Research Project: Row-strict dual immaculate functions

Elizabeth Niese
Marshall University
niese@marshall.edu

Research Community in Algebraic Combinatorics
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Outline

- Motivation
- Introduction to quasisymmetric functions
- Quasisymmetric Schur functions
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- Research project intro
Given an integer partition $\lambda$, the Schur function indexed by $\lambda$ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape $\lambda$.

**Example.** A semi-standard Young tableau of shape $(4, 3, 2, 2)$ corresponding to the monomial $x_1^3 x_2^3 x_3 x_4^2 x_5^2$ is

$$\begin{array}{cccc}
5 & 5 \\
3 & 4 \\
2 & 2 & 4 \\
1 & 1 & 1 & 2
\end{array}$$
Motivation: Schur functions

Given an integer partition $\lambda$, the Schur function indexed by $\lambda$ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape $\lambda$.

**We can also use row strict semi-standard Young tableaux of shape $\lambda'$ to generate $s_\lambda$.**
Motivation: Schur functions

Given an integer partition $\lambda$, the Schur function indexed by $\lambda$ is

$$s_{\lambda} = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape $\lambda$.

Example. A row-strict semi-standard Young tableaux of shape $(4, 3, 2, 2)$ corresponding to the monomial $x_1^2 x_2^2 x_3^2 x_4^4 x_5$ in $s_{(4,4,2,1)}$. 

\[
\begin{array}{ccc}
3 & 4 \\
3 & 4 \\
1 & 2 & 4 \\
1 & 2 & 4 & 5
\end{array}
\]
A function $f(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is **quasisymmetric** if

$$\text{coeff.} (x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = \text{coeff.} (x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k})$$

for all $i_1 < i_2 < \cdots < i_k$.

**Example.** The polynomial

$$f(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$$

is quasisymmetric. The polynomial

$$g(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3$$

is not quasisymmetric.
Let $\text{QSym}_n$ denote the set of all quasisymmetric functions that are homogeneous of degree $n$.

- $\text{QSym}_n$ is a vector space of dimension $2^{n-1}$ for each $n > 0$.
- $\text{QSym} = \bigoplus_{n \geq 0} \text{QSym}_n$ is the ring of quasisymmetric functions.
- The ring of noncommutative symmetric functions (NSym) is dual to $\text{QSym}$.
Bases of QSym

- Monomial basis
- **Schur-like bases:**
  1. Gessel’s Fundamental basis
  2. Quasisymmetric Schur functions (Young, reverse, row-strict)
  3. Dual Immaculate basis
  4. Extended Schur basis
- Power sum basis (type 1 and 2)
- Forgotten basis
A composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence of positive integers that sum to a fixed integer $n$. Write $\alpha \models n$. There is a natural bijection between compositions of $n$ and subsets of $\{1, \ldots, n-1\}$.

**Example.** Let $\alpha = (1, 2, 2, 1)$. Then $\text{Set}(\alpha) = \{1, 3, 5\} \subseteq \{1, \ldots, 5\}$. Similarly, if $A = \{2, 3, 5, 6\} \subseteq \{1, \ldots, 9\}$, then $\beta = \text{comp}(A) = (2, 1, 2, 1, 4)$. The diagram of $\beta$ is

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 |       |
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Young Quasisymmetric Schur functions

Young Column-Strict Composition tableaux

Given a composition \( \alpha \), a composition tableau is a filling, \( F \), of the cells of the diagram of \( \alpha \) such that

1. The leftmost column entries strictly increase from bottom to top.
2. The row entries weakly increase from L to R.
3. The entries satisfy the **triple rule:**
   
   if \( a \geq b \), then \( a > c \)

\[
F = \begin{array}{ccc}
3 & 3 & 4 \\
2 & 2 & 5 \\
1 & 1 & \\
\end{array}
\]

\[
x^F = x_1^2 x_2^2 x_3^2 x_4 x_5
\]
Young Column-Strict Composition tableaux

Given a composition \( \alpha \), a composition tableau is a filling, \( F \), of the cells of the diagram of \( \alpha \) such that

1. The leftmost column entries strictly increase from bottom to top.
2. The row entries weakly increase from L to R.
3. The entries satisfy the **triple rule**:
   
   \[
   \text{if } a \geq b, \text{ then } a > c
   \]

The Young quasisymmetric Schur function indexed by \( \alpha \) is

\[
\hat{S}_\alpha = \sum_F x^F
\]

where the sum is over all composition tableaux of shape \( \alpha \).
**Example.** The column-strict composition tableaux of shape $(1, 2, 1)$:

\[
\begin{array}{ccc}
4 & 4 & 4 \\
2 & 2 & 3 \\
1 & 1 & 3 \\
\end{array}
\begin{array}{ccc}
4 & 3 & 3 \\
2 & 3 & 3 \\
1 & 2 & 2 \\
\end{array}
\]

Then

\[
\hat{S}_{(1,2,1)} = x_1x_2x_3x_4 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2^2x_3
\]
Theorem (Haglund, Luoto, Mason, van Willigenburg, 2011)

Let $\lambda \vdash n$. Then

$$s_{\lambda} = \sum_{\alpha \colon \tilde{\alpha} = \lambda} \hat{S}_{\alpha}.$$ 

Proof sketch

1. 8
2. 5 8 8
3. 4 5 6 7
4. 2 4 5 6
5. 1 1 3 4 6

→

1. 8 8 8
2. 5 5 5 6 6
3. 4 4 6 7
4. 2
5. 1 1 3 4
Definition (Row-strict composition tableaux)

Given a composition $\alpha$, a composition tableau is a filling, $F$, of the cells of the diagram of $\alpha$ such that

1. The leftmost column entries weakly increase from bottom to top.
2. The row entries strictly increase from L to R.
3. The entries satisfy the triple rule:

   If $a > b$, then $a \geq c$

Theorem (Mason-Remmel (2012))

Let $\lambda \vdash n$. Then

$$s_{\lambda} = \sum_{\alpha: \tilde{\alpha} = \lambda'}^{\alpha} R\hat{S}_{\alpha}.$$
Quasisymmetric Schur function facts

Parallels with Schur functions

- Littlewood-Richardson rules for products $s_{\lambda} \cdot \hat{S}_{\alpha}$ and $s_{\lambda} \cdot R\hat{S}_{\alpha}$ (Haglund/Luoto/Mason/van Willigenberg, Ferreira)
- $\hat{S}_{\alpha}$ and $R\hat{S}_{\alpha}$ are characters of 0-Hecke modules. (Tewari/van Willigenburg, Bardwell/Searles)
- Analogues of the RSK algorithm establishing bijections between certain sets of words and pairs of composition tableaux. (Haglund/Mason/Remmel)
- Analogues of the Murnaghan-Nakayama rule: $p_{k} \cdot \hat{S}_{\alpha}$ and $p_{k} \cdot R\hat{S}_{\alpha}$. (LoBue-Tiefenbruck)
- Hook (or super) quasisymmetric Schur functions interpolate between $\hat{S}_{\alpha}$ and $R\hat{S}_{\alpha}$. (Mason/Niese)
Definition (Immaculate tableaux)

Let \( \alpha \models n \). A filling \( U \) of the diagram of \( \alpha \) is an immaculate tableau if

- row entries weakly increase from left to right, and
- the entries in the leftmost column strictly increasing from bottom to top.

The dual immaculate quasisymmetric function indexed by \( \alpha \) is

\[
S^*_\alpha = \sum_{U \in I(\alpha)} x^U.
\]
Example.

\[ S_{(1,2,1)}^* = x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + 2x_1 x_2 x_3 x_4 + x_1 x_3^2 x_4 + x_2 x_3^2 x_4 \\
+ x_1 x_2 x_3^2 + x_1 x_2 x_4^2 + x_1 x_3 x_4^2 + x_2 x_3 x_4^2 \]
Other expansions

Theorem (Allen, Hallam, Mason (2016))

Let $\alpha \models n$. Then

$$S^*_\alpha = \sum_{\beta} c_{\alpha\beta} \hat{S}_\beta$$

where $c_{\alpha\beta}$ is the number of composition tableaux satisfying several conditions.

Theorem (Berg, Bergeron, Saliola, Serrano, Zabrocki (2014))

Let $\lambda \models n$. Then

$$s_\lambda = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^\sigma S^*_{\sigma \cdot \lambda}.$$
Some questions we will seek to answer

There is a natural way to define a row-strict dual immaculate function, $\mathcal{RG}^*_\alpha$.

- Is $\{\mathcal{RG}^*_\alpha : \alpha \models n\}$ a basis of $Q\text{Sym}$?
- What is the expansion of $\mathcal{RG}^*_\alpha$ in the fundamental basis? The $\mathcal{RS}$ basis?
- What are the creation operators in $N\text{Sym}$ that correspond to the dual of $\mathcal{RG}^*_\alpha$?
- How does $\mathcal{RG}^*_\alpha$ refine $s_\lambda$?
Thank you!

Questions?