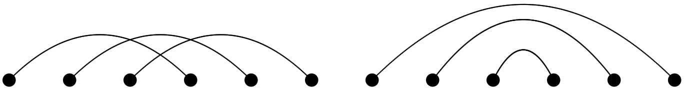
A Combinatorial Problem



Theorem

The number of non-crossing matchings, as well as the number of non-nesting matchings, are equal to the Catalan number.

3-crossing and 3-nesting:



Theorem (Chen, Deng, & Du)

The number of 3-non-crossing matchings is equal to the number of 3-non-nesting matchings.

Crossing and nesting numbers

Let $\text{cros}(M)$ ($\text{nest}(M)$) be the maximal i such that M has an i-crossing (i-nesting).

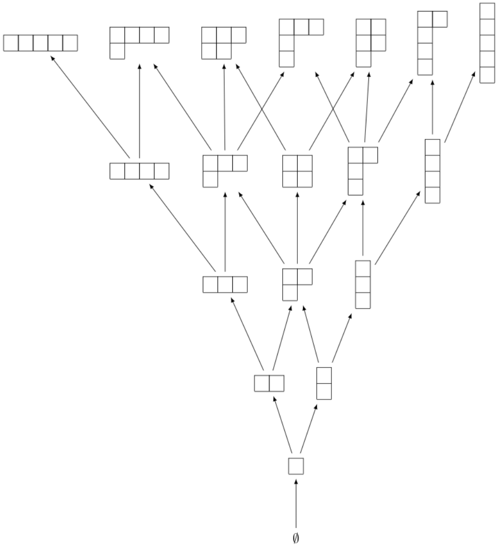
Theorem (CDDSY)

The statistics $\text{cros}(M)$ and $\text{nest}(M)$ have a symmetric joint distribution.

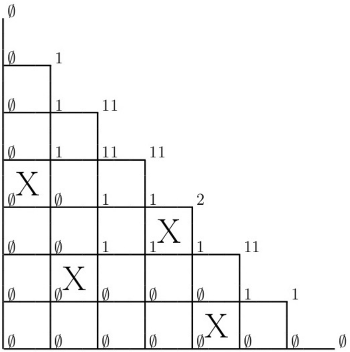
There is a bijection between the set of matchings of $[2n]$ and *oscillating tableaux* of length $2n$ and shape $\emptyset$, where
oscillating tableaux of shape λ:
a walk in the Young lattice starting at $\emptyset$ and ending at λ.

The bijection is constructed using the **row insertion of the RSK algorithm.**

Young Lattice



Triangular Shape and Growth Diagram



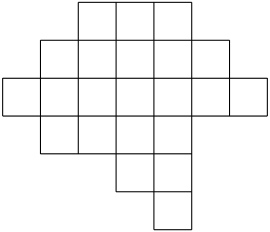
- Diagrams $\Longleftrightarrow$ 01-fillings of Δ
- RSK Algorithm $\Longleftrightarrow$ Growth diagram
- Tableaux sequences $\Longleftrightarrow$ Partitions on the right boundary
- $\mathrm{cros}(M)$, $\mathrm{nest}(M)$ $\Longleftrightarrow$ maximal NE/SE chains

[Krattenthaler] The above can be extended to fillings of Ferrers shapes and other polyominoes

Moon Polyominoes

- *convex* rows and columns
- *comparable* rows and columns

(L-connected)



- Lengths of rows from top to bottom form a unimodal sequence.
- Every cell is filled with an integer in $\mathbb{N}$.

ne-chains in 01-fillings

- A *k*-ne-chain: A set of 1-cells $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k, j_1 < \cdots < j_k$ such that the smallest rectangle containing them is a subset of $\mathcal{M}$.
- For a 01-filling M, $\mathrm{ne}(M)$ is the length of the largest ne-chain.

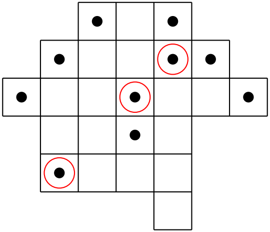
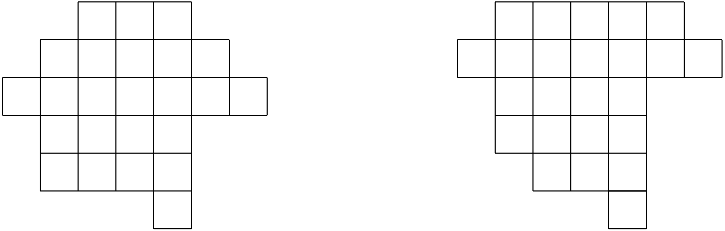


Figure: A 01-filling M of a moon polyomino with $\mathrm{ne}(M) = 3$. The 1's are represented by dots and the 0-cells are left empty. The circled dots form the only 3-chain in M.

Permuting rows

For a moon polyomino $\mathcal{M}$, let $\sigma\mathcal{M}$ be another moon polyomino obtained by permuting the rows of M.



Theorem (Rubey)

The number of 01-fillings with the longest northeast chains of size k and exactly c_i non-zero entries in column i are equal for $\mathcal{M}$ and $\sigma\mathcal{M}$.

Special Cases and Applications

- Chen, Deng, Du, Stanley, & Yan (2007) - Matchings and set partitions
- Backelin, West & Xin (2007) + Krattenthaler (2006) + - Ferrers and reverse Ferrers shapes
- de Mier (2006) – Graphs and multigraphs with given degree sequence
- Jonsson (2007) + Jonsson & Welker (2007) - stack polyominoes, k-triangulation of a polygon
- Jelinet and Mansour (2012) – Pattern avoidance in words and set partitions

Our Project.

An identity from the representation theory of the partition algebra $\mathbb{C}A_k(n)$:

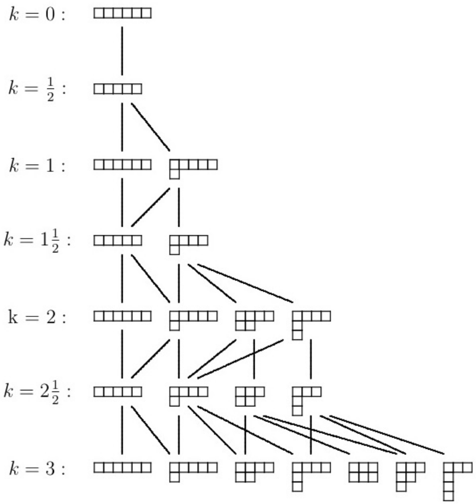
[Halverson & Lewandowski]

Let $n \geq 2k$. Then

$$n^k = \sum_{\lambda \in \Lambda_n^k} f^\lambda m_k^\lambda$$

where Λ_n^k is the set of partitions of n with $|\lambda| - \lambda_1 \leq k$, f^λ is the number of SYT of shape λ, and m_k^λ is the number of vacillating tableaux of shape λ and length $2k$.

Figure 1: Bratteli Diagram for $\mathbb{C}A_k(6)$



Theorem (Halverson & Lewandowski)

There is a bijection between sequences in $\{(i_1, i_2, \ldots, i_k) : 1 \leq i_j \leq n\}$ *and*

$$\bigsqcup_{\lambda \in \Lambda_n^k} \mathcal{SYT}(\lambda) \times \mathcal{VT}_k(\lambda).$$

The bijection:

1. Start with the SYT of shape (n).
2. Recursively, for $j = 1$ to k,
    1. remove i_j using jeu de taquin, then
    2. insert i_j using RSK row insertion.

The Project

What is the Combinatorics behind this bijection.