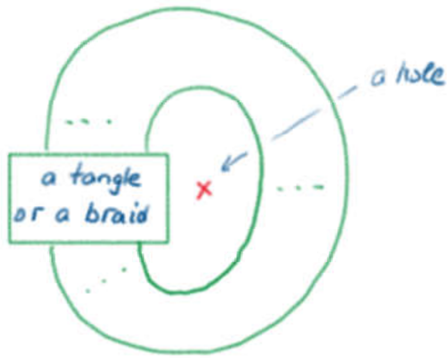
Annular link homology and symplectic algebraic geometry

(joint with A. Oblomkov, with A. Smirnov)

Two versions of "annular homology"



SL_N — Asaeda, Przytycki, Sikora; Beliakova, Putyra, Wehrli $\rightsquigarrow$ A vector space with $\mathbb{C}^\times_q \times SL_N$ action

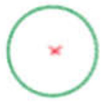
Queffelec, Rose, Sartori; Gorsky, Hogancamp, Wedrich $\rightsquigarrow$ A complex of modules over $\mathbb{C}[\underline{x}, \underline{\theta}] \rtimes S_n$ (not a vector space!)

- What does SL_N homology categorify? (Witten (old paper), Aganagic, ...)
- What is the alg. geom. / sympl. alg. geom. interpretation? (Anno, Aganagic, Webster, ...)

Executive summary / spoiler

Mostly SL_2 case. <u>Notation</u>: V_a — $(a+1)$-dim rep. of SL_2, $\overline{}_{V_a}$ or $\overline{}_{a}$

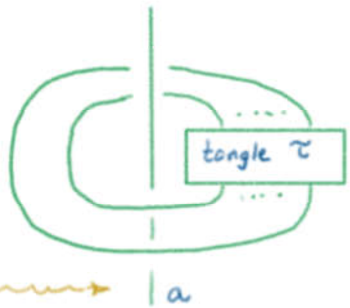
<u>Rem</u>: in SL_2 annular homology



$\rightsquigarrow \langle \underset{s^{-1}}{e_-}, \underset{s}{e_+} \rangle$

in Witten-Reshetikhin-Turaev / Chern-Simons TQFT

$= \left(q^{a+1} + q^{-(a+1)}\right) \Big|_a$ and



$$= \underset{\mathbb{Z}[q^{\pm 1}, s^{\pm 1}]}{\overset{}{}} \; J_\tau^{ann}(q,s)\Big|_{s=q^{a+1}}\Big|_a$$

$\mathbb{Z}[q^{\pm1}, s^{\pm1}] \ni J_\tau^{ann}(q,s)$

Okounkov lives here $\rightsquigarrow |a$

Hence annular SL_2 homology $\mathcal{H}^{ann}(\tau)$ categorifies J_τ^{ann}

$\mathcal{H}^{ann}(\tau)$ was constructed by Rina Anno based on Nakajima quiver varieties

Old Witten - Reshetikhin - Turaev story

(SL_2 case)

V_a - $(a+1)$ - dimensional rep. of SL_2

assume $q = e^{i\pi/K}$

o puncture colored by V_a

$\rightsquigarrow$ a "Hilbert space" $\mathcal{H}(\Sigma, \text{punctures})$

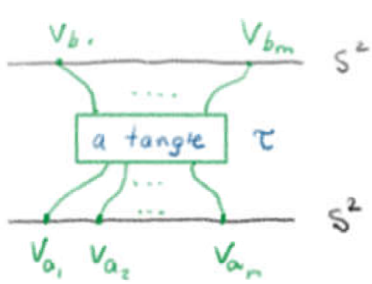
s.t. $\mathcal{H}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$

$\underbrace{(M, \text{tangle})}_{\partial M = \Sigma} \rightsquigarrow |M, \text{tangle}\rangle \in \mathcal{H}(\Sigma, \text{punctures})$

The WRT invariant $Z(M_1 \# M_2, \text{link}) = \langle M_1, \text{tangle} \,|\, M_2, \text{tangle} \rangle_{\mathcal{H}(\Sigma, \text{punctures})}$

hermitian inner product

If $\Sigma = S^2$ and $K \gg a_1, \dots, a_n \rightarrow$ stable limit $\mathcal{H}(S^2, a_1, \dots, a_n) = (V_{a_1} \otimes \cdots \otimes V_{a_n})^{SL_2}$



$\rightsquigarrow \mathcal{H}_1 \xrightarrow{\hat{\tau}} \mathcal{H}_2$

Now q is just a parameter, $\mathcal{H}(S^2, \text{no punctures}) = \mathbb{C}$

$(0,0)$ - tangle : $\mathbb{C} \xrightarrow{\text{Jones pol.}} \mathbb{C}$

Witten asks : how does the quantum gp. R-matrix emerge ?

