Iterated wreath products and foams

Mee Seong Im
United States Naval Academy
Annapolis, MD

Foam Evaluation
ICERM, Brown University
Providence, RI

Joint with
Mikhail Khovanov
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A new application of foams!

We will focus on foams in the representation theory of wreath products.

Induction and restriction

Ground field $\mathbf{k}$.

Inclusion of finite groups $H \subset G$ (or assume $[G : H] < \infty$).

Induction and restriction functors Ind_H^G and Res_G^H between categories of $\mathbf{k}H$-modules and $\mathbf{k}G$-modules are biadjoint.

Natural transformations between compositions of these functors: depicted by planar diagrams of arcs and circles in the plane, regions labelled by G and H:

Figure: Oriented cups and caps represent natural transformations, for induction and restriction between H- and G-modules.

Biadjointness

Biadjointness are encoded by four natural transformations: depicted by the four oriented cup and cap diagrams.

Biadjointness is equivalent to the isotopy invariance of diagrams or arcs and circles built from these diagrams, and the four generating isotopy relations:

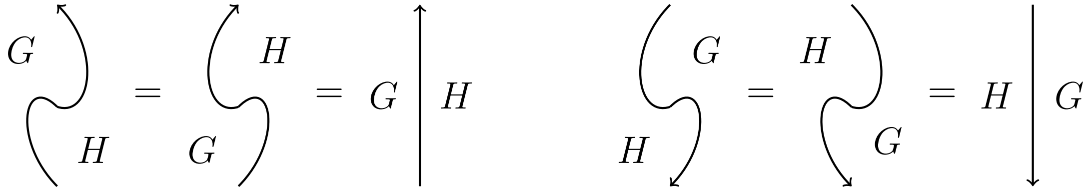


Figure: Biadjointness isotopy relations on compositions of cups and caps.

Natural transformation

The induction functor $F : \mathbf{k}H\text{−mod} \longrightarrow \mathbf{k}G\text{−mod}$ is depicted by a dot on a horizontal line, with intervals to the right and left of the dot labelled by H and G.

The identity natural transformation id_F of F is depicted by a vertical line in the plane.

A natural transformation $\alpha : F \Rightarrow F'$ is depicted by a dot on a vertical line, with intervals below and above the dot labelled by F and F'.

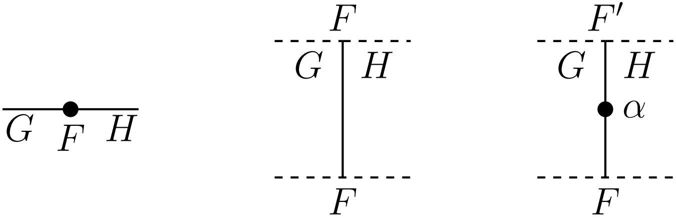


Figure: Functor, identity natural transformation, and natural transformation α.

Extension to foams

Let $H \cong H_1 \times H_2$.

Endofunctors and natural transformations between them in the category of H-modules: reduced to exterior tensor products of those in categories of H_1-modules and H_2-modules.

Diagrammatically, the H-plane that carries information about natural transformations of endofunctors in the category of $\mathbf{k}H$-modules is converted into two parallel planes, one for each term H_1, H_2 in the direct product.

Extension to foams

A natural transformation $\alpha_i : F_i \longrightarrow F_i'$ between endofunctors F_i, F_i' in the category of H_i-modules: depicted by a dot on a vertical line in the H_i-plane for $i = 1, 2$. Bottom and top endpoints of the vertical line denote functors F_i and F_i', respectively.

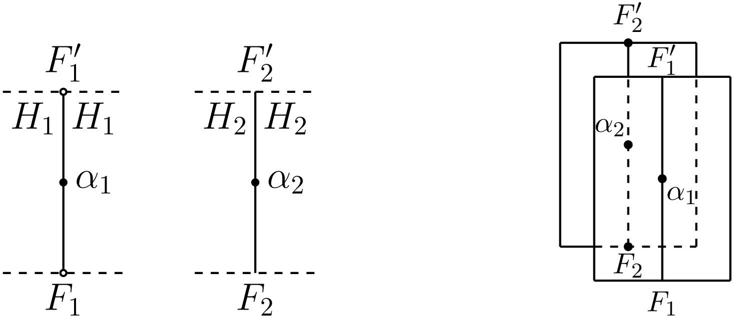


Figure: Diagrams of α_1, α_2, and $\alpha_1 \boxtimes \alpha_2 \ : \ F_1 \boxtimes F_2 \Rightarrow F_1' \boxtimes F_2'$.

Extension to foams

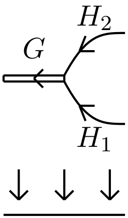
Given inclusion of groups $H_1 \times H_2 \subset G$, denote $\mathrm{Ind}_{H_1 \times H_2}^{G}$ from $H_1 \times H_2$-modules to G-modules by a vertex with H_1, H_2 lines flowing in and G line flowing out.

The restriction functor: depicted by having a G-line split into H_1 and H_2 lines.

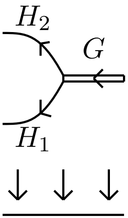
$\Rightarrow$ build diagrams for compositions of these functors.

Graphs come with projections onto $\mathbb{R}$ to keep track of the order of composition of functors.

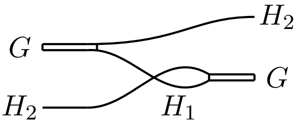
Ind



Res



A composition of one restriction,
one permutation, & one induction functor,
going from $G \times H_2$-mod to $H_2 \times G$-mod.



Foams

Natural transformations between these compositions: depicted by foams that extend between such diagrams.

Identity natural transformation from the induction functor to itself (respectively, from the restriction functor to itself) is depicted by the direct product foam, the graph depicting this functor times the unit interval $[0,1]$:

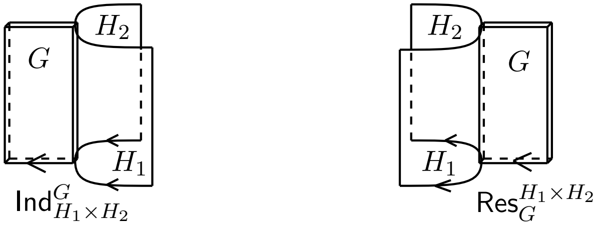


Figure: Identity natural transformations on functors $\mathrm{Ind}_{H_1 \times H_2}^{G}$ and $\mathrm{Res}_{G}^{H_1 \times H_2}$.

