On $\mathfrak{sl}(N)$ link homology with mod $N$ coefficients

Joshua Wang

November 7, 2021
Theorem (Shumakovitch 2004)

Let $L$ be an oriented link with basepoints $q$, $r$.

'Reduced Khovanov homology over $\mathbb{Z}/2$ is basepoint-independent: $\text{Kh}^p_L, q; \mathbb{Z}/2 \equiv \text{Kh}^p_L, r; \mathbb{Z}/2$.

Unreduced Khovanov homology over $\mathbb{Z}/2$ is determined by reduced Khovanov homology: $\text{Kh}^p_L; \mathbb{Z}/2 \equiv \text{Kh}^p_L, q; \mathbb{Z}/2$. 

The basepoint operator $X_q: \text{Kh}^p_L; \mathbb{Z}/2 \rightarrow \text{Kh}^p_L, q; \mathbb{Z}/2$ satisfies $X_q \circ X_q = 0$.

Both statements are false for $\text{Kh}^p_L; \mathbb{Z}$ when $\text{char} \mathbb{Z} \neq 2$. 

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Theorem (Shumakovitch 2004)

Let $L$ be an oriented link with basepoints $q, r \in L$. 

Reduced Khovanov homology over $\mathbb{Z}/2$ is basepoint-independent:

$\text{Kh}^p_{L,q} : \mathbb{Z}/2 \to \mathbb{Z}/2$

Unreduced Khovanov homology over $\mathbb{Z}/2$ is determined by reduced Khovanov homology:

$\text{Kh}^p_{L,q} : \mathbb{Z}/2 \to \mathbb{Z}/2$

The basepoint operator $X_q$:

$X_q : \text{Kh}^p_{L,q} : \mathbb{Z}/2 \to \mathbb{Z}/2$

satisfies $X_q X_q = 0$.

Both statements are false for $\text{Kh}^p_{L,q} : \mathbb{R} \to \mathbb{R}$ when $\text{char } \mathbb{R} \neq 2$. 

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$\mathfrak{sl}(N)$ homology with mod $N$ coefficients

November 7, 2021
Khovanov homology in characteristic 2

**Theorem (Shumakovitch 2004)**

Let $L$ be an oriented link with basepoints $q, r \in L$.

- Reduced Khovanov homology over $\mathbb{Z}/2$ is basepoint-independent:

$$\overline{Kh}(L, q; \mathbb{Z}/2) \cong \overline{Kh}(L, r; \mathbb{Z}/2)$$
Theorem (Shumakovitch 2004)

Let $L$ be an oriented link with basepoints $q, r \in L$.

- **Reduced Khovanov homology over $\mathbb{Z}/2$ is basepoint-independent:**

  $$\overline{Kh}(L, q; \mathbb{Z}/2) \cong \overline{Kh}(L, r; \mathbb{Z}/2)$$

- **Unreduced Khovanov homology over $\mathbb{Z}/2$ is determined by reduced Khovanov homology:**

  $$Kh(L; \mathbb{Z}/2) \cong \overline{Kh}(L, q; \mathbb{Z}/2) \otimes \frac{(\mathbb{Z}/2)[X]}{X^2}$$
Theorem (Shumakovitch 2004)

Let $L$ be an oriented link with basepoints $q, r \in L$.

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The basepoint operator $X_q : \text{Kh}(L; \mathbb{Z}/2) \to \text{Kh}(L; \mathbb{Z}/2)$ satisfies $X_q \circ X_q = 0$. 
**Theorem (Shumakovitch 2004)**

Let $L$ be an oriented link with basepoints $q, r \in L$.

- **Reduced Khovanov homology over $\mathbb{Z}/2$ is basepoint-independent:**

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Both statements are false for $\text{Kh}(L; R)$ when $\text{char}(R) \neq 2$. 

Joshua Wang
Let $L$ be an oriented link with basepoints $q, r \in L$. Let $P$ be prime.

