On $\mathfrak{sl}(N)$ link homology with mod N coefficients

Joshua Wang

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Let $L \amalg O$ be the split union of L and an unknot O. Let $q \in L$ and $r \in O$.

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$$\mathrm{KR}_P(L) \cong \overline{\mathrm{KR}}_P(L \amalg O, r) \cong \overline{\mathrm{KR}}_P(L \amalg O, q) \cong \overline{\mathrm{KR}}_P(L, q) \otimes \frac{(\mathbf{Z}/P)[X]}{X^P}$$

□

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Proof of basepoint-independence uses an operator

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The homology of the subcomplex $X_q^{N-1}\,\mathrm{KRC}_N(D; R)$ is $\overline{\mathrm{KR}}_N(L, q; R)$.

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The relevant modular identity is

$$(N-1)!(N-1)! \bmod N = \begin{cases} 1 & N \text{ is prime} \\ 0 & N \text{ is composite} \end{cases}$$

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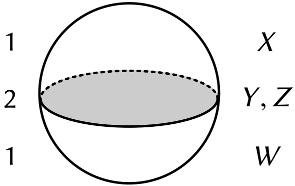
$$Q \in R[F] := \bigotimes_f R[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\mathrm{Sym}}$$

Dotted foams

$F =$ 

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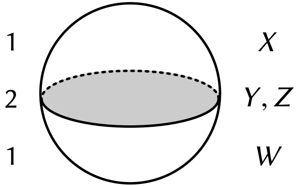

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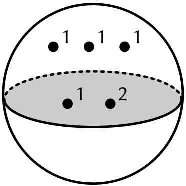
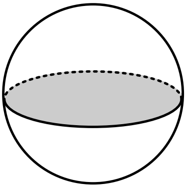


$= X^3(Y+Z)(YZ)$



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$$\mathcal{F}_N(\Gamma; R) = \frac{\widetilde{\mathcal{F}}_N(\Gamma; R)}{\ker \langle -, - \rangle_R} \qquad \langle \boldsymbol{F}, \boldsymbol{G} \rangle_R \coloneqq \langle \boldsymbol{F} \cup \overline{\boldsymbol{G}} \rangle_R$$

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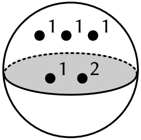
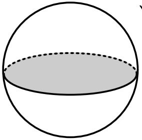
∇ 

$= \nabla \left(X^3(Y+Z)(YZ) \right.$  $\left. \right)$

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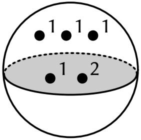
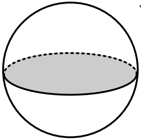
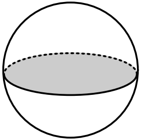
∇ 

$= \nabla \left(X^3(Y+Z)(YZ) \right.$  $\left. \right)$

$= 3X^2(Y+Z)(YZ) + 2X^3(YZ) + X^3(Y+Z)^2$ 

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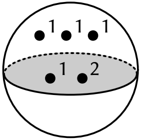
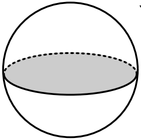
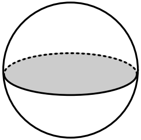
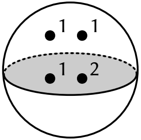
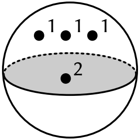
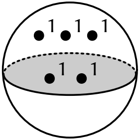
∇ 

$= \nabla \Big(X^3(Y+Z)(YZ)$  $\Big)$

$= 3X^2(Y+Z)(YZ) + 2X^3(YZ) + X^3(Y+Z)^2$ 

$= 3$ 

$+ 2$ 

$+$ 

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Proof.

$$\nabla \left(\sum_c (-1)^{s(F,c)} \frac{P(\boldsymbol{F}, c)}{Q(F, c)} \right) = \sum_c (-1)^{s(F,c)} \frac{\nabla P(\boldsymbol{F}, c)}{Q(F, c)} \qquad \square$$

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$$0 = \sum_i a_i \langle \nabla (\boldsymbol{F}_i \cup \overline{\boldsymbol{G}}) \rangle_R = \sum_i a_i \langle \nabla \boldsymbol{F}_i, \boldsymbol{G} \rangle_R + \sum_i a_i \langle \boldsymbol{F}_i, \nabla \boldsymbol{G} \rangle_R$$

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Let q, r be basepoints on edges of Γ labeled 1. Set $X = X_q$ and $Y = X_r$.

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For each dotted foam $\boldsymbol{F}$, we want to show that

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The proof is just a computation using the Leibniz rule, the identity $X^N = 0$, and Wilson's theorem in elementary number theory.

Proof of basepoint-independence

Proof for $N = 2$.

$$
\begin{aligned}
X\nabla(Y\nabla(X\boldsymbol{F})) &= X\nabla(Y(\boldsymbol{F} + X\nabla\boldsymbol{F})) \\
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(X^2 = 0) \qquad &= X(\boldsymbol{F} + 2Y\nabla\boldsymbol{F}) \\
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□

Proof of basepoint-independence

Proof for $N = 4$.

$$\begin{aligned} X^3\nabla^3(Y^3\nabla^3(X^3\boldsymbol{F})) &= X^3\nabla^3(Y^3(6\boldsymbol{F} + 18X\nabla\boldsymbol{F} + 9X^2\nabla^2\boldsymbol{F} + X^3\nabla^3\boldsymbol{F})) \\ &= X^3(36\boldsymbol{F} + 432Y\nabla\boldsymbol{F} + 540Y^2\nabla^2\boldsymbol{F} + 120Y^3\nabla^3\boldsymbol{F}) \\ &\equiv 0 \bmod 4 \end{aligned}$$

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$$\frac{(N-1)!(N-1)!}{\ell!}\binom{N-1}{\ell}\sum_{i=0}^{\ell}\binom{N-1}{i}\binom{\ell}{i} \bmod N = 0$$

Split link detection (work in progress)

Split link detection for $\mathfrak{sl}(P)$ link homology in characteristic P, where P is prime, generalizing Lipshitz–Sarkar's split link detection result for Khovanov homology in characteristic 2.

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Split link detection for $\mathfrak{sl}(P)$ link homology in characteristic P, where P is prime, generalizing Lipshitz–Sarkar's split link detection result for Khovanov homology in characteristic 2.

Theorem (Lipshitz-Sarkar 2019)

Let q, r be basepoints on an oriented link L, and view $\overline{\mathrm{Kh}}(L, q; \mathbf{Z}/2)$ as a module over $(\mathbf{Z}/2)[X]/X^2$ using X_r. Then $\overline{\mathrm{Kh}}(L, q; \mathbf{Z}/2)$ is a free module over $(\mathbf{Z}/2)[X]/X^2$ if and only if there is a 2-sphere in the complement of L that separates q and r.

Theorem (W. in progress)

Let q, r be basepoints on an oriented link L, and view $\overline{\mathrm{KR}}_P(L, q; \mathbf{Z}/P)$ as a module over $(\mathbf{Z}/P)[X]/X^P$ using X_r. Then $\overline{\mathrm{KR}}_P(L, q; \mathbf{Z}/P)$ is a free module over $(\mathbf{Z}/P)[X]/X^P$ if and only if there is a 2-sphere in the complement of L that separates q and r.

Thanks!

Thanks for listening!