Foams

Seam lines: singular lines in the foams.

A natural transformation $a : \mathrm{Ind} \Rightarrow \mathrm{Ind}$ is denoted by a dot on a seam line, labelled a, and likewise for an endomorphism of the restriction functor.

A central element $c \in Z(\mathbf{k}G)$ of the group algebra $\mathbf{k}G$ is denoted by a dot floating in a facet G labelled c.

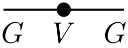
Figure: Natural transformation $a \in \mathrm{End}(\mathrm{Ind})$ and central element c of $\mathbf{k}G$.

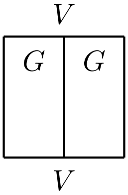
Foams

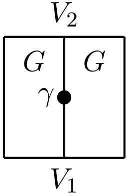
The functor $V \otimes -$ of the tensor product with a representation V of G is denoted by a dot on a line, with label V and the regions to the both sides of the dot labelled G.

Identity natural transformation $V \otimes - \Rightarrow V \otimes -$ is depicted by a vertical line (*defect or seam line*) labelled V.

A homomorphism $\gamma : V_1 \longrightarrow V_2$ of G-modules induces a natural transformation $\gamma : V_1 \otimes - \Rightarrow V_2 \otimes -$ of the functors $V_1 \otimes -$ and $V_2 \otimes -$; it is depicted by a dot on a *defect line* for V, with the defect line label changing from V_1 to V_2.







Iterated wreath product of S_2

G_n: the n-th iterated wreath product of the symmetric group S_2, the group of symmetries of the full binary tree T_n of depth n.

This binary tree has a root, 2^n leaf vertices (children), and all paths from the root to leaf vertices have length n.

Tree T_n has $2^{n+1} - 1$ vertices. Leaf vertices are labelled from 1 to 2^n inductively on n so that the vertices of the left branch are labelled by 1 through 2^{n-1} and those of the right branch are labelled by $2^{n-1} + 1$ through 2^n.

For small values of n, G_n satisfies the following:

- $G_0 = \{1\}$ is the trivial group,
- $G_1 = S_2$ is the symmetric group of order two,
- $G_2 = S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2$ has order 8 and is isomorphic to the dihedral group D_4.

For any n, G_n has order 2^{2^n-1}.

Wreath product

G_n has an index two subgroup naturally isomorphic to $G_{n-1} \times G_{n-1}$:

$$G^{(1)}_{n-1} \; := \; G_{n-1} \times G_{n-1} \xhookrightarrow{\iota_{n-1}} G_n. \tag{1}$$

The embedding consists of symmetries that fix the two branches of the tree, one to the left and the other to the right, of the root. There is a coset decomposition

$$G_n = G^{(1)}_{n-1} \sqcup G^{(1)}_{n-1}\,\beta_n = G^{(1)}_{n-1} \sqcup \beta_n\, G^{(1)}_{n-1}, \tag{2}$$

where β_n is the involution that transposes the left and right branches of T_n, with the coincidence of left and right cosets

$(G_{n-1} \times G_{n-1})\,\beta_n = \beta_n\,(G_{n-1} \times G_{n-1})$ (the left and right cosets are also double cosets).

For $g_1, g_2 \in G_{n-1}$, $(g_1, g_2)\,\beta_n = \beta_n\,(g_2, g_1)$. Moving through β_n switches the order of the two terms in the product $G_{n-1} \times G_{n-1}$.

Wreath product (G_n for $n = 4$)

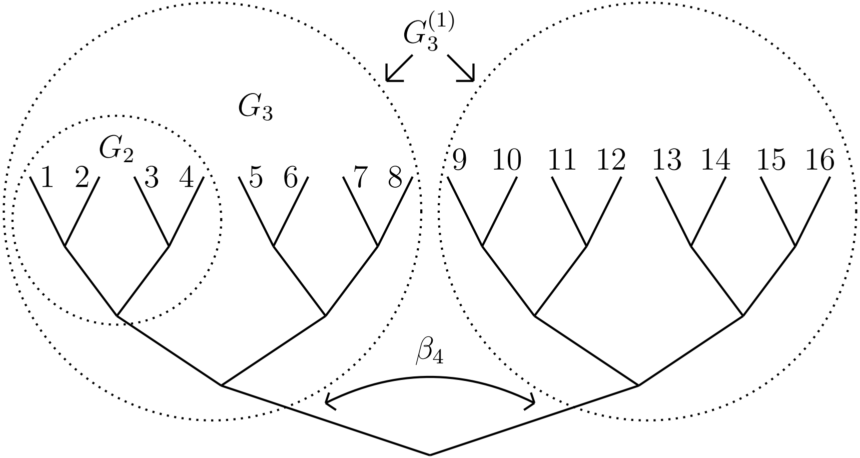


Figure: Tree T_4.

Wreath product

Identify G_n with a subgroup of the symmetric group S_{2^n} using induction on n.

When $n = 0$, both G_0 and $S_{2^0} = S_1$ are the trivial group.

For the induction step, given an inclusion $j_{n-1} : G_{n-1} \hookrightarrow S_{2^{n-1}}$, realize $G_n \subset S_{2^n}$ as the subgroup generated by:

- permutations of $\{1, \ldots, 2^{n-1}\}$ in G_{n-1},
- permutations of $\{2^{n-1} + 1, \ldots, 2^n\}$ in G_{n-1} (obtained by shifting all indices by 2^{n-1}),
- permutation $\beta_n = (1, 2^{n-1} + 1)(2, 2^{n-1} + 2) \cdots (2^{n-1}, 2^n)$.

Identify $\beta_n \in G_n$ with its image in S_{2^n}. The subgroup $G^{(1)}_{n-1}$ is given by products of permutations of the first and the second type on the above list; it's a normal subgroup of index 2, with $\{1, \beta_n\}$ a set of coset representatives.

Center of wreath product

The center of G_n is an order two subgroup,

$$Z(G_n) = \{1, c_n\}, \qquad c_n := (1,2)(3,4)\cdots(2^n - 1, 2^n), \tag{3}$$

see [I.–Oğuz, Lemma 3.6]. c_n moves nearby elements.