**Theorem (W. 2021)**

The reduced $\mathfrak{sl}_P$ link homology over $\mathbb{Z}/P$ is basepoint-independent:

$$KR_{P \cdot L, q} \mathbb{Z}/P - KR_{P \cdot L, r} \mathbb{Z}/P$$

The unreduced $\mathfrak{sl}_P$ link homology over $\mathbb{Z}/P$ is determined by reduced $\mathfrak{sl}_P$ link homology:

$$KR_{P \cdot L, q} \mathbb{Z}/P$$

The basepoint operator $X_q$ satisfies

$$KR_{P \cdot L, q} \mathbb{Z}/P$$

Both statements are false for $KR_{P \cdot L, r} \mathbb{Z}/P$ when char $P = 2$ does not divide $N$.

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$\mathfrak{sl}(N)$ homology with mod $N$ coefficients

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Theorem (W. 2021)

Let $L$ be an oriented link with basepoints $q, r \in L$. Let $P$ be prime.
Theorem (W. 2021)

Let $L$ be an oriented link with basepoints $q, r \in L$. Let $P$ be prime.

• Reduced $\mathfrak{sI}(P)$ homology over $\mathbb{Z}/P$ is basepoint-independent:

$$\overline{KR}_P(L, q; \mathbb{Z}/P) \cong \overline{KR}_P(L, r; \mathbb{Z}/P)$$
Theorem (W. 2021)

Let $L$ be an oriented link with basepoints $q, r \in L$. Let $P$ be prime.

- Reduced $\mathfrak{s}l(P)$ homology over $\mathbb{Z}/P$ is basepoint-independent:

$$\overline{KR}_P(L, q; \mathbb{Z}/P) \cong \overline{KR}_P(L, r; \mathbb{Z}/P)$$

- Unreduced $\mathfrak{s}l(P)$ homology over $\mathbb{Z}/P$ is determined by reduced $\mathfrak{s}l(P)$ homology:

$$KR_P(L; \mathbb{Z}/P) \cong \overline{KR}_P(L, q; \mathbb{Z}/P) \otimes \frac{(\mathbb{Z}/P)[X]}{X^P}$$
Theorem (W. 2021)

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The basepoint operator $X_q : KR_P(L; \mathbb{Z}/P) \rightarrow KR_P(L; \mathbb{Z}/P)$ satisfies $X_q^P = 0$. 

Joshua Wang  \hspace{1cm} $\mathfrak{sl}(N)$ homology with mod $N$ coefficients  \hspace{1cm} November 7, 2021
Theorem (W. 2021)

Let $L$ be an oriented link with basepoints $q, r \in L$. Let $P$ be prime.

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The basepoint operator $X_q : KR_P(L; \mathbb{Z}/P) \to KR_P(L; \mathbb{Z}/P)$ satisfies $X_q^P = 0$.

Both statements are false for $KR_N(L; R)$ when $\text{char}(R)$ does not divide $N$. 
\[ \mathfrak{sl}(P) \text{ link homology in characteristic } P \]

Proof of \( \text{KR}_P(L; \mathbb{Z}/P) \cong \text{KR}_P(L, q; \mathbb{Z}/P) \otimes (\mathbb{Z}/P)[X]/X^P \) from basepoint-independence of reduced \( \mathfrak{sl}(P) \) link homology:
$\mathfrak{sl}(P)$ link homology in characteristic $P$

Proof of $KR_p(L; \mathbb{Z}/P) \cong \overline{KR}_p(L, q; \mathbb{Z}/P) \otimes (\mathbb{Z}/P)[X]/X^P$ from basepoint-independence of reduced $\mathfrak{sl}(P)$ link homology:

(The argument is due to Ozsváth-Rasmussen-Szabó 2013)
$\mathfrak{sI}(P)$ link homology in characteristic $P$

Proof of $\text{KR}_P(L; \mathbb{Z}/P) \cong \overline{\text{KR}}_P(L, q; \mathbb{Z}/P) \otimes (\mathbb{Z}/P)[X]/X^P$ from basepoint-independence of reduced $\mathfrak{sI}(P)$ link homology:

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Proof.

Let $L \sqcup O$ be the split union of $L$ and an unknot $O$. Let $q \in L$ and $r \in O$. 

