Research Project: Row-strict dual immaculate functions

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Research Community in Algebraic Combinatorics
August 5, 2021

Outline

- Motivation
- Introduction to quasisymmetric functions
- Quasisymmetric Schur functions
- Dual immaculate functions
- Research project intro

Motivation: Schur functions

Given an integer partition λ, the Schur function indexed by λ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape λ.

Example. A semi-standard Young tableau of shape $(4,3,2,2)$ corresponding to the monomial $x_1^3 x_2^3 x_3 x_4^2 x_5^2$ is

5	5		
3	4		
2	2	4	
1	1	1	2

Motivation: Schur functions

Given an integer partition λ, the Schur function indexed by λ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape λ.

We can also use *row strict* semi-standard Young tableaux of shape λ' to generate s_λ.

Motivation: Schur functions

Given an integer partition λ, the Schur function indexed by λ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all (column strict) semi-standard Young tableaux of shape λ.

Example. A row-strict semi-standard Young tableaux of shape $(4,3,2,2)$ corresponding to the monomial $x_1^2 x_2^2 x_3^2 x_4^4 x_5$ in $s_{(4,4,2,1)}$.

3	4		
3	4		
1	2	4	
1	2	4	5

Quasisymmetric functions

Definition

A function $f(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is *quasisymmetric* if

$$\text{coeff.}\,({x_1}^{a_1} {x_2}^{a_2} \cdots {x_k}^{a_k}) = \text{coeff.}\,({x_{i_1}}^{a_1} {x_{i_2}}^{a_2} \cdots {x_{i_k}}^{a_k})$$

for all $i_1 < i_2 < \cdots < i_k$.

Example. The polynomial

$$f(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$$

is quasisymmetric. The polynomial

$$g(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3$$

is not quasisymmetric.

QSym

Let QSym_n denote the set of all quasisymmetric functions that are homogeneous of degree n.

- QSym_n is a vector space of dimension 2^{n-1} for each $n > 0$.
- $\mathrm{QSym} = \bigoplus_{n \geq 0} \mathrm{QSym}_n$ is the ring of quasisymmetric functions.
- The ring of noncommutative symmetric functions (NSym) is dual to QSym.

Bases of QSym

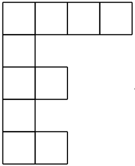
- Monomial basis
- **Schur-like bases:**
  1. Gessel's Fundamental basis
  2. Quasisymmetric Schur functions (Young, reverse, row-strict)
  3. Dual Immaculate basis
  4. Extended Schur basis
- Power sum basis (type 1 and 2)
- Forgotten basis

Compositions

Compositions

A composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence of positive integers that sum to a fixed integer n. Write $\alpha \vDash n$. There is a natural bijection between compositions of n and subsets of $\{1, \ldots, n-1\}$.

Example. Let $\alpha = (1, 2, 2, 1)$. Then $\mathrm{Set}(\alpha) = \{1, 3, 5\} \subseteq \{1, \ldots, 5\}$. Similarly, if $A = \{2, 3, 5, 6\} \subseteq \{1, \ldots, 9\}$, then $\beta = \mathrm{comp}(A) = (2, 1, 2, 1, 4)$. The diagram of β is



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Young Quasisymmetric Schur functions

Young Column-Strict Composition tableaux

Given a composition α, a composition tableau is a filling, F, of the cells of the diagram of α such that

1. The leftmost column entries strictly increase from bottom to top.
2. The row entries weakly increase from L to R.
3. The entries satisfy the **triple rule:**

if $a \geq b$, then $a > c$

b	c
	a

$F =$

3	3	4
2	2	5
1	1	

$$x^F = x_1^2 x_2^2 x_3^2 x_4 x_5$$

Young Quasisymmetric Schur functions

Young Column-Strict Composition tableaux

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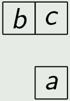


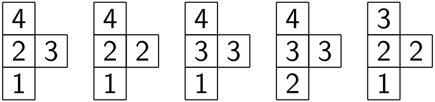
The Young quasisymmetric Schur function indexed by α is

$$\hat{\mathcal{S}}_\alpha = \sum_F x^F$$

where the sum is over all composition tableaux of shape α.