Define $G_{n-k}^{(k)} := (G_{n-k})^{2^k} \subset G_n$ given by permutations that fix all nodes of the full binary tree at distance at most $k-1$ from the root. There is a chain of inclusions

$$\{1\} = G_0^{(n)} \subset G_1^{(n-1)} \subset \ldots \subset G_{n-2}^{(2)} \subset G_{n-1}^{(1)} \subset G_n^{(0)} = G_n. \tag{4}$$

Each inclusion

$$G_{n-k-1}^{(k+1)} \subset G_{n-k}^{(k)} \tag{5}$$

is of an index 2^{2^k} normal subgroup, with the quotient isomorphic $C_2^{2^k}$.

Induction and restriction bimodules

Denote $\mathbf{k}G_n$, viewed as a bimodule over itself, by (n). Denote $\mathbf{k}G_{n-1}^{(1)} := \mathbf{k}(G_{n-1} \times G_{n-1})$ by $(n-1)^{(1)}$, and extend these notations to tensor products of bimodules:

$$(n)_{(n-1)^{(1)}}(n) \ := \ \mathbf{k}G_n \otimes_{\mathbf{k}G_{n-1}^{(1)}} \mathbf{k}G_n, \qquad \mathbf{k}G_n\text{-bimodule.}$$

R and I are biadjoint so the biadjointness maps are:

1. $\alpha_{n-1}^{n} : (n)_{(n-1)^{(1)}}(n) \longrightarrow (n)$, where $x \otimes y \mapsto xy, \ x, y \in (n) = \mathbf{k}G_n$,
2. $\gamma_{n-1}^{n} : (n-1)^{(1)} \longrightarrow {}_{(n-1)^{(1)}}(n)_n(n)_{(n-1)^{(1)}}$, where $x \mapsto x \otimes 1 = 1 \otimes x$, $x \in (n-1)^{(1)}$,
3. $\alpha_{n}^{n-1} : {}_{(n-1)^{(1)}}(n)_n(n)_{(n-1)^{(1)}} \cong {}_{(n-1)^{(1)}}(n)_{(n-1)^{(1)}} \longrightarrow (n-1)^{(1)}$ takes $g \in (n)$ to $p_{n-1}(g) \in (n-1)^{(1)}$ by $p_{n-1}(g) = \begin{cases} g & \text{if } g \in (n-1)^{(1)}, \\ 0 & \text{otherwise.} \end{cases}$
4. $\gamma_{n}^{n-1} : (n) \longrightarrow (n)_{(n-1)^{(1)}}(n)$, where $x \mapsto 1 \otimes x + \beta_n \otimes \beta_n x$, and $x \in (n)$.

The four bimodule maps diagrammatically

R = *restriction* functor, I = *induction* functor.

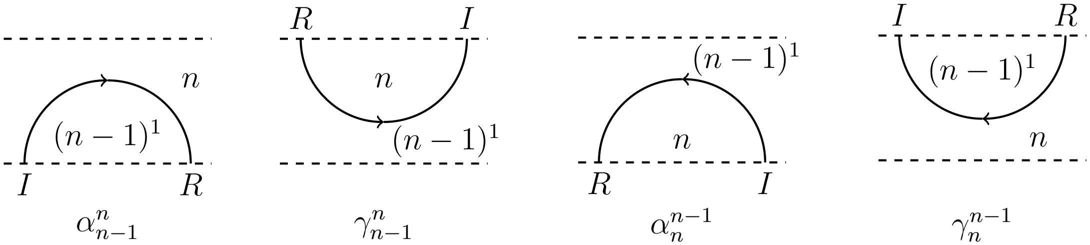


Figure: Diagrams for natural biadjointness transformations.

$\Rightarrow$ These four natural transformations turn I^n_{n-1} and R^{n-1}_n into a cyclic biadjoint pair.

Simple relations

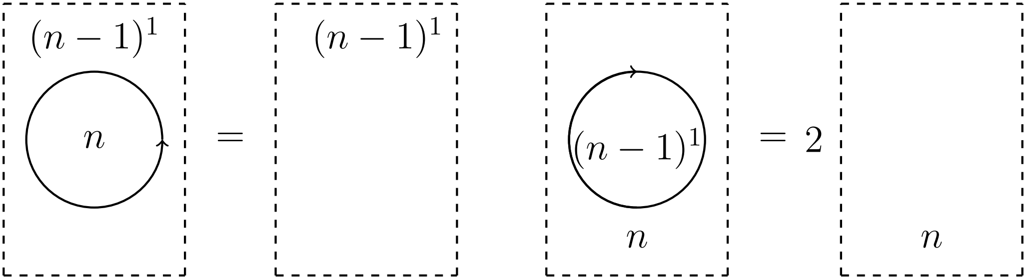


Figure: Simple relations on diagrams.

Planar diagrams to foams

Refine these planar diagrams to a foam description for these and related intertwiners between compositions of I^n_{n-1} and R^{n-1}_n. Denote the induction and restriction functors by trivalent vertices in graphs as shown below.

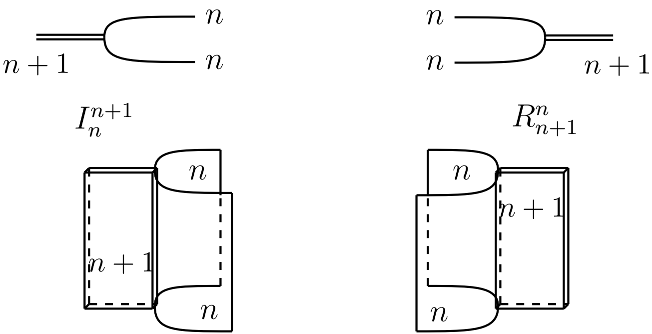


Figure: Induction and restriction functors I^{n+1}_n, R^n_{n+1} and identity natural transformation on them.

Biadjointness transformations

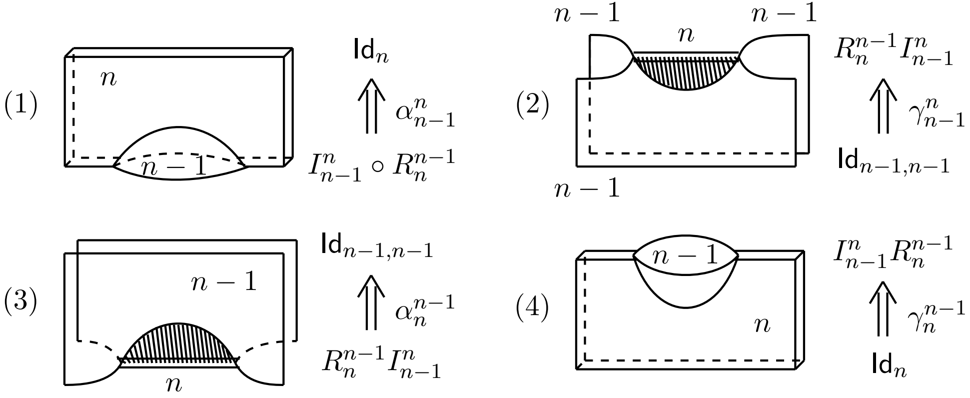


Figure: The four biadjointness transformations for I^n_{n-1}, R^{n-1}_n.

Decomposition of functors

Proposition. There is a canonical decomposition of functors

$$R_n^{n-1} \circ I_{n-1}^n \cong \mathrm{Id} \oplus T_{12}. \tag{6}$$

The composition $R_n^{n-1} \circ I_{n-1}^n$ is given by tensoring with the $G_{n-1}^{(1)}$-bimodule G_n. The proposition follows from the Mackey induction-restriction formula, properties of the wreath product and β_n.

Diagrams for the three functors in this isomorphism are:

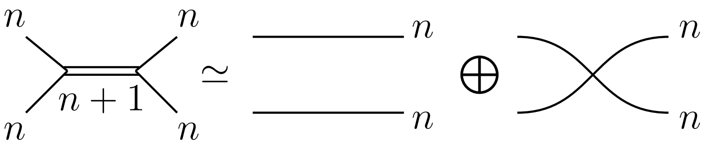


Figure: Diagrams for the three functors in (6).

Direct sum decomposition via foams

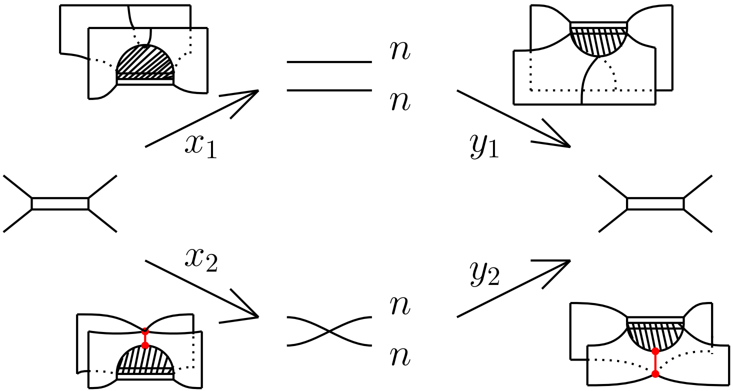


Figure: Maps (foams) describing the direct sum decomposition in (6). The red lines depict facet intersections.

Foams

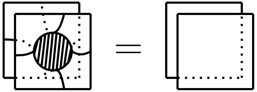
The direct sum decomposition property translates into the following relations:

$$y_1x_1 + y_2x_2 = \text{id}_{RI},$$

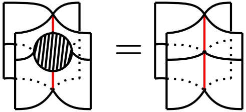
$$x_1y_1 = \text{id}, \qquad x_2y_2 = \text{id},$$

$$x_1y_2 = 0, \qquad x_2y_1 = 0.$$

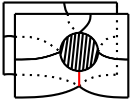
Foam equivalents of these relations are:



$x_1 y_1 = \text{id}$

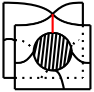


$x_2 y_2 = \text{id}$



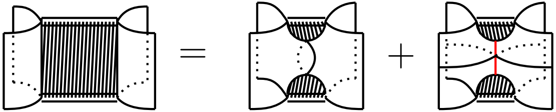
$= 0$

$x_1 y_2 = 0$



$= 0$

$x_2 y_1 = 0$



$$\text{id}_{RI} = y_1 x_1 + y_2 x_2$$

Foams (isomorphisms), conjugation by β_n

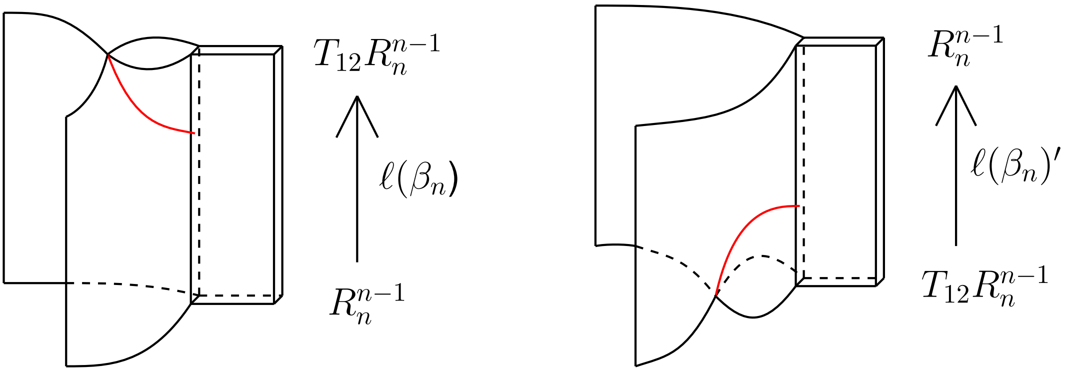


Figure: Intersection seams giving mutually-inverse functor isomorphisms $T_{12}R_n^{n-1} \cong R_n^{n-1}$.

Foams

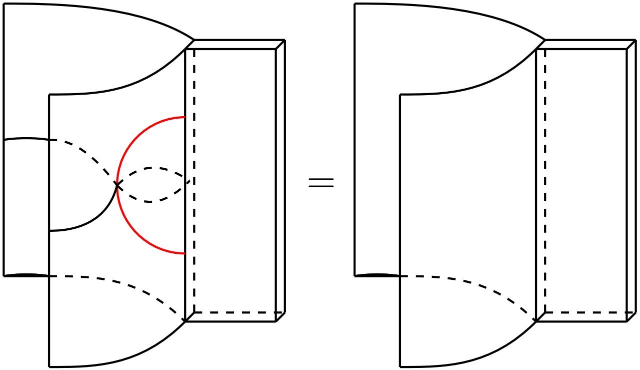


Figure: Relation $\ell(\beta_n)'\ell(\beta_n) = \text{id}$ allows to undo an immersion seam that goes out and back into an $(n, n-1)$-seam.

Foams

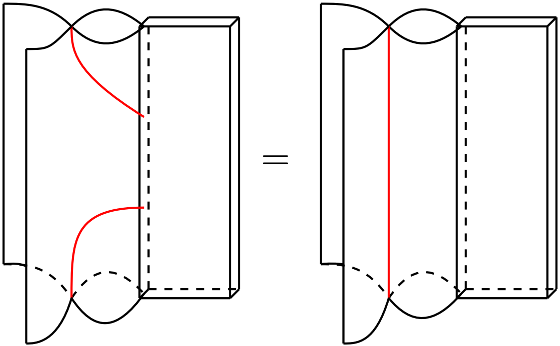


Figure: Relation $\ell(\beta_n)\ell(\beta_n)' = \text{id}$ cancels two adjacent immersion points on an $(n, n-1)$-seam.

Foams

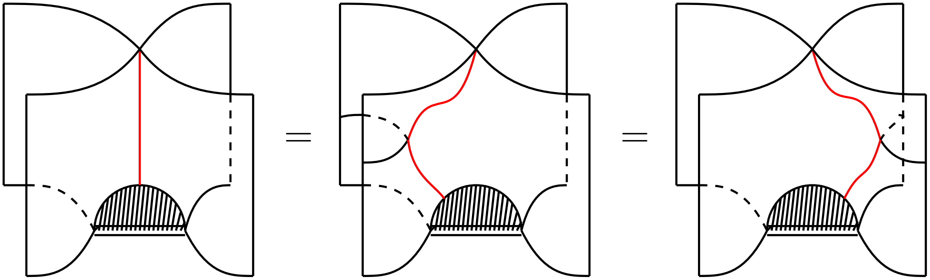


Figure: Deforming an immersion seam and moving its endpoint.

Foams

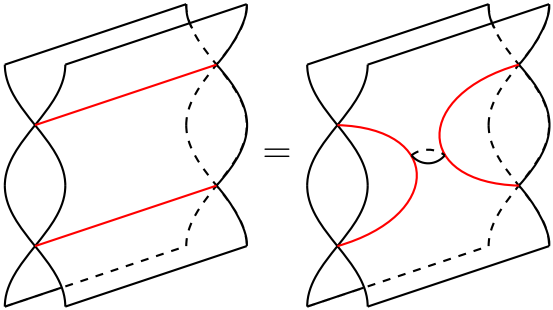


Figure: An isotopy of immersed surfaces in $\mathbb{R}^3$. Intersection lines are shown in red.

Central elements and bubbles

$Z(G_n) \cong S_2$
Central element: $c_n = (1,2)(3,4)\cdots(2^n-1, 2^n)$
Via the inclusion ι_{n-1} this element can be defined inductively as $c_n = \iota_{n-1}(c_{n-1} \times c_{n-1})$.
Denote c_n by a dot on a facet labelled n.

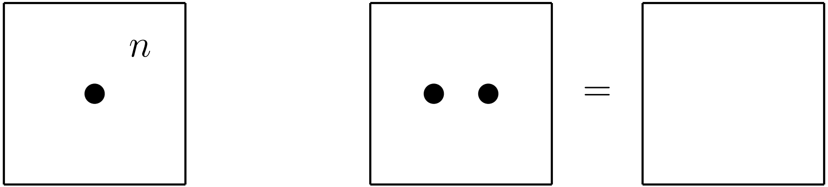


Figure: Central element c_n and a relation on it: the square of the dot is the identity.

Central elements: $c_{n-1} \times 1 + 1 \times c_{n-1}$

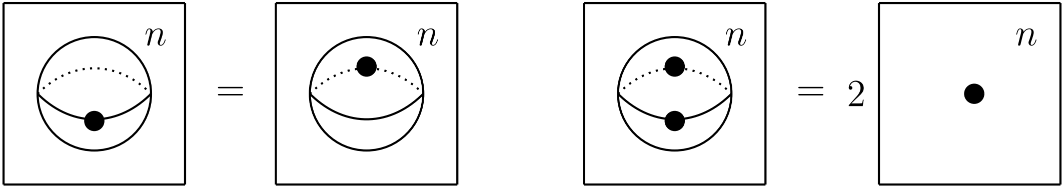


Figure: The simplest relations on c-bubbles.

Central elements

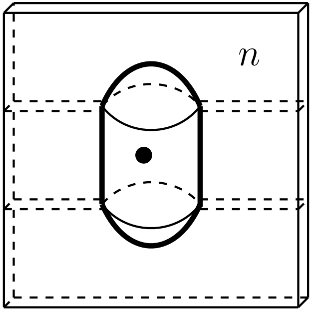


Figure: An example of the composition $\mathbf{k}G_n \to \mathbf{k}G_n \otimes_{\mathbf{k}G_{n-1}^{(1)}} \mathbf{k}G_n \to \mathbf{k}G_n$.

Iterating the bubble construction by splitting the bubble facets

The central element shown below is: $c^{(1)}_{n-2} + c^{(2)}_{n-2} + c^{(3)}_{n-2} + c^{(4)}_{n-2}$, where $c^{(i)}_{n-2}$ = the i-th copy of c_{n-2} in the direct product $G^{\times 4}_{n-2} \subset G_n$.

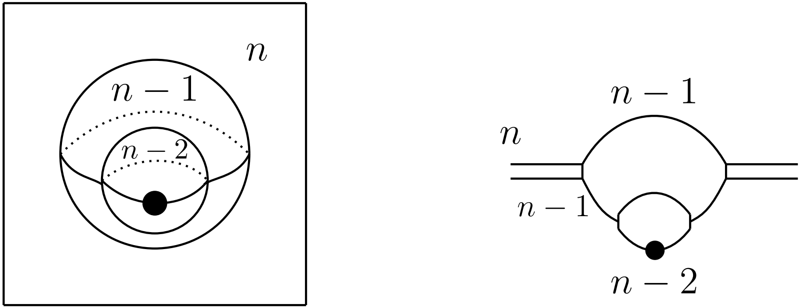


Figure: A more complicated bubble describing a central element. The middle cross-section of this bubble is shown on the right.

Tensoring with induced representations

$H \subseteq G$ is a subgroup and M is a G-module. There is an isomorphism

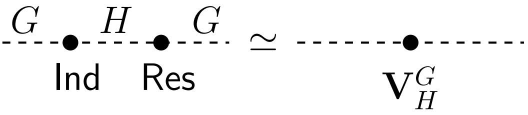
$$\mathrm{Ind}_H^G \circ \mathrm{Res}_G^H(M) \xrightarrow{\sim} \mathrm{Ind}_H^G(\underline{\mathbf{k}}) \otimes M.$$

So

$$\mathrm{Ind}_H^G \circ \mathrm{Res}_G^H \simeq \mathbf{V}_H^G \otimes - \tag{7}$$

is an isomorphism of functors, where $\mathbf{V}_H^G := \mathrm{Ind}_H^G(\underline{\mathbf{k}})$ is the induced representation of G and $\underline{\mathbf{k}}$ is the trivial representation of H.

Tensoring with induced representations



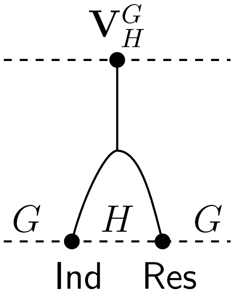


Figure: Left: diagrammatic notations for the two functors. Right: a vertex to denote their isomorphism. The inverse isomorphism is represented by a reflected diagram.

Tensoring with induced representations

$\mathrm{char}(\mathbf{k}) = 0 \Rightarrow$ representations of finite groups over $\mathbf{k}$ are completely reducible.

Given a subrepresentation $V \subseteq \mathbf{V}_H^G$, choose an idempotent endomorphism $e_V \in \mathrm{End}(\mathbf{V}_H^G)$ of projection on V. It can be described by a box labelled e_V on the vertical line depicting the identity natural transformation of the functor $\mathbf{V}_H^G \otimes -$.

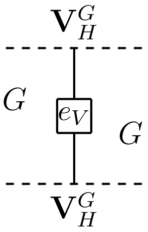


Figure: Idempotent e_V on the endomorphism of $\mathbf{V}_H^G$.

Tensoring with induced representations

The quotient group $G_n/G^{(1)}_{n-1}$ is S_2, and its two-dimensional regular representation, viewed as a representation of G_n, is denoted V_1.

The latter representation is the induced rep from the trivial rep of $G^{(1)}_{n-1}$,

$$V_1 \cong \mathrm{Ind}^{G_n}_{G^{(1)}_{n-1}}(\underline{\mathbf{k}}).$$

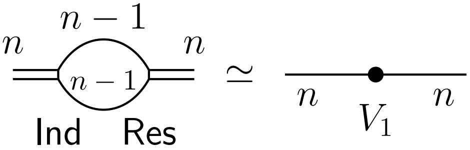


Figure: Functor isomorphism.

Tensoring with the induced representation

Under the quotient map, $\beta_n \in G_n$ is the nontrivial element of S_2, which is denoted by $\underline{\beta}_n$.

Multiplication by β_n is an involutive endomorphism of V_1.

The foam on the right: represents the corresponding endomorphism of Ind $\circ$ Res, under its isom with the tensor product functor. The foam consists of a flip between two $(n-1)$ facets. The intersection interval is shown in red.

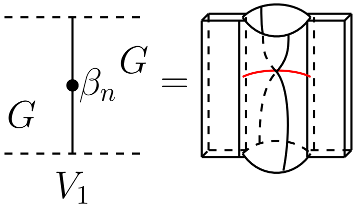


Figure: Foam rep of the endomorphism of the functor $V_1 \otimes - \cong$ Ind $\circ$ Res given by multn by β_n.

Tensoring with the induced representation

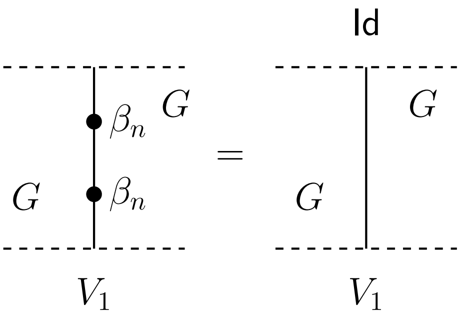


Figure: $\beta_n^2 = 1$, and endomorphism of V_1 it induces squares to identity.

Foams for idempotents

Idempotents $e_+ = \frac{1+\beta_n}{2}$ and $e_- = \frac{1-\beta_n}{2}$ in the group algebra $\mathbf{k}G_n$ give corresponding idempotents, also denoted e_+, e_-, in the quotient algebra $\mathbf{k}S_2 \cong \mathrm{End}_{G_n}(V_1)$.

These idempotents produce direct summands of representation V_1, the trivial V_+ and the sign V_- representations so that

$$V_1 \cong V_+ \oplus V_-.$$

$V_+ \cong \underline{\mathbf{k}}$, our two notations for the trivial representation.

Under functor isomorphism $V_1 \otimes \bullet \cong \mathrm{Ind} \circ \mathrm{Res}$, these idempotents become idempotents in the endomorphism algebra of the latter functor, also denoted e_+ and e_-.

In the foam notation, we represent these idempotents in $\mathrm{End}(\mathrm{Ind} \circ \mathrm{Res})$ by disks, green and blue, respectively, that intersect two opposite seam lines, with labels $+$ and $-$.

Foams for idempotents

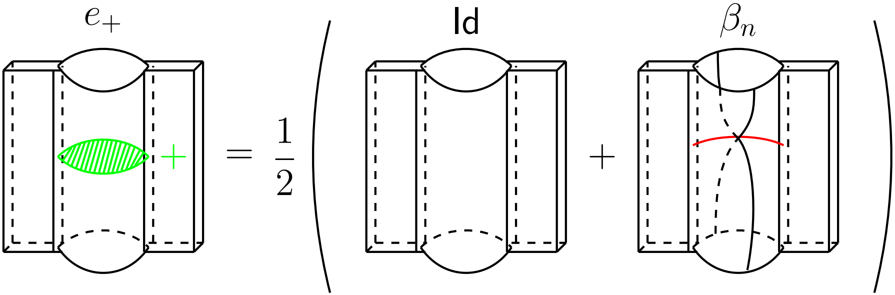


Figure: Idempotent $e_+ = \dfrac{1+\beta_n}{2}$.

Foams for idempotents

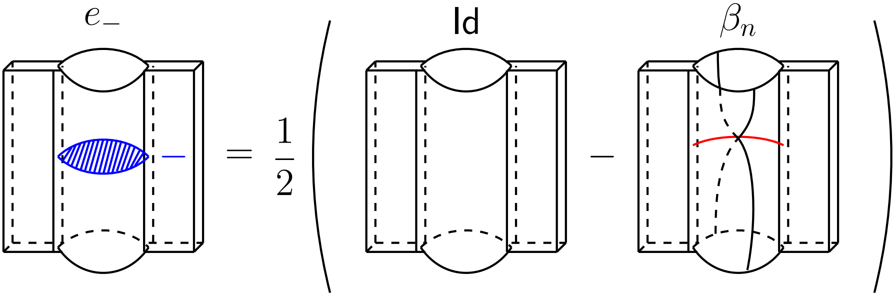


Figure: Idempotent $e_- = \dfrac{1 - \beta_n}{2}$.

Foams for idempotents

Relations:

$$1 = e_+ + e_-, \quad e_+e_- = e_-e_+ = 0, \quad e_+^2 = e_+, \quad e_-^2 = e_-$$

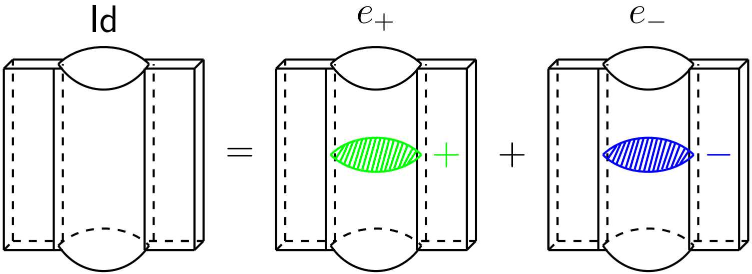


Figure: The sum of two idempotents e_+ and e_- gives the identity foam.

Foams for idempotents

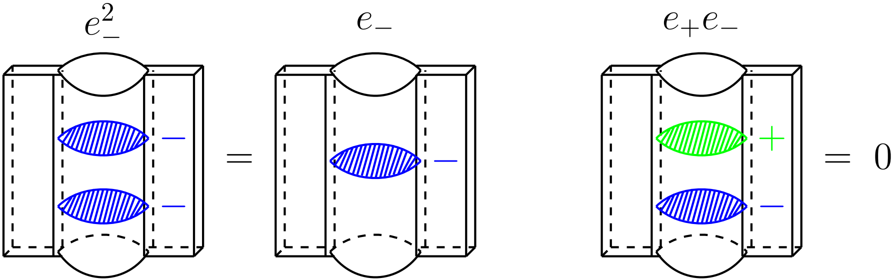


Figure: Left: idempotency relation $e_-^2 = e_-$ via foams. Right: orthogonality relation $e_+e_- = 0$ via foams.

Foams for idempotents

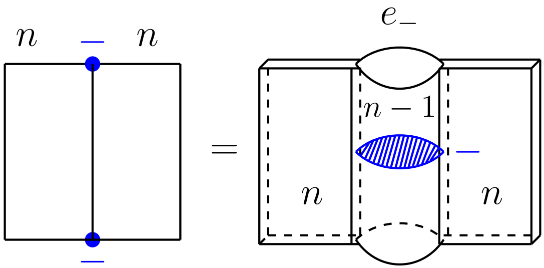


Figure: Converting from the planar to the foam presentation of the identity endomorphism of V_-.

Foams for idempotents

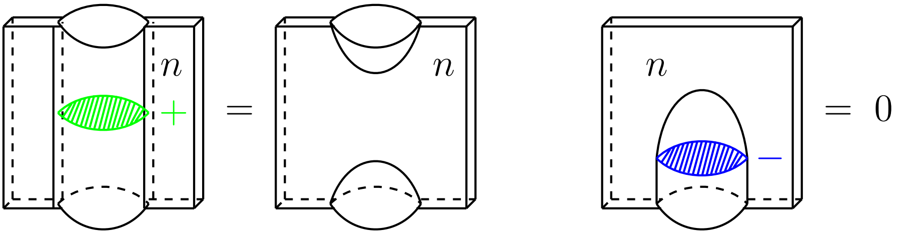
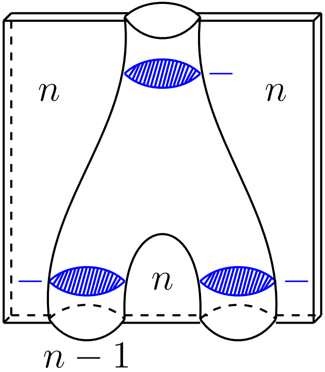


Figure: Left equality: symmetrizer e_+ is the projection onto the trivial representation; the foam interpretation is shown. In the second equality, absence of homs between the trivial and the sign representations implies this relation.

Foams for idempotents



$= \quad 0$

Figure: The only hom between irreducible representations $V_- \otimes V_- \cong V_+$ and V_- is 0. This equality can also be checked by expanding three blue disks and canceling the terms.

Foams for idempotents

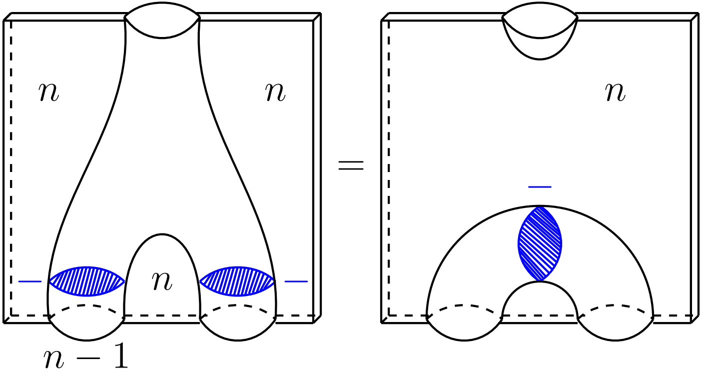


Figure: This relation follows by expanding the “neck” on top left.

Foams for idempotents

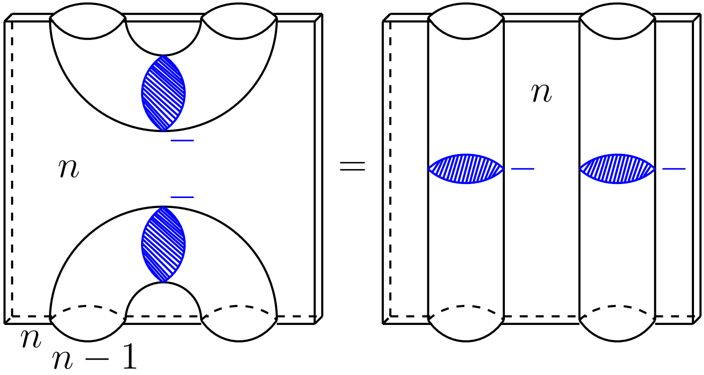


Figure: Converting into the language of tensoring with representations, functor isomorphisms between tensoring with $V_{-} \otimes V_{-}$ and V_{+} given by the two tubes at the top and bottom halves are mutually-inverse on one side.

Foams for idempotents

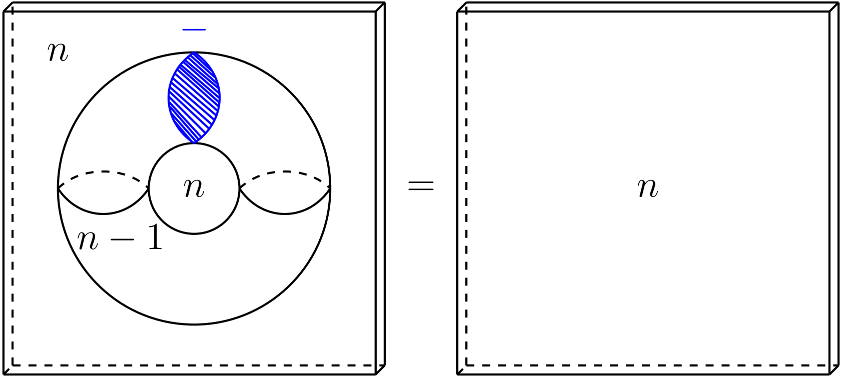


Figure: Horizontal circles indicate that the two $(n-1)$-facets on the left picture constitute a 2-torus inside the foam.

$G_2 = S_2 \wr S_2$ $(n = 2)$

Consider the two diagrams below. Each describes a summand of a composition of restriction and induction functors. In the diagram on the left, restrict from G_n to $G_{n-1} \times G_{n-1}$, then further restrict to $G_{n-2} \times G_{n-2} \times G_{n-1}$. After that, induce back to G_n. The "minus" idempotent is applied for the composition of restriction and induction between G_{n-1} and $G_{n-2} \times G_{n-2}$. In the diagram on the right, a similar functor is described, but the inner induction and restriction is for the other factor of the product $G_{n-1} \times G_{n-1}$.

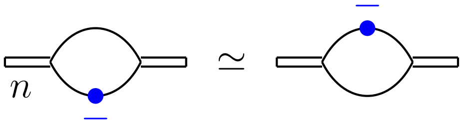


Figure: A functor isomorphism. Denote the functor on the left by $\mathcal{V}$.

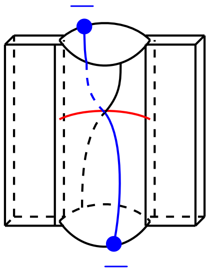


Figure: Foam for the functor isomorphism from previous slide. The blue line depicts the identity endomorphism of the “blue point” functor (direct summand of the induction-restriction functor isomorphic to the tensoring $V_{-} \otimes -$). The black line on the other thin facet is drawn to help see the facet. The two thin facets intersect along the red interval.

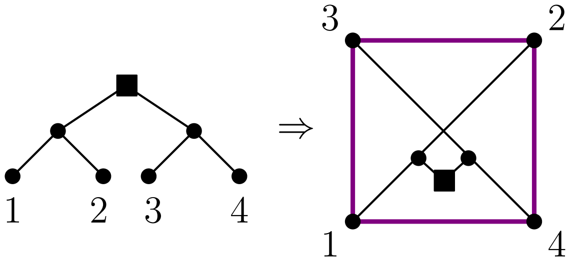


Figure: An identification of symmetries of a depth 2 full binary tree with those of a square. Nodes $1, 2, 3, 4$ of the tree are mapped to vertices of the square.

The table below lists the characters of the five irreducible representations of D_4 and of representation $\mathbf{k}[D_4/H]$.

	1	(12)	$(12)(34)$	(1324)	$(13)(24)$
V	2	0	-2	0	0
V_+	1	1	1	1	1
V_-	1	1	1	-1	-1
V_{-+}	1	-1	1	-1	1
V_{--}	1	-1	1	1	-1
$\mathbf{k}[D_4/H]$	4	2	0	0	0

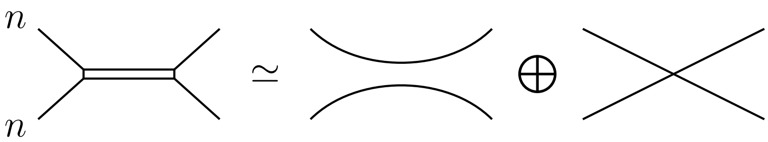


Figure: Direct sum decomposition of Res ∘ Ind into the identity and the transposition functors.

Figure: Left: tensor square of the sign rep V_- is the trivial rep, $V_-^{\otimes 2} \cong V_+$. Right: functor isomorphism $\mathrm{Ind} \circ T_{12} \simeq \mathrm{Ind}$, where T_{12} is the transposition, given by foams reflected in the vertical plane.

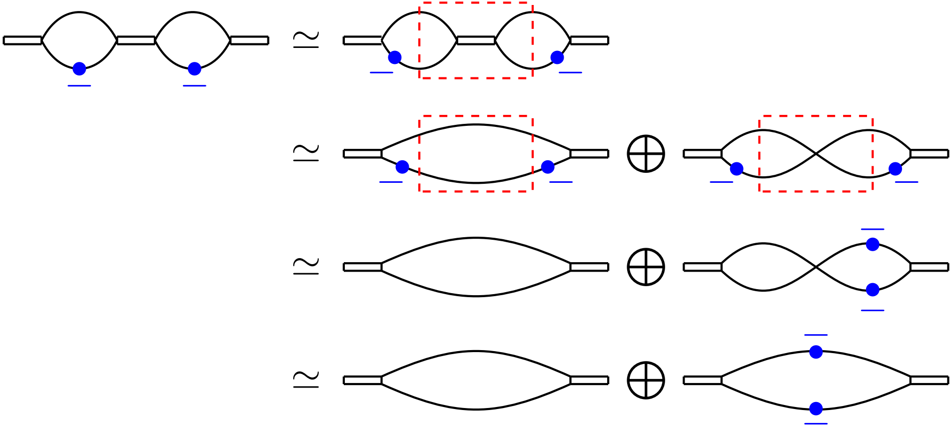


Figure: Use above relations to decompose the square of the functor $\mathcal{V}$, giving us the triv, sign, and two 1-dim'l reps.

Future directions

This is an example of foams very different from the ones we encounter in link homology.

We expect that further developing this type of foams will give a full description of representation theory of iterated wreath products G_n and functors between these representation categories.

Another problem is to find more examples of foams in mathematical nature.

Thank you. Questions?