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$\mathfrak{sI}(N)$ homology with mod $N$ coefficients

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$\mathfrak{s}l(P)$ link homology in characteristic $P$

Proof of $\text{KR}_P(L; \mathbb{Z}/P) \cong \overline{\text{KR}}_P(L, q; \mathbb{Z}/P) \otimes (\mathbb{Z}/P)[X]/X^P$ from basepoint-independence of reduced $\mathfrak{s}l(P)$ link homology:

(The argument is due to Ozsváth-Rasmussen-Szabó 2013)

**Proof.**

Let $L \sqcup O$ be the split union of $L$ and an unknot $O$. Let $q \in L$ and $r \in O$.

$$\text{KR}_P(L) \cong \overline{\text{KR}}_P(L \sqcup O, r) \cong \overline{\text{KR}}_P(L \sqcup O, q) \cong \overline{\text{KR}}_P(L, q) \otimes \frac{(\mathbb{Z}/P)[X]}{X^P}$$
Proof of basepoint-independence uses an operator

\[ \nabla : \text{KR}_N(L; R) \to \text{KR}_N(L; R) \]

defined for any \( N \geq 2 \) when \( \text{char}(R) \mid N \).
The operator $\nabla$

Proof of basepoint-independence uses an operator

\[
\nabla : \text{KR}_N(L; R) \to \text{KR}_N(L; R)
\]

defined for any $N \geq 2$ when $\text{char}(R) | N$. Primality of $N$ arises from the relationship between $\nabla$ and basepoint operators.
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Some notation and background for $\text{KR}_N(L; R)$ for any $N \geq 2$ and any $R$: 
The operator $\nabla$

Proof of basepoint-independence uses an operator

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Some notation and background for $\text{KR}_N(L; R)$ for any $N \geq 2$ and any $R$:

- $\text{KRC}_N(D; R)$ - the $\mathfrak{s}\mathfrak{l}(N)$ chain complex of a diagram $D$
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Proof of basepoint-independence uses an operator

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- $\text{KRC}_N(D; R)$ - the $\mathfrak{sl}(N)$ chain complex of a diagram $D$
- $X_q : \text{KRC}_N(D; R) \to \text{KRC}_N(D; R)$ - the basepoint operator associated to $q \in D$. It is a chain map satisfying $X_q^N = 0$. 
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Proof of basepoint-independence uses an operator

$$\nabla : \text{KR}_N(L; R) \rightarrow \text{KR}_N(L; R)$$

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Some notation and background for $\text{KR}_N(L; R)$ for any $N \geq 2$ and any $R$:

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The homology of the subcomplex $X_q^{N-1} \text{KRC}_N(D; R)$ is $\overline{\text{KR}}_N(L, q; R)$. 
The operator $\nabla$

If $\text{char}(R) \mid N$, there is a chain map $\nabla : KRC_N(D; R) \to KRC_N(D; R)$. 

If $N$ is prime or composite, the relevant modular identity is $p_n \equiv p_0 \pmod{N}$. 

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$\mathfrak{sl}(N)$ homology with mod $N$ coefficients

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The operator $\nabla$

If $\text{char}(R) \mid N$, there is a chain map $\nabla : \text{KRC}_N(D; R) \rightarrow \text{KRC}_N(D; R)$. It preserves $h$-grading and decreases $q$-grading by 2 (opposite of $X_q$).
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**Proposition (W. 2021)**

Let $q, r \in D$.

\[
X_q^{N-1} \text{KRC}_N(D; R) \xrightarrow{X_q^{N-1}\nabla^{N-1}} X_q^{N-1} \text{KRC}_N(D; R) \quad \xleftarrow{X_r^{N-1}\nabla^{N-1}} X_r^{N-1} \text{KRC}_N(D; R)
\]
The operator $\nabla$

If $\text{char}(R) \mid N$, there is a chain map $\nabla : \text{KRC}_N(D; R) \to \text{KRC}_N(D; R)$. It preserves $h$-grading and decreases $q$-grading by 2 (opposite of $X_q$).

