Alternating Sign Matrices and Plane Partitions

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Four families of objects

I. Alternating sign matrices

II. Cyclically symmetric lozenge tilings with a central triangular hole

III. Totally symmetric self-complementary plane partitions

IV. Alternating sign triangles

I. Alternating sign matrices (ASMs)

Alternating sign matrices =ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

All permutation matrices are ASMs !

The number of $n \times n$ ASMs

Conjecture (Mills, Robbins, Rumsey 1980s). The number of $n \times n$ alternating sign matrices is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} =: A_n.$$

Robbins 1991: "*These conjectures are of such compelling simplicity that it is hard to know how any mathematician can bear the pain of living without understanding why they are true.*"

• Zeilberger then provided the first proof of the conjecture in 1996. The paper has 84 pages and an army of "proof checkers" was required before the proof was believed to be true.

• In the same year, Greg Kuperberg came up with a shorter proof that is based on techniques developed by physicists.

II. **Cyclically symmetric lozenge tilings with a central triangular hole**

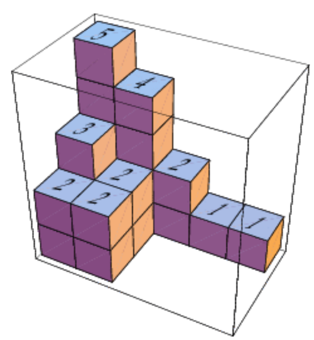
What is a plane partition?

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1,2,\ldots,a\} \times \{1,2,\ldots,b\} \times \{1,2,\ldots,c\}$$

with

$$(i,j,k) \in PP \Rightarrow (i',j',k') \in PP \quad \forall (i',j',k') \leq (i,j,k).$$



$a = 4, b = 3, c = 5$

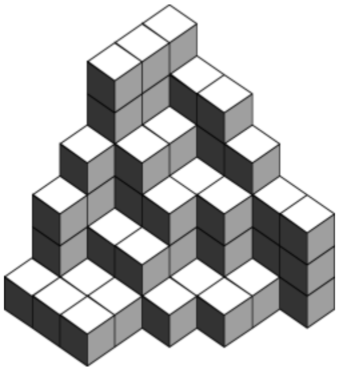
Cyclically symmetric plane partitions = CSPPs

An $n \times n \times n$ plane partition PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitions is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



A determinant in Andrews' proof

In his proof, Andrews shows that the number of CSPPs of order n is given by the following determinant

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)$$

and then proves that

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)=\prod_{i=0}^{n-1}\frac{(3i+2)(3i)!}{(n+i)!}.$$

Then, probably out of curiousity, he also considered the following more general determinant:

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{k+i+j}{i}\right):=D_n(k)$$

$D_n(k)$ for small values of n

$$2$$

$$k+5$$

$$(k+4)(k+5)$$

$$\frac{1}{12}(k+4)^2(k+9)(k+11)$$

$$\frac{1}{72}(k+4)^2(k+6)(k+9)(k+11)^2$$

$$\frac{(k+4)^2(k+6)^2(k+11)^2(k+13)(k+15)(k+17)}{8640}$$

$$\frac{(k+4)^2(k+6)^2(k+8)(k+10)(k+11)(k+13)(k+15)^2(k+17)^2}{518400}$$

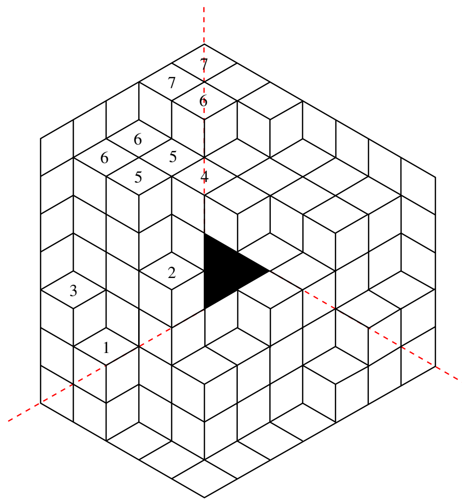
$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^2(k+15)^2(k+17)^3(k+19)(k+21)(k+23)}{870912000}$$

$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^3(k+12)(k+15)(k+17)^3(k+19)^2(k+21)^2(k+23)^2}{731566080000}$$

Big surprise:

$$D_n(2) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Combinatorial interpretation for $D_n(2)$ due to C. Krattenthaler



Cyclically symmetric lozenge tiling of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central triangular hole of size 2.

To obtain the combinatorial interpretation for any k, replace 2 by k!

An explicit bijection between ASMs and these lozenge tilings is not known !