Young Quasisymmetric Schur functions

Example. The column-strict composition tableaux of shape $(1,2,1)$:



Then

$$\hat{\mathcal{S}}_{(1,2,1)} = x_1x_2x_3x_4 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2^2x_3$$

Quasisymmetric Schur functions

Theorem (Haglund,Luoto, Mason, van Willigenburg, 2011)

Let $\lambda \vdash n$. Then

$$s_\lambda = \sum_{\alpha : \widetilde{\alpha} = \lambda} \hat{\mathcal{S}}_\alpha \, .$$

Proof sketch

8				
5	8	8		
4	5	6	7	
2	4	5	6	
1	1	3	4	6

$\rightarrow$

8	8	8		
5	5	5	6	6
4	4	6	7	
2				
1	1	3	4	

Row-strict quasisymmetric Schur functions

Definition (Row-strict composition tableaux)

Given a composition α, a composition tableau is a filling, F, of the cells of the diagram of α such that

1. The leftmost column entries weakly increase from bottom to top.
2. The row entries strictly increase from L to R.
3. The entries satisfy the **triple rule:**

   if $a > b$, then $a \geq c$

b	c
	a

Theorem (Mason-Remmel (2012))

Let $\lambda \vdash n$. Then

$$s_\lambda = \sum_{\alpha : \widetilde{\alpha} = \lambda'} \mathcal{R}\hat{\mathcal{S}}_\alpha \, .$$

Quasisymmetric Schur function facts

Parallels with Schur functions

- Littlewood-Richardson rules for products $s_\lambda \cdot \hat{\mathcal{S}}_\alpha$ and $s_\lambda \cdot \mathcal{R}\hat{\mathcal{S}}_\alpha$ (Haglund/Luoto/Mason/van Willigenberg, Ferreira)
- $\hat{\mathcal{S}}_\alpha$ and $\mathcal{R}\hat{\mathcal{S}}_\alpha$ are characters of 0-Hecke modules. (Tewari/van Willigenburg, Bardwell/Searles)
- Analogues of the RSK algorithm establishing bijections between certain sets of words and pairs of composition tableaux. (Haglund/Mason/Remmel)
- Analogues of the Murnaghan-Nakayama rule: $p_k \cdot \hat{\mathcal{S}}_\alpha$ and $p_k \cdot \mathcal{R}\hat{\mathcal{S}}_\alpha$. (LoBue-Tiefenbruck)
- Hook (or super) quasisymmetric Schur functions interpolate between $\hat{\mathcal{S}}_\alpha$ and $\mathcal{R}\hat{\mathcal{S}}_\alpha$. (Mason/Niese)

Dual Immaculate Basis (combinatorial approach)

Definition (Immaculate tableaux)

Let $\alpha \vDash n$. A filling U of the diagram of α is an *immaculate tableau* if

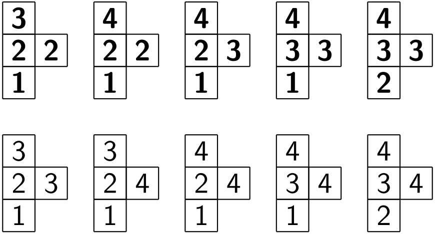
- row entries weakly increase from left to right, and
- the entries in the leftmost column strictly increasing from bottom to top.

The dual immaculate quasisymmetric function indexed by α is

$$\mathfrak{S}^*_\alpha = \sum_{U \in I(\alpha)} x^U.$$

Dual Immaculate Basis

Example.



$$\mathfrak{S}^*_{(1,2,1)} = x_1x_2^2x_3 + x_1x_2^2x_4 + 2x_1x_2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2x_3^2 + x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2$$

Other expansions

Theorem (Allen, Hallam, Mason (2016))

Let $\alpha \vDash n$. Then

$$\mathfrak{S}^*_\alpha = \sum_\beta c_{\alpha\beta}\, \hat{\mathcal{S}}_\beta$$

where $c_{\alpha\beta}$ is the number of composition tableaux satisfying several conditions.

Theorem (Berg, Bergeron, Saliola, Serrano, Zabrocki (2014))

Let $\lambda \vdash n$. Then

$$s_\lambda = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^\sigma \mathfrak{S}^*_{\sigma\cdot\lambda}.$$

Some questions we will seek to answer

There is a natural way to define a row-strict dual immaculate function, $\mathcal{R}\mathfrak{S}^*_\alpha$.

- Is $\{\mathcal{R}\mathfrak{S}^*_\alpha : \alpha \vDash n\}$ a basis of QSym?
- What is the expansion of $\mathcal{R}\mathfrak{S}^*_\alpha$ in the fundamental basis? The $\mathcal{R}\hat{\mathcal{S}}$ basis?
- What are the creation operators in NSym that correspond to the dual of $\mathcal{R}\mathfrak{S}^*_\alpha$?
- How does $\mathcal{R}\mathfrak{S}^*_\alpha$ refine s_λ?

Thank you!

Questions?