**Proposition (W. 2021)**

Let $q, r \in D$.

$$X_q^{N-1} \text{KRC}_N(D; R) \xrightarrow{X_q^{N-1}\nabla^{N-1}} X_r^{N-1} \text{KRC}_N(D; R)$$

Then $$(X_q^{N-1}\nabla^{N-1}) \circ (X_r^{N-1}\nabla^{N-1}) = \begin{cases} \text{Id} & \text{N is prime} \\ 0 & \text{N is composite} \end{cases}$$
The operator $\nabla$

If $\text{char}(R) \mid N$, there is a chain map $\nabla : KRC_N(D; R) \to KRC_N(D; R)$. It preserves $h$-grading and decreases $q$-grading by 2 (opposite of $X_q$).

**Proposition (W. 2021)**

Let $q, r \in D$.

\[
\begin{align*}
X_q^{N-1} KRC_N(D; R) & \xleftarrow{X_q^{N-1} \nabla^{N-1}} X_r^{N-1} KRC_N(D; R) \\
X_r^{N-1} \nabla^{N-1} & \xrightarrow{X_r^{N-1} KRC_N(D; R)} X_q^{N-1} KRC_N(D; R)
\end{align*}
\]

Then \[
(X_q^{N-1} \nabla^{N-1}) \circ (X_r^{N-1} \nabla^{N-1}) = \begin{cases} 
\text{Id} & N \text{ is prime} \\
0 & N \text{ is composite}
\end{cases}
\]

The relevant modular identity is

\[
(N - 1)! (N - 1)! \mod N = \begin{cases} 
1 & N \text{ is prime} \\
0 & N \text{ is composite}
\end{cases}
\]
Version of $\mathfrak{sl}(N)$ link homology: Robert–Wagner evaluation of closed foams
The operator $\nabla$

Version of $\mathfrak{sl}(N)$ link homology: Robert–Wagner evaluation of closed foams.

$\nu$ a vertex of the cube $\mapsto$ MOY graph $D_\nu \mapsto$ $R$-module $\mathcal{F}_N(D_\nu; R)$
The operator $\nabla$

Version of $\mathfrak{sl}(N)$ link homology: Robert–Wagner evaluation of closed foams

$v$ a vertex of the cube $\leadsto$ MOY graph $D_v \leadsto R$-module $\mathcal{F}_N(D_v; R)$

$$KRC_N(D; R) = \bigoplus_v \mathcal{F}_N(D_v; R)$$
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Version of $\mathfrak{sl}(N)$ link homology: Robert–Wagner evaluation of closed foams

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$$KRC_N(D; R) = \bigoplus_\nu \mathcal{F}_N(D_\nu; R)$$

Both $X_q$ and $\nabla$ are defined on the individual $\mathcal{F}_N(D_\nu; R)$, and commute with the edge maps.
The operator $\nabla$

Version of $\mathfrak{sl}(N)$ link homology: Robert–Wagner evaluation of closed foams

$v$ a vertex of the cube $\leadsto$ MOY graph $D_v \leadsto$ $R$-module $\mathcal{F}_N(D_v; R)$

$$KRC_N(D; R) = \bigoplus_v \mathcal{F}_N(D_v; R)$$

Both $X_q$ and $\nabla$ are defined on the individual $\mathcal{F}_N(D_v; R)$, and commute with the edge maps. It suffices to prove

$$(X_q^{N-1}\nabla^{N-1}) \circ (X_r^{N-1}\nabla^{N-1}) = \begin{cases} 
\text{Id} & N \text{ is prime} \\
0 & N \text{ is composite}
\end{cases}$$

on $\mathcal{F}_N(D_v; R)$. 
Dotted foams

MOY graph $\Gamma$. 

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Dotted foams

MOY graph $\Gamma$. The $R$-module $\mathcal{F}_N(\Gamma; R)$ is a quotient of

$$\tilde{\mathcal{F}}_N(\Gamma; R) = \text{free } R\text{-module with basis } \text{dotted foams } F : \emptyset \to \Gamma$$
Dotted foams

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Dots lie on facets of the foam.
Dotted foams

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Dots lie on facets of the foam. Each facet $f$ has a label $\ell(f) \in \{0, \ldots, N\}$. 
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Dotted foams

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A dot of weight $w$ \quad with elementary symmetric polynomial $e_w$ in $\ell(f)$ formal variables associated to $f$
Dotted foams

MOY graph $\Gamma$. The $R$-module $\widetilde{\mathcal{F}}_N(\Gamma; R)$ is a quotient of

$$\widetilde{\mathcal{F}}_N(\Gamma; R) = \text{free } R\text{-module with basis dotted foams } F: \varnothing \to \Gamma$$

Dots lie on facets of the foam. Each facet $f$ has a label $\ell(f) \in \{0, \ldots, N\}$. Each dot $d$ on $f$ has a weight $w(d) \in \{1, \ldots, \ell(f)\}$.

A dot of weight $w \leftrightarrow$ with elementary symmetric polynomial $e_w$ in $\ell(f)$ formal variables associated to $f$

\[
\{\text{dots on } f\} \leftrightarrow P_f = \prod_{d \in f} e_{w(d)} \in \mathbb{Z}[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\text{Sym}}
\]
Dotted foams

MOY graph $\Gamma$. The $R$-module $\tilde{\mathcal{F}}_N(\Gamma; R)$ is a quotient of

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Dots lie on facets of the foam. Each facet $f$ has a label $\ell(f) \in \{0, \ldots, N\}$. Each dot $d$ on $f$ has a weight $w(d) \in \{1, \ldots, \ell(f)\}$.

A dot of weight $w \leftrightarrow \text{wth elementary symmetric polynomial } e_w$

in $\ell(f)$ formal variables associated to $f$

$$\{\text{dots on } f\} \leftrightarrow P_f = \prod_{d \in f} e_{w(d)} \in \mathbb{Z}[\langle X_f \rangle_1, \ldots, \langle X_f \rangle_{\ell(f)}]^{\text{Sym}}$$

$$\{\text{dots on } F\} \leftrightarrow P = \prod_f P_f \in \bigotimes_f \mathbb{Z}[\langle X_f \rangle_1, \ldots, \langle X_f \rangle_{\ell(f)}]^{\text{Sym}}$$
Dotted foams

MOY graph $\Gamma$. The $R$-module $\mathcal{F}_N(\Gamma; R)$ is a quotient of

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Dots lie on facets of the foam. Each facet $f$ has a label $\ell(f) \in \{0, \ldots, N\}$. Each dot $d$ on $f$ has a weight $w(d) \in \{1, \ldots, \ell(f)\}$.

A dot of weight $w$ $\longleftrightarrow$ with elementary symmetric polynomial $e_w$ in $\ell(f)$ formal variables associated to $f$

$$\{\text{dots on } f\} \longleftrightarrow P_f = \prod_{d \in f} e_{w(d)} \in \mathbb{Z}[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\text{Sym}}$$

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We think of $F = PF$ where $F$ has no dots.
Dotted foams

MOY graph $\Gamma$. The $R$-module $\tilde{\mathcal{F}}_N(\Gamma; R)$ is a quotient of

$$\tilde{\mathcal{F}}_N(\Gamma; R) = \text{free } R\text{-module with basis dotted foams } F : \emptyset \to \Gamma$$

Dots lie on facets of the foam. Each facet $f$ has a label $\ell(f) \in \{0, \ldots, N\}$. Each dot $d$ on $f$ has a weight $w(d) \in \{1, \ldots, \ell(f)\}$.

A dot of weight $w$ \[ \longleftrightarrow \] with elementary symmetric polynomial $e_w$ in $\ell(f)$ formal variables associated to $f$

$$\{\text{dots on } f\} \longleftrightarrow P_f = \prod_{d \in f} e_{w(d)} \in \mathbb{Z}[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\text{Sym}}$$

$$\{\text{dots on } F\} \longleftrightarrow P = \prod_f P_f \in \bigotimes_f \mathbb{Z}[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\text{Sym}}$$

We think of $F = PF$ where $F$ has no dots. Define $QF \in \tilde{\mathcal{F}}_N(\Gamma; R)$ for any

$$Q \in R[F] := \bigotimes_f R[(X_f)_1, \ldots, (X_f)_{\ell(f)}]^{\text{Sym}}$$
Dotted foams

\[ F = \begin{array}{ccc}
0 & 1 & X \\
2 & Y, Z & YZ \\
1 & W &
\end{array} \]

\[ R[F] = R[X, Y + Z, YZ, W] \]
Dotted foams

\[ F = \begin{array}{c}
1 \\
2 \\
1
\end{array} \]

\[ X, Y, Z \]

\[ W \]

\[ R[F] = R[X, Y + Z, YZ, W] \]

\[ F = \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
2 \\
1
\end{array} \]

\[ X^3(Y + Z)(YZ) \]
Dotted foams

Given dotted foams $F, G: \emptyset \rightarrow \Gamma$, the union $F \cup \overline{G}$ is a closed dotted foam.
Dotted foams

Given dotted foams $F, G: \emptyset \to \Gamma$, the union $F \cup \overline{G}$ is a closed dotted foam. The Robert–Wagner evaluation of a closed dotted foam

$$\langle F \cup \overline{G} \rangle \in \mathbb{Z}[X_1, \ldots, X_N]^\text{Sym}$$

is a homogeneous symmetric polynomial.
Dotted foams

Given dotted foams $F, G : \emptyset \rightarrow \Gamma$, the union $F \cup \overline{G}$ is a closed dotted foam. The Robert–Wagner evaluation of a closed dotted foam

$$\langle F \cup \overline{G} \rangle \in \mathbb{Z}[X_1, \ldots, X_N]^\text{Sym}$$

is a homogeneous symmetric polynomial. Let $\langle F \cup \overline{G} \rangle_R$ be the image under $\mathbb{Z}[X_1, \ldots, X_N] \rightarrow R$ given by $X_i \mapsto 0$. 

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$$\mathcal{F}_N(\Gamma; R) = \frac{\tilde{\mathcal{F}}_N(\Gamma; R)}{\ker \langle -, - \rangle_R} \quad \langle F, G \rangle_R := \langle F \cup \overline{G} \rangle_R$$
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Joshua Wang
$\mathfrak{sl}(N)$ homology with mod $N$ coefficients
November 7, 2021
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Construction of $\nabla$

Define $\nabla : R[X_1, \ldots, X_k] \to R[X_1, \ldots, X_k]$ by $\nabla = \sum_{i=1}^{k} \frac{\partial}{\partial X_i}$. 
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Define $\tilde{\nabla} : \tilde{\mathcal{F}}_N(\Gamma; R) \rightarrow \tilde{\mathcal{F}}_N(\Gamma; R)$ by $\nabla(PF) = \nabla(P)F$. 
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Define $\nabla : \mathbb{F}_N(\Gamma; R) \to \mathbb{F}_N(\Gamma; R)$ by $\nabla(PF) = \nabla(P)F$. 

\[ \nabla \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \nabla \begin{bmatrix} X^3(Y + Z)(YZ) \end{bmatrix} \]
Construction of $\nabla$

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$$\nabla = \nabla \left( X^3(Y + Z)(YZ) \right)$$

$$= 3X^2(Y + Z)(YZ) + 2X^3(YZ) + X^3(Y + Z)^2$$
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- If $F : \Gamma \to \Gamma'$ and $G : \Gamma' \to \Gamma''$, then $\nabla(F \cup G) = \nabla(F) \cup G + F \cup \nabla(G)$.
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**Proposition (W. 2021)**

*Let $F$ be a closed dotted foam. Then $\langle \nabla F \rangle = \nabla\langle F \rangle$.***
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**Proposition (W. 2021)**

Let $F$ be a closed dotted foam. Then $\langle \nabla F \rangle = \nabla \langle F \rangle$.

A coloring $c$ associates to each facet $f$ a subset of $\{X_1, \ldots, X_N\}$ of size $\ell(f)$

$$
\langle F \rangle = \sum_c (-1)^{s(F,c)} \frac{P(F, c)}{Q(F, c)} \\
\quad P(F, c) = \text{evaluate } P = \prod_f P_f \text{ according to } c \\
\quad Q(F, c) = \text{product of } (X_i - X_j) \text{ and inverses}
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**Construction of \( \nabla \)**

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**Proposition (W. 2021)**

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**Proof.**

\[
\nabla \left( \sum_c (-1)^{s(F,c)} \frac{P(F, c)}{Q(F, c)} \right) = \sum_c (-1)^{s(F,c)} \frac{\nabla P(F, c)}{Q(F, c)}
\]
Corollary

If \( \text{char}(R) \mid N \), then \( \langle \nabla F \rangle_R = 0 \) for all closed dotted foams \( F \).
Construction of $\nabla$

**Corollary**

If $\text{char}(R) \mid N$, then $\langle \nabla F \rangle_R = 0$ for all closed dotted foams $F$.

$\langle \nabla F \rangle_R$ is the image of the homogeneous polynomial $\langle \nabla F \rangle$ under

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Only nontrivial case to check: \( \deg \langle \nabla F \rangle = 0 \).
Construction of $\nabla$

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*If* $\text{char}(R) \mid N$, *then* $\langle \nabla F \rangle_R = 0$ *for all closed dotted foams* $F$.

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\langle F \rangle = m(X_1 + \cdots + X_N) \implies \langle \nabla F \rangle = \nabla \langle F \rangle = mN.
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Corollary

If char(\(R\)) \(\mid\) \(N\), then \(\langle \nabla F \rangle_R = 0\) for all closed dotted foams \(F\).

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If char(\(R\)) \(\mid\) \(N\), then \(\nabla\) descends to a map \(\tilde{\mathcal{F}}_N(\Gamma; R) \to \mathcal{F}_N(\Gamma; R)\).
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If \( \text{char}(R) \mid N \), then \( \nabla \) descends to a map \( \mathcal{F}_N(\Gamma; R) \rightarrow \mathcal{F}_N(\Gamma; R) \).

If \( \sum_i a_i \langle F_i, G \rangle_R = 0 \) for all \( G \), then \( \sum_i a_i \langle \nabla F_i, G \rangle_R = 0 \) for all \( G \):
Construction of $\nabla$

**Corollary**

If $\text{char}(R) \mid N$, then $\langle \nabla F \rangle_R = 0$ for all closed dotted foams $F$.

$\langle \nabla F \rangle_R$ is the image of the homogeneous polynomial $\langle \nabla F \rangle$ under

$$Z[X_1, \ldots, X_N]^\text{Sym} \rightarrow R \quad X_i \mapsto 0.$$

Only nontrivial case to check: $\deg \langle \nabla F \rangle = 0$.

$$\langle F \rangle = m(X_1 + \cdots + X_N) \implies \langle \nabla F \rangle = \nabla \langle F \rangle = mN.$$

**Corollary**

If $\text{char}(R) \mid N$, then $\nabla$ descends to a map $F_N(\Gamma; R) \rightarrow F_N(\Gamma; R)$.

If $\sum_i a_i \langle F_i, G \rangle_R = 0$ for all $G$, then $\sum_i a_i \langle \nabla F_i, G \rangle_R = 0$ for all $G$:

$$0 = \sum_i a_i \langle \nabla (F_i \cup \overline{G}) \rangle_R = \sum_i a_i \langle \nabla F_i, G \rangle_R + \sum_i a_i \langle F_i, \nabla G \rangle_R.$$
Proof of basepoint-independence

Let $q, r$ be basepoints on edges of $\square$ labeled 1. Set $X = X_q$ and $Y = X_r$.

$\nabla = 1_F N \cdot 1_R$; $\nabla = 1_F N \cdot 1_R$

For each do/tshed foam $F$, we want to show that $X = X_q$ and $Y = X_r$.

The proof is just a computation using the Leibniz rule, the identity $X = 0$, and Wilson's theorem in elementary number theory.
Proof of basepoint-independence

Let $q, r$ be basepoints on edges of $\Gamma$ labeled 1. Set $X = X_q$ and $Y = X_r$.

For each foam $\mathcal{F}_N(\Gamma; R)$, we want to show that $X^{N-1} \nabla^{N-1} p Y^{N-1} \nabla^{N-1} p X^{N-1} \mathcal{F}_N(\Gamma; R)$ is prime if $N$ is prime and composite if $N$ is composite.

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Proof of basepoint-independence

Let $q, r$ be basepoints on edges of $\Gamma$ labeled 1. Set $X = X_q$ and $Y = X_r$.

\[
\begin{array}{c}
\chi^{N-1}\nabla^{N-1}
\end{array}
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\end{array}
\]

\[
X^{N-1}\mathcal{F}_N(\Gamma; R) \leftrightarrow Y^{N-1}\mathcal{F}_N(\Gamma; R)
\]

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The proof is just a computation using the Leibniz rule, the identity $\chi^N = 0$, and Wilson’s theorem in elementary number theory.
Proof for $N = 2$.

\[
X \nabla (Y \nabla (XF)) = X \nabla (Y(F + X \nabla F)) \\
= X(F + X \nabla F + Y(\nabla F + \nabla F + X \nabla^2 F)) \\
(X^2 = 0) = X(F + 2Y \nabla F) \\
\equiv XF \mod 2
\]
Proof of basepoint-independence

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\equiv XF \mod 2
\]

Proof of $N = 3$.

\[
X^2 \nabla^2 (Y^2 \nabla^2 (X^2 F)) = X^2 \nabla^2 (Y^2 \nabla (2XF + X^2 \nabla F)) \\
= X^2 \nabla^2 (2Y^2 F + 4XY^2 \nabla F + X^2 Y^2 \nabla^2 F) \\
(X^3 = 0) \quad = X^2 \nabla (4YF + (6X^2 + 8XY) \nabla F + (6XY^2) \nabla^2 F) \\
(X^3 = 0) \quad = X^2 (4F + 24Y \nabla F + 12Y^2 \nabla^2 F) \\
\equiv X^2 F \mod 3
\]
Proof of basepoint-independence

Proof for $N = 4$.

\[
X^3 \nabla^3 (Y^3 \nabla^3 (X^3 F)) = X^3 \nabla^3 (Y^3 (6F + 18X \nabla F + 9X^2 \nabla^2 F + X^3 \nabla^3 F)) \\
= X^3 (36F + 432Y \nabla F + 540Y^2 \nabla^2 F + 120Y^3 \nabla^3 F) \\
\equiv 0 \mod 4
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\]
\[
\equiv 0 \pmod{4}
\]

In general, the coefficient of $X^{N-1}F$ is

\[
(N - 1)! (N - 1)! \pmod{N} = \begin{cases} 
1 & N \text{ is prime} \\
0 & N \text{ is composite}
\end{cases}
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Proof of basepoint-independence

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In general, the coefficient of $X^{N-1}F$ is

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(N - 1)!(N - 1)! \mod N = \begin{cases} 
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\end{cases}
\]

The coefficient of $X^{N-1}Y^\ell \nabla^\ell F$ for $1 \leq \ell \leq N - 1$ is

\[
\frac{(N - 1)!(N - 1)!}{\ell!} \binom{N - 1}{\ell} \sum_{i=0}^{\ell} \binom{N - 1}{i} \binom{\ell}{i} \mod N = 0
\]
Split link detection (work in progress)

Split link detection for $\mathfrak{sl}(P)$ link homology in characteristic $P$, where $P$ is prime, generalizing Lipshitz–Sarkar’s split link detection result for Khovanov homology in characteristic 2.
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Theorem (Lipshitz-Sarkar 2019)

Let $q, r$ be basepoints on an oriented link $L$, and view $\overline{\text{Kh}}(L, q; \mathbb{Z}/2)$ as a module over $\mathbb{Z}/2[X]/X^2$ using $X_r$.  

Theorem (W. in progress)

Let $q, r$ be basepoints on an oriented link $L$, and view $\overline{\text{KR}}_P(L, q; \mathbb{Z}/P)$ as a module over $\mathbb{Z}/P[X]/X^2$ using $X_r$. 

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Thanks for listening!