("Gauge theories, vertex models, and quantum groups", 1989)

Bases in $\mathcal{H}(S^2, \text{punctures}) = \underbrace{(V_{a_1} \otimes \cdots \otimes V_{a_n})}^{SL_2}$

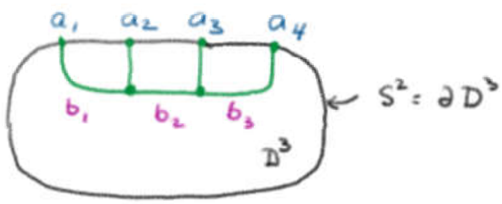
$= \mathrm{Hom}(V_{a_1} \otimes \cdots \otimes V_{a_n}, \underbrace{V_0}_{\text{trivial}})$

An "old-fashioned" basis :

A basis vector $(\underbrace{\beta_1}_{=\alpha_1}, \beta_2, \ldots, \underbrace{\beta_{n-1}}_{=\alpha_n})$:

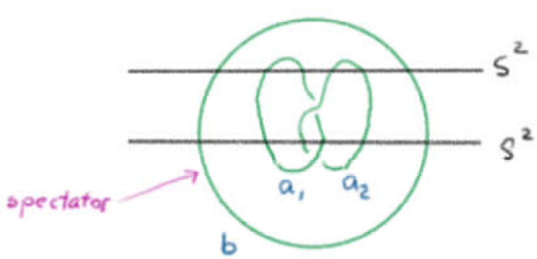
$$\left.\begin{array}{l} \mathrm{Hom}(V_{a_1} \otimes V_{a_2}, V_{b_2}) \\ \mathrm{Hom}(V_{b_2} \otimes V_{a_3}, V_{b_3}) \\ \vdots \\ \mathrm{Hom}(V_{b_{n-1}} \otimes V_{a_n}, V_0) \end{array}\right\} \begin{array}{l} \text{all spaces are 1-dimensional} \\ \text{there are canonical normalizations} \end{array}$$

Picture :

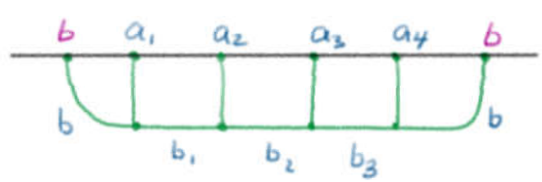


<u>Witten's idea</u> : put the link inside a "spectator" unknot V_b

and assume $b \gg a_1, \ldots, a_n$



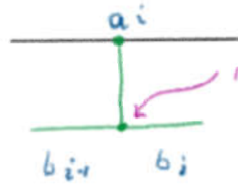
A basis with a spectator :



Use the differences

$b_1 \quad b_2 \quad b_3$

> Use the differences
> $m_i = b_i - b_{i-1}$
> instead of b_i
> and interpret m_i's as
> weights of "virtual" V_{a_i}



$m_i = b_i - b_{i-1}$ $\Rightarrow$ $-a_i \leq m_i \leq a_i$

$a_i - m_i \in 2\mathbb{Z}$

If $b \gg a_1, \dots, a_n$, then

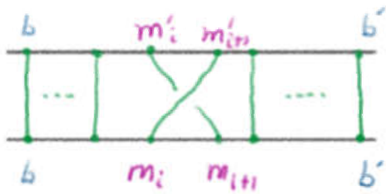
$$\left(V_b \otimes V_{a_1} \otimes \cdots \otimes V_{a_n} \otimes V_{b'}\right)^G \cong \underbrace{W_{b'-b}}_{\text{weight subspace}}\left(V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n}\right)$$

$\leftarrow$ spectators V_b, $V_{b'}$ disappeared

A basis vector is depicted as
(α's are omitted)

