Combinatorics of Convex Polytopes

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Definitions

Definition

A convex *polytope* is the convex hull of a finite set of points in R^d.

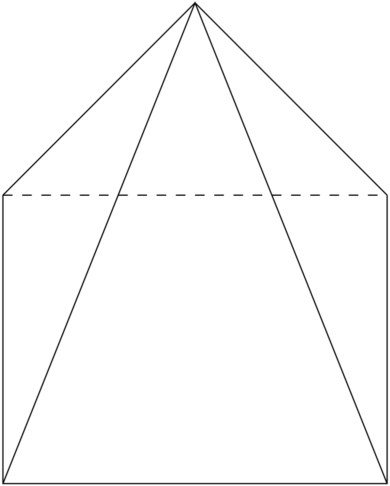
Definition

A *face* of a polytope is the intersection of the polytope with a supporting hyperplane.

Definition

The *f-vector* of a polytope is the vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces of the d-polytope P.

Example



$f_0 = 5$
$f_1 = 8$
$f_2 = 5$

Which integer vectors are f-vectors of convex polytopes?

Euler 1752, Schläfli 1852, Poincaré 1899

Dimension d:

$$\sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d$$

Also: this is the only linear equation satisfied by the f-vectors of all polytopes.

Characterizations

Steinitz 1906

Known: Dimension 3

Billera and Lee 1980, Stanley 1980

Known: Simplicial polytopes (all faces are simplices), and simple polytopes (dual to simplicial)

Unknown: Everything else

Flag Vectors

A chain of faces

$$\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq P$$

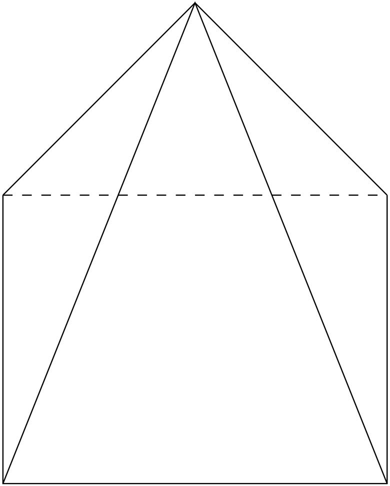
is an S-flag, where $S = \{\dim(F_i) : 1 \leq i \leq k\}$.

$f_S = f_S(P)$ is the number of S-flags of P.

$(f_S)_{S \subseteq \{0,1,\ldots,d-1\}} \in \mathrm{Z}^{2^d}$ is the *flag vector* of P.

No new information for 3-dimensional polytopes.
No new information for simplicial/simple polytopes.

Example



$f_\emptyset = 1$
$f_0 = 5$
$f_1 = 8$
$f_2 = 5$
$f_{01} = 16$
$f_{02} = 16$
$f_{12} = 16$
$f_{012} = 32$

Linear inequalities for f-vectors and flag vectors

Bayer and Billera 1983

The affine span of flag vectors of d-polytopes.

Some known inequalities

- 4-polytopes
- nonnegativity of *cd*-index (linearly equivalent to flag vector)
- polytopes with few vertices
- linear inequalities among *cd* coefficients
- toric h-vector (linear function of flag vector)
- combinations of these

Some inequalities on f-vectors can be derived from these.

Some questions

For what polytopes is the f-vector unimodal?
Not in general for dimension ≥ 8.

What are the smallest closed convex set and smallest closed cone containing all f-vectors of d-polytopes?

Pairs of face numbers

dimension 4

Known: Characterization of all pairs (f_i, f_j) for 4-polytopes.
Known: Characterization of (f_0, f_{03}) for 4-polytopes.
Unknown: Other pairs involving flag numbers of 4-polytopes.

higher dimensions

Known: Characterization of (f_0, f_1) for 5-polytopes and 6-polytopes.
Known: Characterization of (f_0, f_{d-1}) when each entry is large relative to d.

Special classes of polytopes

Cubical

Affine span
Closed cone of g-vectors equals positive orthant? (Contains it.)
Barycentric subdivisions of cubical polytopes

Centrally symmetric

Lower bound theorem for centrally symmetric aimplicial.
Existence of neighborly centrally symmetric.

Cyclic and "Ordinary" polytopes

Known: f-vectors and log-concavity for cyclic, f-vectors for ordinary

Other special classes

- Quasisimplicial
- Multiplicial
- Minkowski sums (zonotopes)
- Matroid polytopes
- Generalized permutahedra
- Neighborly or k-neighborly polytopes
- 0/1 polytopes
- Sign matrix polytopes
- Products of polytopes

Tools

- Barycentric subdivision/Order complex
- Stanley-Reisner rings
- Toric varieties
- Shellings
- Rigidity

Let's prove some theorems!