Computer Bounds for Kronheimer-Mrowka Foam Evaluation

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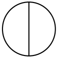
The Four-Color Theorem

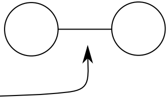
Four-Color Theorem (Appel and Haken, 1976)

Any map can be colored using no more than four colors in such a way that no two adjacent regions share the same color.

Reformulate the Four-Color Theorem in terms of *webs*:

- A *web* is an unoriented planar trivalent graph.
- $\text{Tait}(K) =$ number of 3-colorings of the edges of K.

$\text{Tait}($  $) = 6$

$\text{Tait}($  $) = 0$

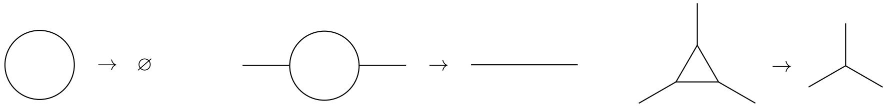
Four-Color Theorem, Reformulated

For any bridgeless web K, we have $\text{Tait}(K) > 0$.

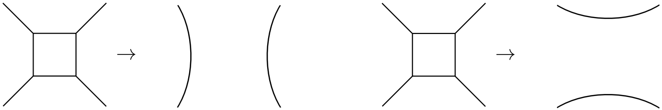
Reducible webs

A web is *reducible* if it can be reduced to the empty web by recursively applying the following local replacements:

circle elimination bigon elimination triangle elimination

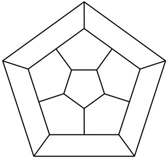


square elimination



Reducible webs

- A web in which all the faces are disks, bigons, triangles, or squares is reducible.
- A web in which none of the faces are disks, bigons, triangles, or squares is nonreducible.
- The simplest nonreducible web is the dodecahedral web:



Four-Color Theorem for Reducible Webs

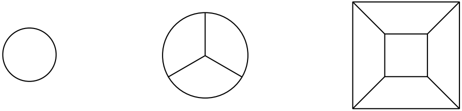
For any reducible web K, we have $\text{Tait}(K) > 0$.

The category Foams

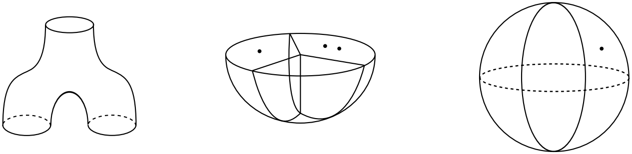
Define a category Foams:

- The objects are *webs*:

$\varnothing$

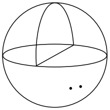


- The morphisms are *foams* (singular cobordisms between webs):



Closed foams

A *closed foam* $F \subset \mathbb{R}^3$ is a cobordism $F : \varnothing \to \varnothing$. Example:

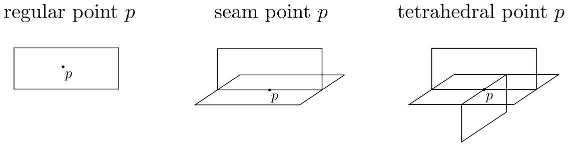


6 facets

4 seams

2 tetrahedral points

Three local models near a point $p \in F$:



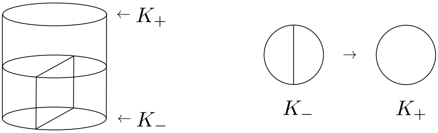
facets = connected components of set of regular points

seams = connected components of set of seam points

Facets may be decorated with a finite number of *dots*

Foams with boundary

A *foam with boundary* $F \subset \mathbb{R}^2 \times [a, b]$ is a cobordism $F : K_- \to K_+$ for webs K_- and K_+. Example:



Local models $K_- \times [a, a+\epsilon)$ and $K_+ \times (b-\epsilon, b]$ near bottom and top

A *half-foam* H is a cobordism $H : \varnothing \to K$. Example:

The gauge-theoretic functor $J^\sharp$: Foams $\to$ $\mathrm{Vect}_{\mathbb{F}}$

Kronheimer and Mrowka use singular instanton homology to define a functor $J^\sharp$: Foams $\to$ $\mathrm{Vect}_{\mathbb{F}}$, where $\mathbb{F}$ is the field of two elements.