For simplicity (and for categorification) choose $\alpha_1, \dots, \alpha_n = 2$

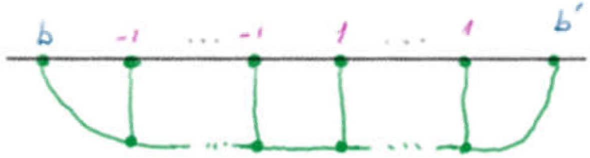
Witten:



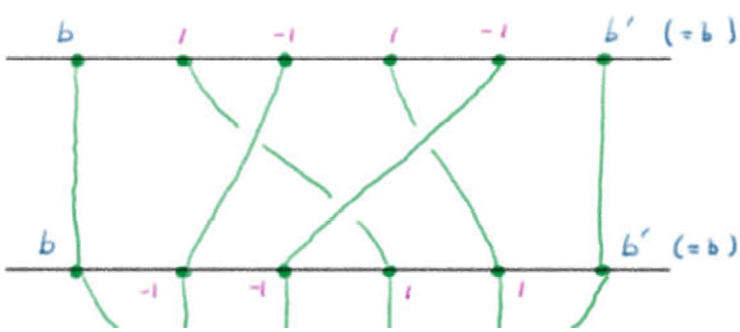
is the entry of the $SL(2)_q$ R-matrix
in the limit $q^\beta \to \infty$

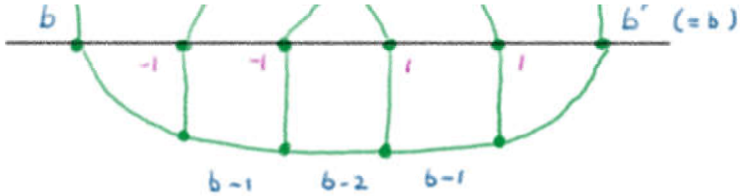
A "better" basis (compare with M. Khovanov's Ph.D thesis and Chen-Khovanov

Begin with

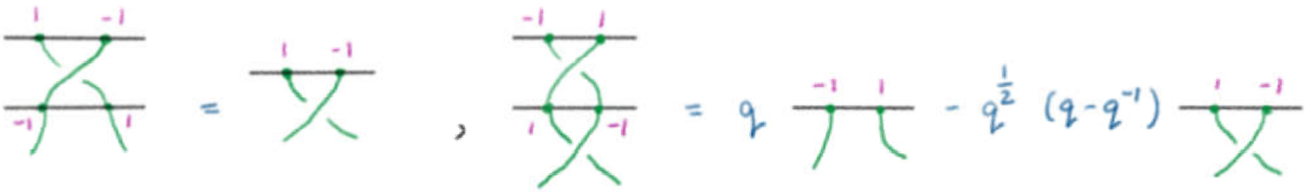


and then apply a short positive braid



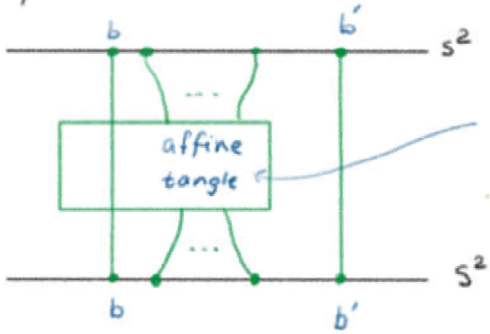


Then R-matrix entries appear for any b :



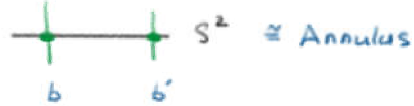
Canonical basis comes from using crossingless matchings instead of short pos. braids

A spectator strand allows an inclusion of an affine tangle:

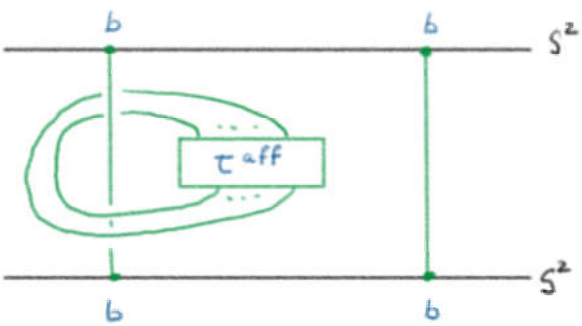


τ is inside $\mathrm{Ann} \times \underbrace{I}_{\text{end-points}}$

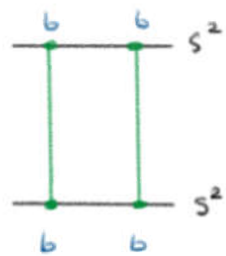
$\mathrm{Ann} \cong S^2 \setminus 2$ points (cut by the spectator strands)



