$\mathfrak{sl}_2$-actions on Soergel Bimodules

Ben Elias

UOregon

featuring joint work with You Qi (UVA)

Foam Evaluation at ICERM 11/2021

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3) As an $\mathfrak{sl}_2$-module, R is

$$1 \qquad x \qquad x^2 \qquad x^3 \qquad \ldots$$

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$$\begin{tikzcd} 1 & x \arrow[l, bend left, "1"] \arrow[r, bend left, "1"] & x^2 \arrow[l, bend left, "2"] \arrow[r, bend left, "2"] & x^3 \arrow[l, bend left, "3"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

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4) Divided powers $x^{(k)} := \dfrac{x^k}{k!}$ are defined integrally for $x \in \{d, z\}$. $d^{(k)} : x^{\ell} \mapsto \binom{k+\ell}{k} x^{k+\ell}$

Let $R = R_n := \mathbb{Z}[x_1, \quad , x_n]$.

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④ Morphisms between these.

IN SHORT:

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Eg Khovanov homology often uses the Frobenius algebra $\Bbbk[x]/x^2$ rather than the

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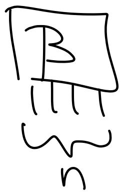
For safety I'll stick to triply-graded link homology.

Triply graded knot homology is defined using complexes of Bott-Samelson bimodules.



braid $\underset{\text{Rouquier}}{\rightsquigarrow}$ complex of (R_n, R_n)-bimodules

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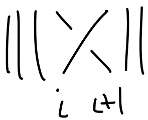

$\longmapsto \quad F_i = \left(\underline{B_i} \xrightarrow{\mu} R(1) \right)$

Stacking $\longmapsto \otimes$

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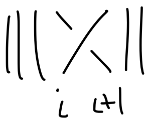

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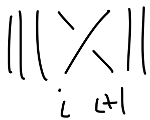 $\longmapsto \quad F_i = (\underline{B_i} \xrightarrow{\mu} R(1))$

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<u>Note</u>: HH_0 is quotient by ideal

$(x_i - y_i)$ which is $\mathfrak{sl}_2$-closed

left $\qquad$ right

$d(x_i-y_i) = x_i^2 - y_i^2 = (x_i-y_i)(x_i+y_i)$

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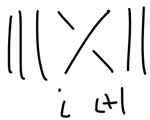
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I want to advocate that we work in an $\mathfrak{sl}_2$-enriched setting,

i.e keep track of extra structure

Rest of talk: no knots, just $BSBim \circlearrowleft \mathfrak{sl}$

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Khovanov-Qi, Sussan, E, etc: d is differential used for cat'fn at a root of unity

oops, this was first

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but no one seemed to look for z. Negative degree maps not cohomology operations.

We found while exploring hard Lefschetz/Hodge-Riemann phenomena related to d

Where I'm going: in ordinary Hecke category / SBim,

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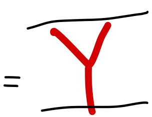
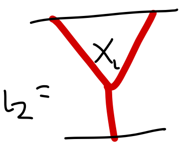
$\iota_1 =$ 

$\iota_2 =$ 

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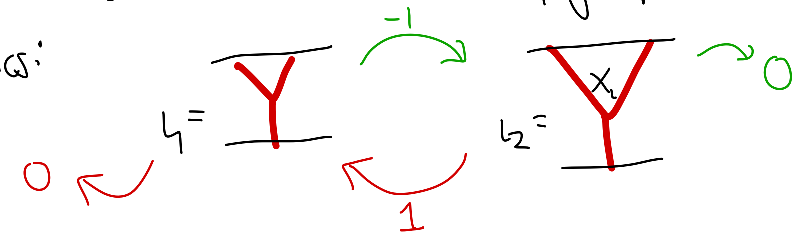

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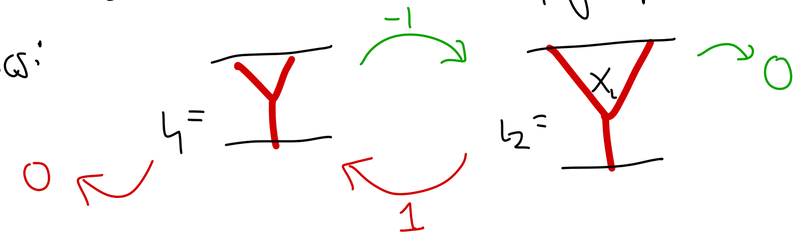

These span a copy of

$$\mathbb{C}^2 \subset \mathrm{Hom}(B_i, B_i \otimes B_i)$$

$\circlearrowleft \mathfrak{sl}_2$

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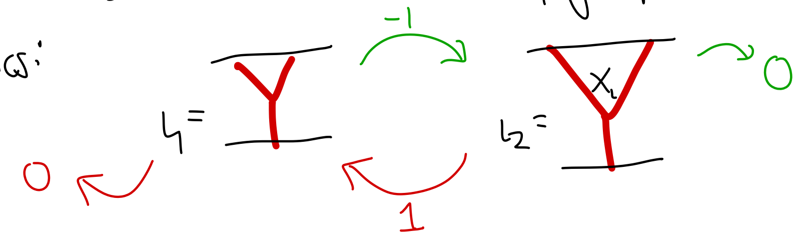

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Normally, $\mathrm{Hom}(X, B_i \otimes B_i) \cong \mathrm{Hom}(X, B_i(1)) \oplus \mathrm{Hom}(X, B_i(-1))$ via composition w/ proj + incl maps

Where I'm going: in ordinary Hecke category / SBim,

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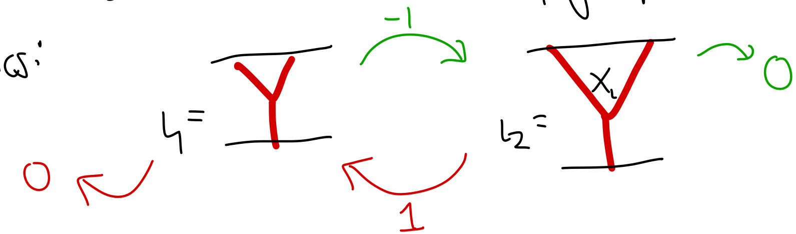

These span a copy of $\mathbb{C}^2 \subset \mathrm{Hom}(B_1, B_1 \otimes B_1)$ with $\mathfrak{sl}_2$ acting

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Now $\mathrm{Hom}(X, B_1 \otimes B_1) \cong \mathrm{Hom}(X, B_1) \otimes \mathbb{C}^2$ as $\mathfrak{sl}_2$-repns

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(Some new kind of Hopfological algebra?)

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Three categories:

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, leftrightarrow, dashed, bend left] \arrow[dr, leftrightarrow, dashed] & & \mathcal{H}^{\mathfrak{sl}_2} \arrow[dl, leftrightarrow, dashed] \\
& \mathcal{H}_{\smiley} &
\end{tikzcd}$$

$\mathfrak{sl}_2 \circlearrowright \mathcal{H}$ (acts on morphisms)

$\mathcal{H}^{\mathfrak{sl}_2} \circlearrowleft (\mathrm{Rep}\,\mathfrak{sl}_2, \otimes)$ changes objects

$\mathfrak{sl}_2 \circlearrowright \mathcal{H}_{\smiley} \circlearrowleft \mathrm{Rep}\,\mathfrak{sl}_2$

In $\mathcal{H}$, $B_1 B_2 B_1 \cong B_w \oplus B_1$

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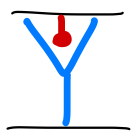
$\iota_2 \cdot B_2 \to B_2 B_1 B_2$  is killed by d, z $\Rightarrow$ $B_2 \subset B_2 B_1 B_2$ is $\mathfrak{sl}_2$-subobject

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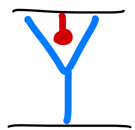
$\iota_2 \cdot B_2 \to B_2 B_1 B_2$  is killed by d, z $\Rightarrow$ $B_2 \subset B_2 B_1 B_2$ is $\mathfrak{sl}_2$-subobject

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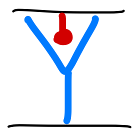
$\iota_2 : B_2 \to B_2 B_1 B_2$  is killed by d, z $\Rightarrow$ $B_2 \subset B_2 B_1 B_2$ is $\mathfrak{sl}_2$-subobject

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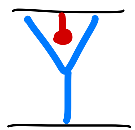
$\iota_2 : B_2 \to B_2 B_1 B_2$  is killed by d, z $\Rightarrow$ $B_2 \subset B_2 B_1 B_2$ is sl_2-subobject

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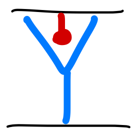
$\iota_2 : B_2 \to B_2B_1B_2$  is killed by d, z $\Rightarrow$ $B_2 \subset B_2B_1B_2$ is $\mathfrak{sl}_2$-subobject

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$\rightsquigarrow$ $0 \to B_2 \to B_2B_1B_2 \to B_w \to 0$ in $\mathcal{H}^{\mathfrak{sl}_2}$, splits in $\mathcal{H}$ and even in $\mathcal{H}^{\bar{b}}$

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WARNING: I didn't equip indecomp. Soergel bim. B_w w/ sl_2-structure $(D_w \neq B_w)$

Don't use ordinary ideas about Karoubi envelopes!!

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Well, not <u>that</u> crazy...