Stanley 2009: *"This is one of the most intriguing open problems in the area of bijective proofs."*

A (complicated) bijection (Fischer & Konvalinka, 2020)

ASM_n = set of $n \times n$ ASMs

$\mathrm{CSLT}(2)_n$ = cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central triangular hole of size 2

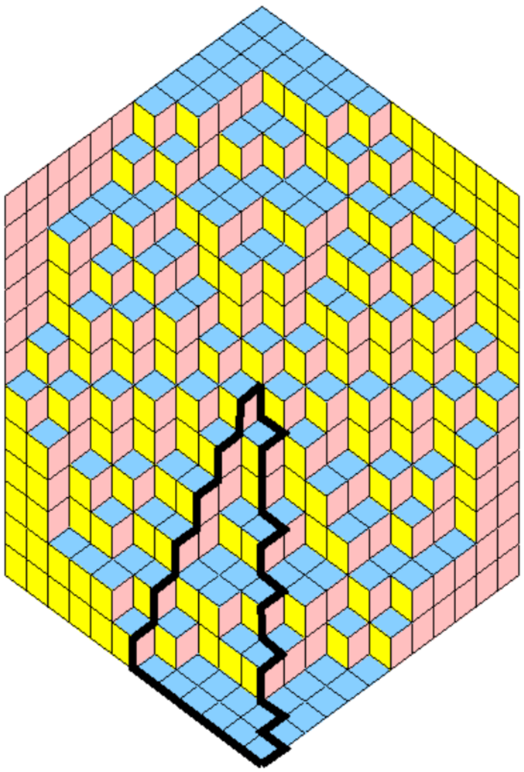
We have constructed a bijection between the following sets:

$$\mathrm{CSLT}(2)_{n-1} \times \mathrm{ASM}_n \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{CSLT}(2)_n$$

Implies inductively the equinumerosity of ASM_n and $\mathrm{CSLT}(2)_n$.

III. Totally symmetric self-complementary plane partitions (TSSCPPs)

Totally symmetric self-complementary plane partitions



- **Totally symmetric:**
$(i, j, k) \in PP \Rightarrow \sigma(i, j, k) \in PP \ \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Figure by Di Francesco / Zinn-Justin

Another surprise!

The number of TSSCPPs in a $2n \times 2n \times 2n$ box is (again) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

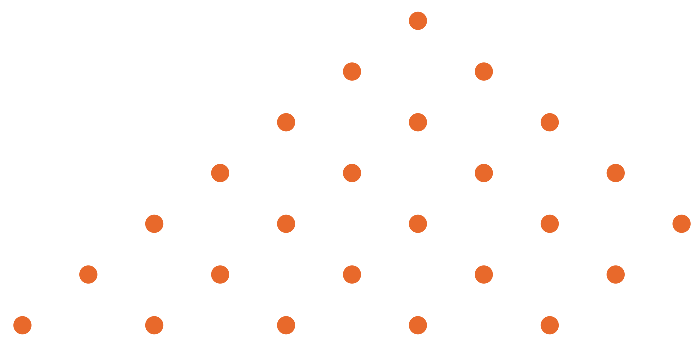
Conjectured by Mills, Robbins and Rumsey in 1986, proved by Andrews in 1994.

Where are the bijections ?

- . . . between TSSCPPs and ASMs.

- . . . between TSSCPPs and cyclically symmetric lozenge tilings of a hexagon with a central triangular hole of size 2.

(m,n,k)**-Gog trapezoid:** arrangement of positive integers of the following form



$n =$ number of rows $= 7$

$k =$ number of $\nearrow$-diagonals $= 5$

such that

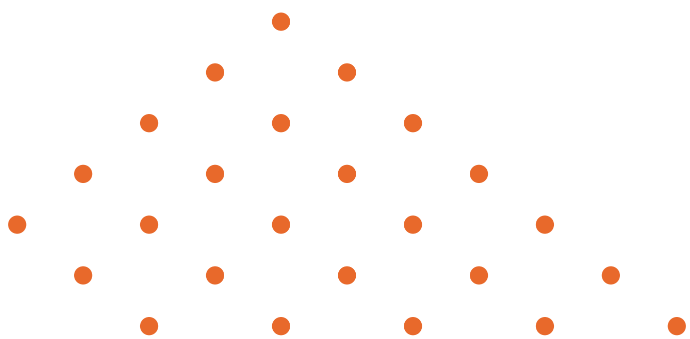
- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- rows are strictly increasing,
- entries in the i-th $\searrow$-diagonal are bounded from above by $m + i$.

Example: $(0, 7, 5)$-Gog trapezoid

$$
\begin{array}{ccccccccccc}
 & & & & & 4 & & & & & \\
 & & & & 3 & & 5 & & & & \\
 & & & 3 & & 4 & & 5 & & & \\
 & & 2 & & 3 & & 4 & & 5 & & \\
 & 1 & & 2 & & 3 & & 5 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 6 & & \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

(m,n,k)-**Magog trapezoid:** arrangement of positive integers of the following form



$n =$ number of rows $= 7$

$k =$ number of $\searrow$-diagonals $= 5$

such that

- $\nearrow$- and $\searrow$-diagonals are weakly increasing,
- entries in the i-th $\nearrow$-diagonal are bounded from above by $m + i$.

Example: $(0, 7, 5)$-Magog trapezoid

				1						
			1		2					
		1		2		3				
	1		1		2		3			
1		1		2		3		4		
	1		2		3		4		6	
		2		3		3		5		6

Generalized conjecture

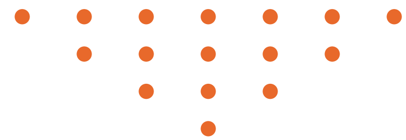
Conjecture (Krattenthaler, Mills, Robbins, Rumsey). There is the same number of (m, n, k)-Gog trapezoids as there is of (m, n, k)-Magog trapezoids.

So far, there is not even a "computational" proof of this conjecture !

IV. Alternating sign triangles

Alternating sign triangles =ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows



such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

Theorem (Ayyer, Behrend, and Fischer, 2020). The number of ASTs with n rows is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

Back to Andrews' determinant

$$D_n(k) = \det_{0 \le i,j \le n-1} \left(\delta_{i,j} + \binom{k+i+j}{i} \right)$$

Recall:

• $D_n(2)$ is the number of $n \times n$ ASMs as well as the number of ASTs with n rows.

• $D_n(k)$ is the number of cyclically symmetric lozenge tilings of a hexagon with central triangular hole of size k.

Is there a combinatorial realization of $D_n(k)$ in terms of ASM-like objects ?

Alternating sign trapezoids

For $n \geq 1, l \geq 2^*$, an *(n, l)-alternating sign trapezoid* is an array of 1's, -1's and 0's with n centered rows and l elements in the bottom row, arranged as follows

such that the following conditions are satisfied.

(1) In each row and column, the non-zero entries alternate.

(2) All row sums are 1.

(3) The topmost non-zero entry in each column is 1.

(4) The column sums are 0 for the middle $l - 2$ columns.

*Can be extended to $l = 1$.

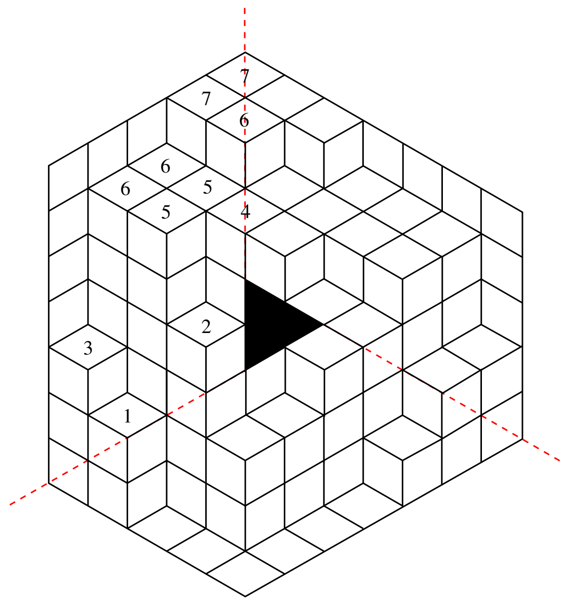
Example

A $(5,4)$-alternating sign trapezoid.

$$\begin{array}{ccccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & -1 & & 1 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & -1 & & 0 & & 1 & & -1 & & 1 & & & & \\
 & & & 0 & & 0 & & 0 & & 1 & & -1 & & 1 & & & & & & & \\
 & & & & 1 & & 0 & & -1 & & 1 & & & & & & & & & &
\end{array}$$

Alternating sign trapezoids and cyclically symmetric lozenge tilings of a holey hexagon

Theorem (Behrend, Fischer 2018). There is the same number of (n,l)-alternating sign trapezoids as there is of cyclically symmetric lozenge tilings of a hexagon with side lengths $n+l-1, n, n+l-1, n, n+l-1, n$ that has a central triangular hole of size $l-1$.



What's the plan ?

Find bijections in special cases!

More concretely: There exist various pairs of statistics on the four classes of objects that have the same joint distribution.

Find bijections when fixing such statistics to take certain (extreme) values.

Thanks for listening !

Special thanks to the organizers for this great format !