In particular, a (0,0) affine tangle:



$= J^{ann}_{\tau}(q, q^{b+1})$



Categorification

Since $\mathcal{H}($  $S^2) \cong W_{b'-b}(\underbrace{V_1 \otimes \cdots \otimes V_1}_{n \text{ times}})$

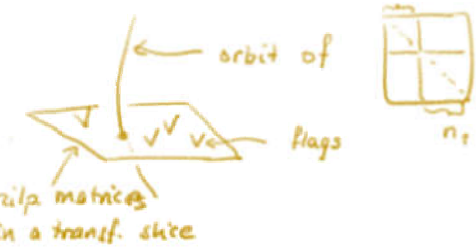
use (the symplectic dual of) the corresponding Nakajima quiver variety (R. Anno) of A_N type

$\mathcal{M}_{-} = (T^* Fl \times S^{eq}_{\cdot} \quad) /\!/ GL_n$

← orbit of 

of A_N type

$$\mathcal{M}_Q = \Big(T^* FL \times \underbrace{\mathcal{S}'^{\,eq}_{(n_-,n_+)}}_{\substack{\text{equivariant Slodowy slice} \\ \cong\, \mathcal{S}'_{(n_-,n_+)} \times GL_n}} \Big) /\!\!/ GL_n$$



Equivariance: $\mathbb{C}^\times_q \times \mathbb{C}^\times_t \times \underbrace{\mathbb{C}^\times_s}_{\text{new}} \curvearrowright \mathcal{M}_Q$

$\mathbb{C}^\times_q \times \mathbb{C}^\times_t \xrightarrow{(-1,1)} \mathbb{C}^\times_{\hbar}$ scales $T^* FL$ and $\mathcal{S}'_{(n_-,n_+)} \times GL_n$

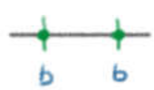
$\mathbb{C}^\times_s \curvearrowright \mathcal{S}'_{(n_-,n_+)} \times GL_n$ by right mult. by $\left(\begin{array}{c|c} \lambda I & 0 \\ \hline 0 & \lambda^{-1} I \end{array}\right)$

If $n_- = n_+$ then $\mathbb{C}^\times_s \subset SL_{2,s} \curvearrowright SL^{eq}_{(n_-,n_+)}$

SL_2 action on SL_2 annular homology

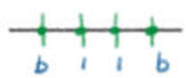
The category: $\boxed{D\,Coh^{\mathbb{C}^\times_\hbar \times \mathbb{C}^\times_s}(\mathcal{M}_Q)}$

<u>Ex 1</u> $n=0$



$\mathcal{M} = 1\,pt$

<u>Ex 2</u> $n=2$, $\underbrace{n_- = n_+ = 1}_{b' = b - n_- + n_+ = b}$,



$\mathcal{S}'^{\,eq}_{(1,1)} = T^* GL_2$, $SL_{(1,1)} = \mathfrak{gl}_2$

$\boxed{\mathcal{M}_Q = T^*\mathbb{P}^1} \circlearrowright \mathbb{C}^\times_\hbar \times SL_{2,s}$

$\qquad\qquad\qquad\qquad \cup$

$\qquad\qquad\qquad\qquad \mathbb{C}^\times_s$

$$\begin{array}{c} T^*\mathbb{P}^1 \\ \iota \uparrow \downarrow \pi \\ \mathbb{P}^1 \end{array}$$

Anno:

$\rightsquigarrow \; - \otimes \iota_* \mathcal{O}(-1)$

$\rightsquigarrow \; Hom(\iota_* \mathcal{O}(-1), -)$

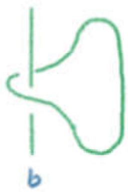
$\rightsquigarrow \; - \otimes \pi^* \mathcal{O}(1)$