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1) splits over $\overline{\mathbb{Q}}$, R. 2) subquots free of rank 1 over R

Thm (E-Q): $\mathrm{Hom}_{\mathcal{D}_P}(BS, BS)$ has $\mathfrak{sl}_2$-filtration satisfying

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Base ring $R = \mathbb{Z}[x_1, x_2] \cong \nabla(0) \otimes \nabla(0)$ as $\mathfrak{sl}_2$-mod

Any $(R, \mathfrak{sl}_2)$-mod which is free of rank 1 as R-mod is $\cong \nabla(a_1) \otimes \nabla(a_2)$ as $\mathfrak{sl}_2$-mod

$\bigotimes \nabla(a_i)$ has a f.d. subrepn $\Longleftrightarrow$ $a_i \leq 0 \ \forall i$.

Ex: $\mathrm{Hom}(B_1, B_1 B_1)$ has quotient $\nabla(0) \otimes \nabla(-1)$.

NOT OBVIOUS that $V_1 \subset \nabla(0) \otimes \nabla(1)$ should lift.

Ex: $\mathrm{Hom}(R, B_1 B_1)$ has quotient $\nabla(0) \otimes \nabla(0)$. $V_0 \subset \nabla(0) \otimes \nabla(0)$ does NOT lift.

No f.d. subrepn — and also no split maps, no summands $R \overset{\oplus}{\subset} B_1 B_1$

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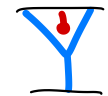
Ex: $\mathrm{Hom}(B_2, BS(212))$ had subrepn V_0 spanned by L_2 

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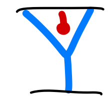
Ex: $\mathrm{Hom}(B_2, BS(212))$ had subrepn V_0 spanned by ι_2 

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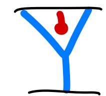
Ex: $\mathrm{Hom}(B_2, BS(212))$ had subrepn V_0 spanned by ι_2 

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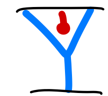
Ex: $\mathrm{Hom}(B_2, BS(212))$ had subrepn V_0 spanned by ι_2 

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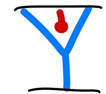
Ex: $\mathrm{Hom}(B_2, BS(212))$ had subrepn V_0 spanned by i_2 

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Conjecture says: Could find i_2 by computing $\mathrm{Ker}(d)$

Easy linear algebra

i_w $\mathrm{Ker}(d)$ modulo $\mathrm{Im}(i_2)$

Rest of talk: HIGH LEVEL BS

trying to motivate $\mathcal{H}$☺ and think about analogue of homotopy category.

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<u>Def</u> An <u>equivariant object</u> is a pair (M, φ), $M \in \mathrm{Ob}(\mathcal{C})$, φ compatible isom.

$\underline{\text{Ex}}$: $\mathbb{Z}/2\mathbb{Z} = G \circlearrowright \mathrm{Vect}_{\mathbb{C}}$, $F_\tau = F_{id} = \mathbb{1}$ Clearly $V \cong F_\tau(V)$ for all V, but $V \xrightarrow{\varphi_\tau} V$ must satisfy

$$\begin{tikzcd} V \arrow[r, "\varphi_\tau"] \arrow[rr, bend right, "\varphi_1"'] & V \arrow[r, "\mathbb{1}(\varphi_\tau)"] & V \end{tikzcd}$$

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morphisms = $f \in \mathrm{Hom}_{\mathcal{C}}(M,N)$ st
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i.e. $f \in Hom_{\mathcal{C}}(M,N)^G$ under G-action from before

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Ex $\mathbb{Z}/2\mathbb{Z} \circlearrowright S = \mathbb{C}$ via sign rep $\quad (V,\varphi)\otimes S = (V, -\varphi)$

$\varphi^2 = \mathrm{Id}_V \qquad (-\varphi)^2 = \mathrm{Id}_V$

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Another Idea: Interpolate using third category $\mathcal{C}_{☺}$

$\operatorname{Ob}(\mathcal{C}_{☺}) = \operatorname{Ob}(\mathcal{C}^G)$

$\operatorname{Mor}(\mathcal{C}_{☺}) \text{ "="} \operatorname{Mor}(\mathcal{C})$

Recall $\operatorname{Hom}_{\mathcal{C}^G} = (\operatorname{Hom}_{\mathcal{C}_{☺}})^G$

$G \curvearrowright \mathcal{C}_{☺} \circlearrowleft \operatorname{Rep} G$

Idea: Interpolate using third category $\mathcal{C}_{☺}$

$$G \circlearrowright \mathcal{C}_{☺} \circlearrowleft \operatorname{Rep} G$$

Objects are (M, φ). $\operatorname{Hom}_{\mathcal{C}_{☺}}((M,\varphi),(N,\psi)) = \operatorname{Hom}_{\mathcal{C}}(M,N) \circlearrowleft G$

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What would be the infinitessmal version of this? $\mathfrak{g} \circlearrowright \mathcal{C}$ means what?

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First guess same objects (since everything is invariant) but $\operatorname{Hom}_{\mathcal{C}^{\mathfrak{g}}} = (\operatorname{Hom}_{\mathcal{C}})^{\mathfrak{g}}$.

But where's the action of $\operatorname{Rep} \mathfrak{g}$?

Better guess: If $X \in \mathcal{C}$ has stabilizer $S_X \subset G$, then to get an equiv object w/ underlying object $\bigoplus_{X' \text{ in orbit of } X} X'$, Better guess: If $X \in \mathcal{C}$ has stabilizer $S_X \subset G$, then to get an equiv object w/ underlying object $\bigoplus X'$, need to specify a character of S_X (ie. a map $S_X \to \mathrm{Aut}(X)$)

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Even when G acts trivially on objects, still need objects $X \boxtimes V$ to accont for all equiv structures.

so should set
$$\mathrm{Ob}(\mathcal{C}^{G}) = \{X \otimes V\} \quad \text{w/ } \oplus$$

This whole concept generalizes well to any (f.d.) Hopf algebra

$H \circlearrowright \mathcal{C}$ then can construct $\mathcal{C}_{☺}$ with objects $X \boxtimes V$, and $\mathcal{C}^H$ where the

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ANOTHER EXAMPLE: $H = k[d]/d^2$, a Hopf algebra in SuperVect.

If $H \circlearrowright \mathcal{C}$, $\mathcal{C}$ is a dg-category

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A-mod	$\underline{Ch}$(A-mod)	Ch(A-mod)	K(A-mod)	D(A-mod)
abelian		abelian	triang.	triang

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Focus in here:

$A\text{-mod}$	$\underline{\mathrm{Ch}}(A\text{-mod})$	$\mathrm{Ch}(A\text{-mod})$
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If you take a chain complex and look at all (not necc chain) maps, you get morphisms in here

it's just like $\quad H \curvearrowright \mathcal{C} \qquad \mathcal{C} \qquad \mathcal{C}^H$

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$H \circlearrowright \mathcal{C}$	$\mathcal{C}$	$\mathcal{C}^H$

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More generally, if B is a dg-algebra,

B-mod	B-$\underline{\text{dgmod}}$	B-dgmod
(um, who cares)	all B-linear maps allowed	morphisms commute with d
	morphisms are a chain complex	
abelian	enriched	abelian

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enriched

B-dgmod

morphisms commute with d

abelian

METAPHOR

$\mathcal{H}$

$\mathcal{H}_{\ddot{\smile}}$

$\mathcal{H}^{sl_2}$

(oh, I guess we cared.)

but maybe we should have been working here all along

Maybe we're using the wrong Groth group!

Let A be an algebra. We have:

internal hom

A-mod	$\underline{Ch}$(A-mod)	Ch(A-mod)	K(A-mod)	D(A-mod)
abelian	enriched	abelian	triang.	triang

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Khovanov's Hopfological algebra, developed further by Qi. H a f.d. Hopf algebra

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how does this work?

$\boxed{H = \Bbbk[d]/d^2}$

$$\begin{tikzcd}
\mathrm{Ch}(A\text{-mod}) \arrow[r, "\text{kill nulhom}"] \arrow[d, "\text{forget}"] & K(A) \arrow[d] \\
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$$\mathrm{Ch}(\mathrm{Vect}) = H\text{-mod}$$

Key fact:

f nullhomotopic $\Leftrightarrow$ factors thru a free complex

$$\bigoplus \; 0 \to M \xrightarrow{\mathrm{id}_M} M \to 0$$

ie proj mod for H

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H\text{-mod} & &
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Key fact: f is qisom in $K(A) \iff \text{forget}(f)$ is isom in $K(\mathrm{Vect})$.

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$\oplus\ 0 \to M \xrightarrow{1_{d_M}} M \to 0$

i.e. proj mod for H

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ie proj mod for H

For general f.d Hopf H, do same thing.

$$B\#H\text{-mod} \xrightarrow[\substack{\text{kill}\\ H\text{-proj}}]{} K_H(B) \xrightarrow[\substack{\text{invert}\\ \mathrm{For}^{-1}(\text{isom})}]{} D_H(B)$$

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$\| $
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KILL MODULES W/O ANY H-(homology/torsion/?)

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Hom spaces are ∞-dim repns (filtered by $\otimes$ of vermas)

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Pinch me, I'm dreaming, possibly drooling.

THANKS

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AND

THANKS TO THE ORGANIZERS