Conjecture (Kronheimer and Mrowka)

For any web K, we have $\dim J^\sharp(K) = \mathrm{Tait}(K)$. *(True if K is reducible.)*

Theorem (Kronheimer and Mrowka)

For any bridgeless web K, we have $\dim J^\sharp(K) > 0$.

Four-Color Theorem, Reformulated

For any bridgeless web K, we have $\mathrm{Tait}(K) > 0$.

If the Conjecture is true, the Theorem implies the Four-Color Theorem!

The combinatorial functor $J^\flat : \text{Foams} \to \text{Vect}_{\mathbb{F}}$

Kronheimer and Mrowka consider a possible combinatorial replacement $J^\flat : \text{Foams} \to \text{Vect}_{\mathbb{F}}$ for $J^\sharp : \text{Foams} \to \text{Vect}_{\mathbb{F}}$.

For a closed foam $F : \varnothing \to \varnothing$, Kronheimer and Mrowka defined $J^\flat(F) \in \mathbb{F}$ using combinatorial reduction rules that they conjectured would yield a well-defined result.

Khovanov and Robert proved that this is the case by giving an explicit formula for $J^\flat(F)$.

Conjecture (Kronheimer and Mrowka)

For all closed foams F we have $J^\flat(F) = J^\sharp(F)$.

The combinatorial functor $J^\flat : \text{Foams} \to \text{Vect}_{\mathbb{F}}$

For a web K, define the $\mathbb{F}$-vector space $J^\flat(K)$ using the *universal construction*:

- Define an infinite-dimensional $\mathbb{F}$-vector space

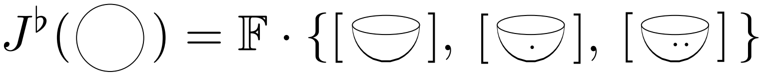
$$V(K) = \mathbb{F} \cdot \{\text{all half-foams } H : \varnothing \to K\}$$

- Define a bilinear form $(-,-) : V(K) \otimes V(K) \to \mathbb{F}$ by

$$(H_1, H_2) = J^\flat(H_1 \cup_K \overline{H}_2) \in \mathbb{F}$$

- Define $J^\flat(K) = V(K)/V(K)^\perp$ (always finite-dimensional)

Example:



Previously known results

Based on results due to Khovanov and Robert (for $J^\flat$) and Kronheimer and Mrowka (for $J^\sharp$), we have that

$$\dim J^\flat(K) \leq \text{Tait}(K) \leq \dim J^\sharp(K).$$

For reducible webs K these three numbers coincide:

$$\dim J^\flat(K) = \text{Tait}(K) = \dim J^\sharp(K).$$

For nonreducible webs K, the only computations of $\dim J^\flat(K)$ and $\dim J^\sharp(K)$ are for the dodecahedral web W_1, which has $\text{Tait}(W_1) = 60$:

Theorem (Kronheimer and Mrowka)

For the dodecahedral web W_1, we have $58 \leq \dim J^\flat(W_1) \leq 60$.

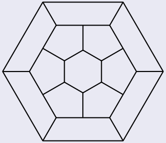
Theorem (Kronheimer and Mrowka)

For the dodecahedral web W_1, we have $60 \leq \dim J^\sharp(W_1) \leq 68$.

New results

Theorem (B.)

For the following nonreducible webs we have $\dim J^\flat(K) = \mathrm{Tait}(K)$*:*



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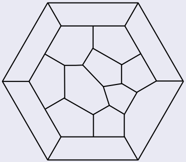

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Proof.

For each web K use a computer program to generate a list of half-foams $\{H_1, \ldots, H_n\}$ with top boundary K. Define $\mathbb{F}$-vector spaces

$$W(K) = \mathbb{F} \cdot \{H_1, \cdots, H_n\}$$
$$\overline{W}(K) = \mathbb{F} \cdot \{J^\flat(H_1), \cdots, J^\flat(H_n)\} \subseteq J^\flat(K)$$

From a computer calculation, we find that

$$\mathrm{Tait}(K) = \dim(W(K)/W(K)^\perp) \leq \dim \overline{W}(K) \leq \dim J^\flat(K) \leq \mathrm{Tait}(K).$$

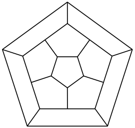
So $\dim J^\flat(K) = \mathrm{Tait}(K)$. □

Remark

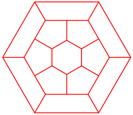
For *any* web K, same procedure yields a lower bound $\ell(K)$ on $\dim J^\flat(K)$:

$$\ell(K) := \dim(W(K)/W(K)^\perp) \leq \dim \overline{W}(K) \leq \dim J^\flat(K) \leq \mathrm{Tait}(K).$$

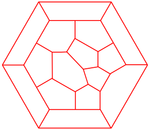
Example nonreducible webs



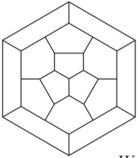
W_1



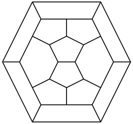
W_2



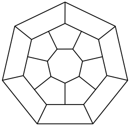
W_3



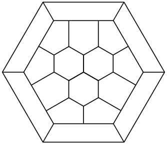
W_4



W_5



W_6



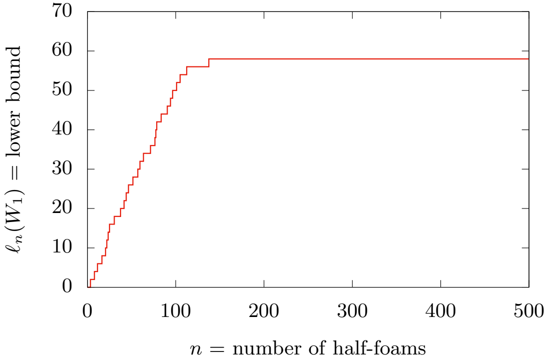
W_7

Lower bounds $\ell(K)$ on dim $J^\flat(K)$

Web K	$\ell(K)$	Tait(K)
W_1	58	60
W_2	120	120
W_3	162	162
W_4	178	180
W_5	188	192
W_6	248	252
W_7	308	312

Empirical evidence that dim $J^\flat(W_1) = 58$

- For the dodecahedral web W_1 we have $\mathrm{Tait}(W_1) = 60$, but our lower bound on dim $J^\flat(W_1)$ is only 58.
- We find 58 generators after examining 156 half-foams, but we find no additional generators after examining 6700 half-foams. This suggests that dim $J^\flat(W_1) = 58$.



Details of computer program

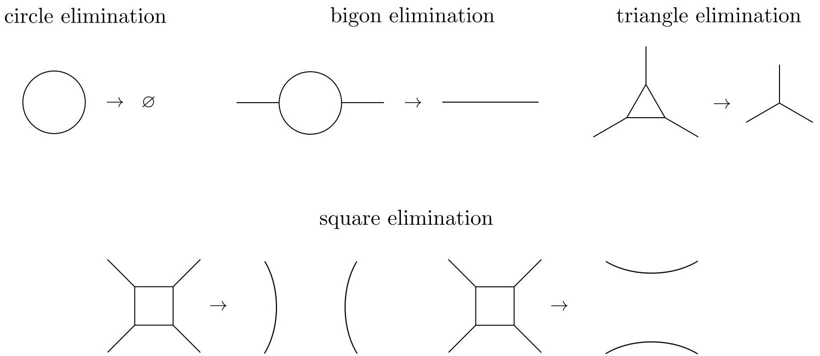
Two issues:

- How do we generate the list of half-foams?
- How do we calculate $J^\flat(F) \in \mathbb{F}$ for closed foams F?

Recursive algorithm for K reducible

If K is reducible, then there is a recursive algorithm for producing a generating set of half-foams for $J^\flat(K)$.

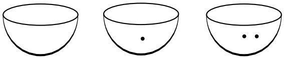
Recall that a web is *reducible* if it can be reduced to the empty web by recursively applying the following local replacements:



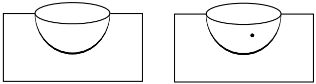
Recursive algorithm for K reducible

Define elementary cobordisms corresponding to each replacement:

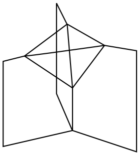
circle elimination



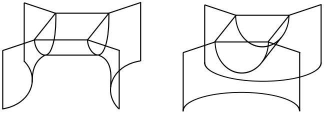
bigon elimination



triangle elimination

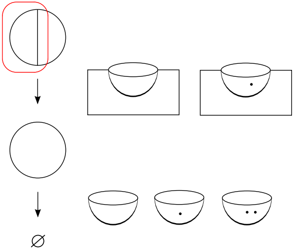


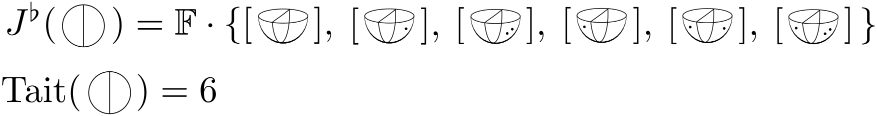
square elimination



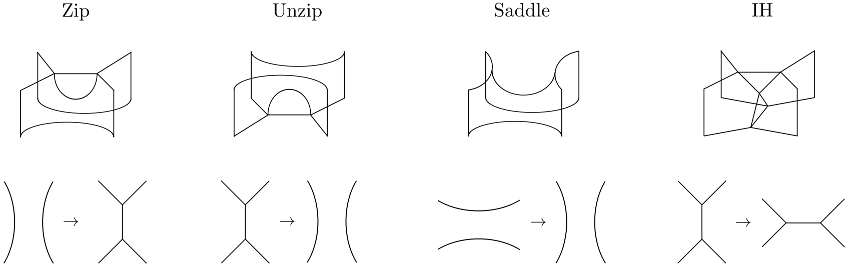
Recursive algorithm for K reducible

Example: theta web



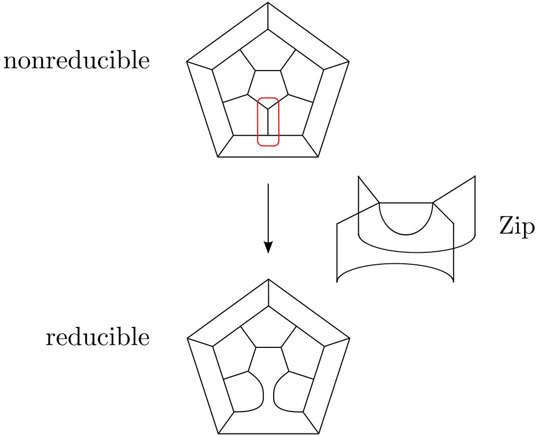


Define four elementary cobordisms:



Algorithm for K nonreducible

Example: dodecahedral web W_1



More empirical evidence that $\dim J^\flat(W_1) = 58$

For the dodecahedral web W_1 we have:

Zip cobordisms	3960 half-foams	$\dim J^\flat(W_1) \geq 58$
Unzip cobordisms	1080 half-foams	$\dim J^\flat(W_1) \geq 58$
Saddle cobordisms	3960 half-foams	$\dim J^\flat(W_1) \geq 58$
IH cobordisms	2160 half-foams	$\dim J^\flat(W_1) \geq 58$
All four types	11,160 half-foams	$\dim J^\flat(W_1) \geq 58$

Can also compute lower bounds on $\dim J^\flat(K)$ for *reducible* webs K, for which we know $\dim J^\flat(K) = \mathrm{Tait}(K)$.

For all the reducible webs I have checked, this lower bound is $\mathrm{Tait}(K)$.

Closed foam evaluation formula for $J^\flat(F) \in \mathbb{F}$

Define rings $R \subset R' \subset R''$:

$$
\begin{aligned}
R &= \mathbb{F}[E_1, E_2, E_3] \\
R' &= \mathbb{F}[X_1, X_2, X_3] \\
R'' &= R'[(X_1 + X_2)^{-1}, (X_1 + X_3)^{-1}, (X_2 + X_3)^{-1}]
\end{aligned}
$$

$$E_1 = X_1 + X_2 + X_3, \quad E_2 = X_1X_2 + X_1X_3 + X_2X_3, \quad E_3 = X_1X_2X_3$$

Closed foam F with a facet 3-coloring $c : \text{facets}(F) \to \{1, 2, 3\}$:

$$
\begin{aligned}
P(F, c) &= \prod_{f \in \text{facets}(F)} X_{c(f)}^{d(f)} \in R' \\
Q(F, c) &= \prod_{1 \leq i < j \leq 3} (X_i + X_j)^{\chi(F_{ij}(c))/2} \in R''
\end{aligned}
$$

$$
\begin{aligned}
d(f) &= \text{number of dots on facet } f \\
F_{ij}(c) &= \text{closure of union of all facets colored either } i \text{ or } j \text{ by } c
\end{aligned}
$$

Closed foam evaluation formula

Khovanov-Robert closed foam evaluation formula:

$$\langle F \rangle = \sum_{c \in \mathrm{adm}(F)} \frac{P(F,c)}{Q(F,c)} \in R = \mathbb{F}[E_1, E_2, E_3]$$

$$\mathrm{adm}(F) = \{\text{3-colorings of facets of } F\}$$

Given ring homomorphism $\psi : R \to S$, define $\langle F \rangle_\psi := \psi(\langle F \rangle) \in S$

Homomorphism $R \to \mathbb{F}$, $E_1, E_2, E_3 \mapsto 0$:

$$J^\flat(F) = \langle F \rangle|_{E_1 = E_2 = E_3 = 0} \in \mathbb{F}$$

Homomorphism $\phi : R \to \mathbb{F}[E]$, $E_1, E_2 \mapsto 0$, $E_3 \mapsto E$:

$$\langle F \rangle_\phi = \langle F \rangle|_{E_1 = E_2 = 0, E_3 = E} \in \mathbb{F}[E]$$

Foam degree

Define the *degree* of a foam F:

$$\deg(F) = 2d(F) - 2\chi(F) - \chi(s(F)) \in \mathbb{Z}$$

Here

$$d(F) = \text{total number of dots decorating } F$$
$$s(F) = \text{union of seam points and tetrahedral points of } F$$

View $R = \mathbb{F}[E_1, E_2, E_3]$ as a graded ring:

$$\deg E_1 = 2, \qquad \deg E_2 = 4, \qquad \deg E_3 = 6$$

Degree of foam F is related to degree of polynomial $\langle F \rangle \in \mathbb{F}[E_1, E_2, E_3]$:

Theorem (Khovanov and Robert)

If F is a closed foam with $\langle F \rangle \neq 0$ then $\deg\langle F \rangle = \deg F$.

Quantum gradings

The $\mathbb{F}$-vector spaces $J^\flat(K)$ are $\mathbb{Z}$-graded:

- Define the grading of the vector $J^\flat(H) \in J^\flat(K)$ to be $\deg(H)$
- Define $\operatorname{qdim} J^\flat(K) = \sum_k q^k \dim J^\flat(K)_k$

Lower bounds $\ell_q(K)$ on $\operatorname{qdim} J^\flat(K)$:

K	$\ell(K)$	Tait(K)	$\ell_q(K)$
W_1	58	60	$9q^{-3} + 20q^{-1} + 20q + 9q^3$
W_2	120	120	$3q^{-5} + 2q^{-4} + 16q^{-3} + 6q^{-2} + 29q^{-1} + 8 + 29q + 6q^2 + 16q^3 + 2q^4 + 3q^5$
W_3	162	162	$2q^{-5} + 7q^{-4} + 13q^{-3} + 21q^{-2} + 24q^{-1} + 28 + 24q + 21q^2 + 13q^3 + 7q^4 + 2q^5$
W_4	178	180	$q^{-6} + 11q^{-4} + 10q^{-3} + 29q^{-2} + 19q^{-1} + 38 + 19q + 29q^2 + 10q^3 + 11q^4 + q^6$
W_5	188	192	$4q^{-5} + 31q^{-3} + 59q^{-1} + 59q + 31q^3 + 4q^5$
W_6	248	252	$20q^{-4} + 62q^{-2} + 84 + 62q^2 + 20q^4$
W_7	308	312	$4q^{-5} + 5q^{-4} + 41q^{-3} + 15q^{-2} + 79q^{-1} + 20 + 79q + 15q^2 + 41q^3 + 5q^4 + 4q^5$

Evaluation in other rings

For a closed foam F, the evaluation formula gives $\langle F \rangle \in \mathbb{F}[E_1, E_2, E_3]$

$$J^\flat(F) = \langle F \rangle|_{E_1=E_2=E_3=0} \in \mathbb{F}$$
$$\langle F \rangle_\phi = \langle F \rangle|_{E_1=E_2=0, E_3=E} \in \mathbb{F}[E]$$

The universal construction extends these to functors

$$J^\flat : \text{Foams} \to \text{Vect}_{\mathbb{F}}$$
$$\langle - \rangle_\phi : \text{Foams} \to \mathbb{F}[E]-\text{Mod}$$

For any web K, one can show

$$\dim J^\flat(K) \leq \text{Tait}(K)$$
$$\text{rank}\, \langle K \rangle_\phi = \text{Tait}(K) \qquad (\langle K \rangle_\phi \text{ is a free } \mathbb{F}[E]\text{-module})$$

Relationship between $\langle K \rangle_\phi$ and $J^\flat(K)$

Using a Smith decomposition of $(-,-)$, one can show

$$\langle K \rangle_\phi = \mathbb{F}[E] \cdot \{g_1, \cdots, g_r\} = \mathbb{F}[E] \cdot \{\tilde{g}_1, \cdots, \tilde{g}_r\}$$

where $r = \mathrm{Tait}(K)$ and g_i and $\tilde{g}_j$ are homogeneous generators such that

$$(g_i, \tilde{g}_j) = E^{m_i} \delta_{ij}.$$

Then

$$J^\flat(K) = \mathbb{F} \cdot \{g_i \mid m_i = 0\}$$

$$\mathrm{qrank}\, \langle K \rangle_\phi = \sum_{i=1}^{r} q^{\deg(g_i)}, \qquad \mathrm{qdim}\, J^\flat(K) = \sum_{i=1 \mid m_i = 0}^{r} q^{\deg(g_i)}$$

$$\dim J^\flat(K) \leq \mathrm{rank}\, \langle K \rangle_\phi = \mathrm{Tait}(K), \qquad \mathrm{qdim}\, J^\flat(K) \leq \mathrm{qrank}\, \langle K \rangle_\phi$$

Computer constraints on qrank $\langle K \rangle_\phi$

For each example web K, the computer program generates a set of half-foams that span a free $\mathbb{F}[E]$-submodule $N(K) \subseteq \langle K \rangle_\phi$ such that

$$\operatorname{rank} N(K) = \operatorname{rank} \langle K \rangle_\phi = \operatorname{Tait}(K)$$

This does not necessarily mean $N(K) = \langle K \rangle_\phi$: homogeneous generators of $N(K)$ may be shifted upward in grading by multiples of $\deg(E) = 6$ relative to corresponding generators of $\langle K \rangle_\phi$

Example: $K = \varnothing$, $\operatorname{Tait}(K) = 1$

$$\langle K \rangle_\phi = \mathbb{F}[E], \qquad \operatorname{qrank} \langle K \rangle_\phi = q^0 = 1 \qquad (\text{since } \deg(1) = 0)$$

$$N(K) = \mathbb{F}[E] \cdot E, \qquad \operatorname{qrank} N(K) = q^6 \qquad (\text{since } \deg(E) = 6)$$

Computer constraints on qrank $\langle K \rangle_\phi$

The computer program yields quantum numbers $\ell_q(K)$ and $r_q(K)$ that strongly constrain the possibilities for qdim $J^\flat(K)$ and qrank $\langle K \rangle_\phi$:

- $\ell_q(K)$ is the lower bound on qdim $J^\flat(K)$
- $r_q(K) :=$ qrank $N(K)$, where $N(K) \subseteq \langle K \rangle_\phi$ has full rank

We always have $\ell_q(K) \leq r_q(K)$.

Example: dodecahedral web W_1

The computer calculations give

$$\ell_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 9q^3 = \text{lower bound on qdim}\, J^\flat(W_1)$$
$$r_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 11q^3 = \text{qrank}\, N(W_1)$$

Only two possibilities:

- $N(W_1) = \langle W_1 \rangle_\phi$, which implies

$$\dim J^\flat(W_1) = 58$$
$$\text{qdim}\, J^\flat(W_1) = \ell_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 9q^3$$
$$\text{qrank}\, \langle W_1 \rangle_\phi = r_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 11q^3$$

- $N(W_1) \subsetneq \langle W_1 \rangle_\phi$, which implies

$$\dim J^\flat(W_1) = 60$$
$$\text{qdim}\, J^\flat(W_1) = 10q^{-3} + 20q^{-1} + 20q + 10q^3$$
$$\text{qrank}\, \langle W_1 \rangle_\phi = 10q^{-3} + 20q^{-1} + 20q + 10q^3$$

Computer constraints on qrank $\langle K\rangle_\phi$

K	$\ell(K)$	Tait(K)	$\ell_q(K) + (r_q(K) - \ell_q(K))$
W_1	58	60	$9q^{-3} + 20q^{-1} + 20q + 9q^3 + (2q^3)$
W_2	120	120	$3q^{-5} + 2q^{-4} + 16q^{-3} + 6q^{-2} + 29q^{-1} + 8 + 29q + 6q^2 + 16q^3 + 2q^4 + 3q^5$
W_3	162	162	$2q^{-5} + 7q^{-4} + 13q^{-3} + 21q^{-2} + 24q^{-1} + 28 + 24q + 21q^2 + 13q^3 + 7q^4 + 2q^5$
W_4	178	180	$q^{-6} + 11q^{-4} + 10q^{-3} + 29q^{-2} + 19q^{-1} + 38 + 19q + 29q^2 + 10q^3 + 11q^4 + q^6 + (q + q^5)$
W_5	188	192	$4q^{-5} + 31q^{-3} + 59q^{-1} + 59q + 31q^3 + 4q^5 + (q + 2q^3 + q^5)$
W_6	248	252	$20q^{-4} + 62q^{-2} + 84 + 62q^2 + 20q^4 + (2q^2 + 2q^4)$
W_7	308	312	$4q^{-5} + 5q^{-4} + 41q^{-3} + 15q^{-2} + 79q^{-1} + 20 + 79q + 15q^2 + 41q^3 + 5q^4 + 4q^5 + (q + 2q^3 + q^5)$

$$\ell_q(K) = \text{lower bound on qdim } J^\flat(K)$$
$$r_q(K) = \text{qrank } N(K)$$

$\ell_q(K) \leq r_q(K)$, and $r_q(K) - \ell_q(K)$ gives degrees of missing generators

Question 1

Observation

qdim $J^\flat(K)$ *is symmetric under* $q \to 1/q$ *for all webs* K

Question

Is $\text{qrank}\,\langle K\rangle_\phi$ *is symmetric under* $q \to 1/q$ *for all webs* K*?*

(Yes for all reducible webs K)

Theorem (B.)

If $\text{qrank}\,\langle K\rangle_\phi$ *is symmetric under* $q \to 1/q$ *then*

$$\dim J^\flat(K) = \text{Tait}(K), \qquad \text{qdim}\, J^\flat(K) = \text{qrank}\,\langle K\rangle_\phi.$$

Question 2

Observation

rank $\langle K \rangle_\phi = \text{Tait}(K)$ *is divisible by* $3! = 6$ *for all webs* K*, except the circle web and the empty web (since we can always permute a given coloring)*

quantum numbers:

$$[n] = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)} \qquad [n]! = [n][n-1]\cdots[1]$$

Question

Is qrank $\langle K \rangle_\phi$ *divisible by* $[3]! = (q^2 + 1 + q^{-2})(q + q^{-1})$ *for all webs* K*?*

(Yes for all reducible webs K)

If yes, then $\dim J^\flat(W_1) = 60$ for dodecahedral web W_1

Question 3

Observation

Based on computer calculations of $r_q(K) = \operatorname{qrank} N(K)$*, we find that* $\operatorname{qrank} \langle K \rangle_\phi$ *contains*

- *only odd powers of* q *for webs* W_1 *and* W_5
- *only even powers of* q *for web* W_6
- *both even and odd powers of* q *for webs* W_2, W_3, W_4, *and* W_7

Question

Under what conditions does $\operatorname{qrank} \langle K \rangle_\phi$ *contain only even, or only odd, powers of* q*?*

Question 4

Observation

The $\mathbb{F}[E]$*-module* $\langle K \rangle_\phi$ *has rank* $\mathrm{Tait}(K)$ *and has a natural quantum rank.*

Question

Can one one view $\mathrm{qrank}\,\langle K \rangle_\phi$ *as resulting from a natural assignment of quantum gradings to Tait colorings of* K*?*

Concluding remarks

Summary:

- Can compute lower bounds for $\dim J^\flat(K)$ and $\operatorname{qdim} J^\flat(K)$ for nonreducible webs K, in some cases these give exact results
- For dodecehderal web W_1, which has $\mathrm{Tait}(W_1) = 60$, either $\dim J^\flat(W_1) = 58$ or $\dim J^\flat(W_1) = 60$
- Results suggest that $\dim J^\flat(W_1) = 58$; if so, then the combinatorial functor $J^\flat$ is not the same as the gauge-theoretic functor $J^\sharp$
- Can also define a functor $\langle - \rangle_\phi : \mathrm{Foams} \to \mathbb{F}[E]-\mathrm{Mod}$ and constrain possibilities for $\operatorname{qrank} \langle K \rangle_\phi$ and $\operatorname{qdim} J^\flat(K)$

Questions:

- Can quantum grading information be used to find an example nonreducible web K for which we can prove $\dim J^\flat(K) < \mathrm{Tait}(K)$?
- Is there a quantum grading on $J^\sharp(K)$?