$\rightsquigarrow \quad - \otimes \pi^* \mathcal{O}(1)$

<u>Exercise</u>: $\pi^*\mathcal{O}(1)\,\hat{q}^{2}\hat{t}^{-2} \longrightarrow \pi^*\mathcal{O}(-1)$ is the resolution of $\iota_* \mathcal{O}(-1)$

$\rightsquigarrow \ \mathrm{Hom}_{T^*\mathbb{P}^1}(\iota_*\mathcal{O}(-1), \iota_*\mathcal{O}(-1)) = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(1)\,\hat{q}^{2}\hat{t}^{-2} \to \mathcal{O}(-1), \mathcal{O}(-1))$

$= H(\mathcal{O}(-2))\,\hat{q}^{2}\hat{t}^{-1} \oplus H(\mathcal{O}(0)) = \boxed{\mathbb{C}\,\hat{q}^{2} \oplus \mathbb{C}}$



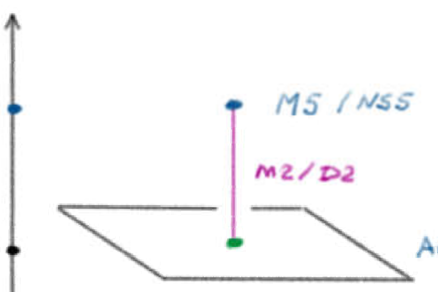
$\rightsquigarrow \ \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(-1))\,\hat{q}^{2}\hat{t}^{-1} \oplus H(\mathcal{O}(1)) = \underbrace{\boxed{\mathbb{C}\,\hat{s} \oplus \mathbb{C}\,\hat{s}^{-1}}}_{\text{with the action of } SL_{2,s}}$

This can be extended to links in $S^2 \times S^1$ with b and b' unknots along S^1

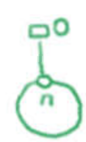
<u>String theory / sympl. alg. geometry</u>

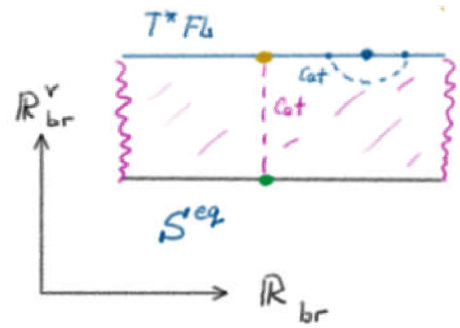
$M^3 = ($ b  b$'$ $) \times$  , $CY^3 = T^*M^3 = \underbrace{T^*\mathbb{C}^*}_{\mathbb{C}^* \times \underbrace{\mathbb{C}}} \times \underbrace{\mathbb{C}}_{T^*\mathbb{R}_{br} = \underbrace{\mathbb{R}_{br}}_{M5/NS5} \times \mathbb{R}^\vee_{br}}$

$\mathbb{C}^* \cong S^2$ with 2 punctures $\qquad \mathbb{R}_{br}$ - braid time



$\times\ \mathbb{R}_{br}$

A strip of M2 / D2 branes is described by a 3d B-model with target 



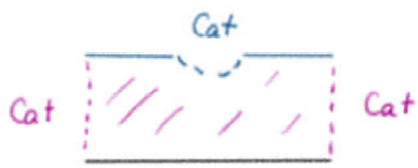
$$Cat = MF^{GL_n}\left(gl_n \times T^*FL \times S^{eq};\ W = Tr\, X(\mu_{FL} - \mu_S)\right)$$

$$\cong DCoh(\mathfrak{m}_Q)$$

$$Cat = MF^{GL_n}\left(gl_n \times T^*FL \times T^*FL;\ W = Tr\, X(\mu_{FL_2} - \mu_{FL_1})\right)$$

$$\cong DCoh^{GL_n}(\text{derived Steinberg variety})$$

"Pair of pants"



$$\begin{tikzcd} & Br_n^{aff} \arrow[d, "\text{Bezrukavnikov-Riche}"] & \\ Cat \times & Cat \arrow[r] & Cat \end{tikzcd}$$

$$\rightsquigarrow$$

$$Cat \times Br_n^{aff} \xrightarrow{\text{Anno}} Cat$$

Tangles: a cup $\cup$ comes from 

and $\|$ is "invisible" in the presence of S^{eq}, that is, in Cat

Categorified $S^2 \times S^1$ with two spectator unknots: