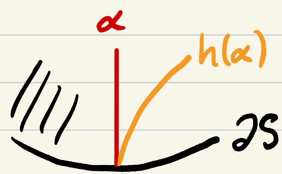
Floer homology and right-veering monodromy

with Yi Ni and Steven Sivek



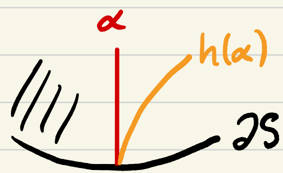
Right-veering monodromy

Let $K \subseteq Y$ be a fibered knot, with fiber S and monodromy $h : S \to S$.

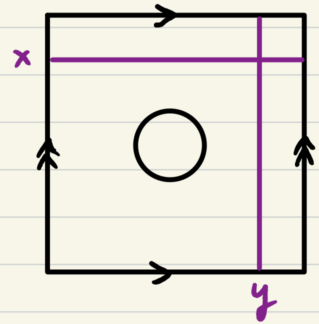
$Y \setminus n(K) \cong$ mapping torus $M_h := S \times [0,1] \big/ _{(x,1) \sim (h(x),0)}$

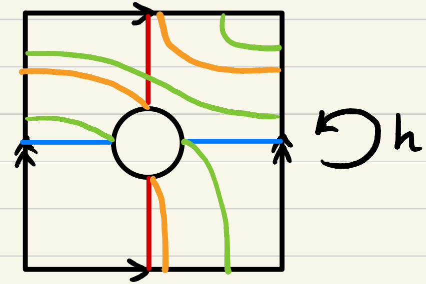
(S,h) an open book $\longleftrightarrow$ contact structure ξ on Y.

h is <u>right-veering</u> if it sends every arc $\alpha \subseteq S$ to right.



<u>Thm</u> (Honda-Kazez-Matić): (Y,ξ) tight $\iff$ every compatible fibered knot $K \subseteq Y$ is right-veering.

<u>Ex</u>: S =  = 



$h = D_x D_y$: RV, right-handed trefoil T_+

$h = D_x^{-1} D_y^{-1}$: LV, left-handed trefoil T_-

$h = D_x^{-1} D_y$: neither RV nor LV, figure-8 E

The knot Floer filtration

Suppose $K \subseteq Y$ a genus-g fibered knot. Consider mirror $K \subseteq -Y$.

Alexander filtration: $\emptyset \subseteq \boxed{\mathcal{F}_{-g} = \mathbb{F}\langle x \rangle} \subseteq \mathcal{F}_{1-g} \subseteq \ldots \subseteq \mathcal{F}_g = \widehat{CF}(-Y)$

$\widehat{HFK}(-Y, K, i) = H_*(\mathcal{F}_i / \mathcal{F}_{i-1})$. Spectral sequence $\underset{\substack{\rotatebox{90}{$\cong$} \\ E_1}}{\widehat{HFK}(-Y,K)} \Rightarrow \underset{\substack{\rotatebox{90}{$\cong$} \\ E_\infty}}{\widehat{HF}(-Y)}$

Contact invariant $c(\xi) := [x] \in \widehat{HF}(-Y)$.

Suppose $c(\xi) = 0$ (as when ξ is overtwisted).

Def: $b(K \subseteq Y) = \min\{k \mid [x] = 0 \text{ in } H_*(\mathcal{F}_k)\} + g \geq 1$.

Thm (B-Vela-Vick): If K not RV then $b(K \subseteq Y) = 1$.

Right-veering detection

$\emptyset \subseteq \boxed{\mathcal{F}_{-g} = \mathbb{F}\langle x\rangle} \subseteq \mathcal{F}_{1-g} \subseteq \ldots \subseteq \mathcal{F}_g = \widehat{CF}(-Y)$, $\widehat{HFK}(-Y, K, i) = H_*(\mathcal{F}_i / \mathcal{F}_{i-1})$.

$b(K \subseteq Y) = \min\{k \mid [x] = 0 \text{ in } H_*(\mathcal{F}_k)\} + g \geq 1$.

<u>Thm</u> (B-Vela-Vick): If K not RV then $b(K \subseteq Y) = 1$.

<u>Thm</u> (B-Ni-Sivek): If $b(K \subseteq Y) = 1$ then K not RV. (Main Theorem)

<u>Cor 1</u>: (Y, ξ) tight $\Leftrightarrow$ every compatible fibered knot $K \subseteq Y$ satisfies $b(K \subseteq Y) > 1$.

<u>Note</u>: $b(K \subseteq Y) = 1 \Leftrightarrow \exists$ nontrivial $d_1 : \widehat{HFK}(-Y, K, 1-g) \to \widehat{HFK}(-Y, K, -g)$

$\emptyset \subseteq \boxed{\mathcal{F}_{-g} = \mathbb{F}\langle x\rangle} \subseteq \mathcal{F}_{1-g} \subseteq \dots \subseteq \mathcal{F}_g = \widehat{CF}(-Y)$, $\quad \widehat{HFK}(-Y,K,i) = H_*(\mathcal{F}_i/\mathcal{F}_{i-1})$.

$b(K\subseteq Y) = \min\{k \mid [x]=0 \text{ in } H_*(\mathcal{F}_k)\} + g \geq 1$.

Thm (B-Vela-Vick): If K not RV then $b(K\subseteq Y)=1$.

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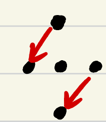
T_+:  T_-:  E: 

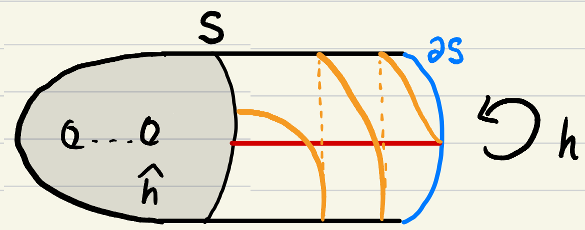
Fractional Dehn twist coefficient

$FDTC(h) \in \mathbb{Q}$ quantifies how RV h is.

$FDTC(h) > 0 \Rightarrow RV$, $FDTC(h) < 0 \Rightarrow LV$

Nielsen-Thurston rep.

measures twisting near ∂S in free isotopy from h to $\hat{h}$



Here, $FDTC(h) \in (2,3)$.

Q (Hubbard et al.): $K \subseteq S^3$ fibered, slice, hyperbolic $\Rightarrow FDTC(h) = 0$?

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Cor 2: $K \subseteq S^3$ fibered, $|\tau(K)| < g(K)$, thin $\Rightarrow$ K neither RV nor LV

$\Rightarrow FDTC(h) = 0$

Pf: $\tau(K)$ = Alexander grading on $E_\infty \cong \widehat{HF}(S^3) \cong \mathbb{F}$

$\Rightarrow \exists$ nontrivial $d_1 : \widehat{HFK}(S^3, K, g) \to \widehat{HFK}(S^3, K, g-1)$

$\Leftrightarrow b(K \subseteq S^3) = 1 \Leftrightarrow K$ not RV. Apply to mirror $K \subseteq -S^3$. $\square$

Cor 2: $K \subseteq S^3$ fibered, $|\tau(K)| < g(K)$, thin $\Rightarrow$ K neither RV nor LV

Cor 3: $K \subseteq S^3$ fibered, $|\tau(K)| < g(K)$, thin $\Rightarrow$ $S^3_r(K)$ has taut foliation $\forall r \in \mathbb{Q}$.

Cor 4: If $K \subseteq S^3$ fibered, alternating, then $S^3_r(K)$ has a taut foliation $\Leftrightarrow$ $S^3_r(K)$ is not an L-space.

Note: Of the 50 prime non-alternating knots with ≤ 10 crossings, 26 are fibered, quasi-alternating, with $|\tau(K)| < g(K)$.

Cor 2: $K \subseteq S^3$ fibered, $|\tau(K)| < g(K)$, thin $\Rightarrow$ K neither RV nor LV

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Cor 5*: Suppose $K_0 \subseteq Y_0$ and $K_1 \subseteq Y_1$ are fibered of the same genus. If they are ribbon homology concordant, then $FDTC(h_0) = FDTC(h_1)$.

Symplectic Floer homology

Suppose Σ a closed surface, $\varphi : \Sigma \to \Sigma$ an area-preserving diffeo.

Let M_φ be the mapping torus of φ.

Def: $HF^+(M_\varphi, \text{top}-1) := \bigoplus_{\langle c_1(s), [\Sigma] \rangle = 2g(\Sigma) - 4} HF^+(M_\varphi, s)$

Thm (Lee-Taubes, Kutluhan-Lee-Taubes): If $g(\Sigma) \geq 3$, then

$$HF^+(M_\varphi, \text{top}-1) \cong HF^{\text{symp}}(\varphi) = H_*(\underset{\substack{\uparrow \\ \text{generated by } \text{Fix}(\varphi)}}{CF^{\text{symp}}(\varphi)})$$

The proof, Part I

Thm (B-Ni-Sivek): If $b(K \subseteq Y) = 1$ then K not RV.

Pf: Suppose $b(K \subseteq Y) = 1$ but the monodromy $h: S \to S$ is RV

$\Rightarrow$ monodromy h^{-1} of mirror $K \subseteq -Y$ is not RV

$\Rightarrow b(K \subseteq -Y) = 1$ by B-Vela-Vick.

$$\Rightarrow \exists \text{ nontrivial } d_1 : \widehat{HFK}(Y, K, g) \to \widehat{HFK}(Y, K, g-1)$$

$$d_1 : \widehat{HFK}(-Y, K, g) \to \widehat{HFK}(-Y, K, g-1)$$

$\therefore$ K "looks like" its mirror from POV of HFK in grading $g, g-1$.

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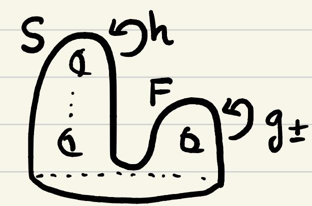
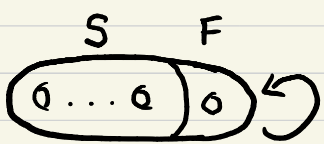
Let $T_\pm$ be trefoils, with monodromies $g_\pm : F \to F$.

$K \# T_\pm$ has fibration with fiber $S \#_b F$ =



$Y_0(K \# T_\pm)$ = mapping torus of



$h \cup g_\pm$.

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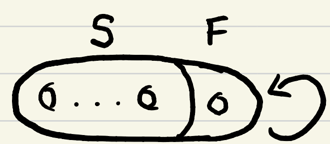
$Y_0(K \# T_\pm)$ = mapping torus of 
$h \cup g_\pm$.

Prop: $HF^+(Y_0(K\#T_+), \mathrm{top}-1) \cong HF^+(Y_0(K\#T_-), \mathrm{top}-1)$ (slight lie)

Thm (B-Ni-Sivek): If $b(K \subseteq Y) = 1$ then K not RV.

Pf: Suppose $b(K \subseteq Y) = 1$ but the monodromy $h : S \to S$ is RV

Let $T_\pm$ be trefoils, with monodromies $g_\pm : F \to F$.

Prop: $HF^+(Y_0(K \# T_+), \text{top}-1) \cong HF^+(Y_0(K \# T_-), \text{top}-1)$

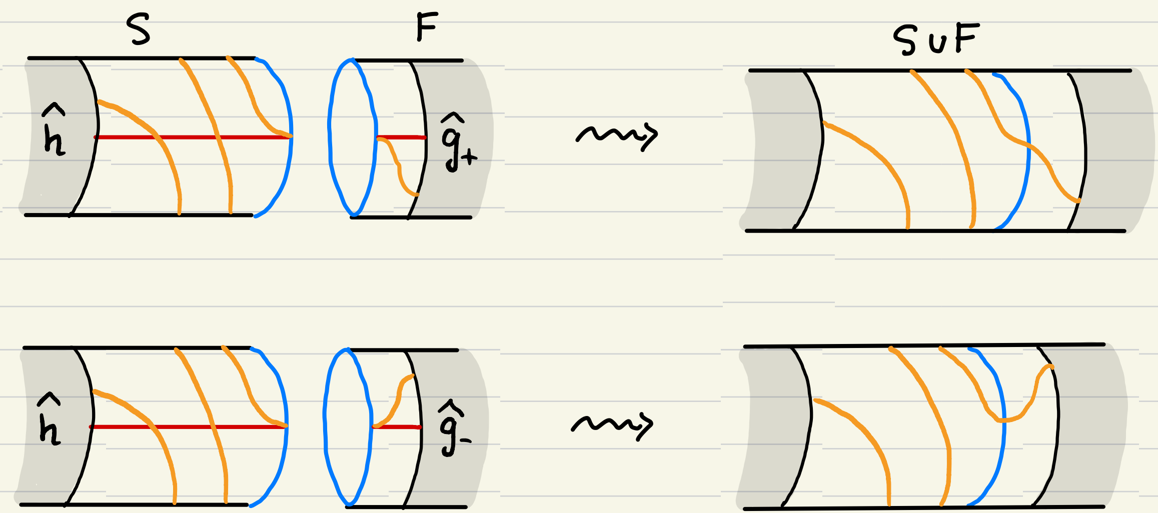
$$\begin{array}{ccccc} & HF^+(Y_0(K \# T_+), \text{top}-1) & \cong & HF^+(Y_0(K \# T_-), \text{top}-1) \\ & \wr\wr & & \wr\wr \\ \Longrightarrow & HF^{symp}(h \cup g_+) & \cong & HF^{symp}(h \cup g_-) \end{array}$$

But **Prop**: $\dim HF^{symp}(h \cup g_+) = 2 + \dim HF^{symp}(h \cup g_-)$, $\Rightarrow\Leftarrow$ $\square$

The proof, Part II

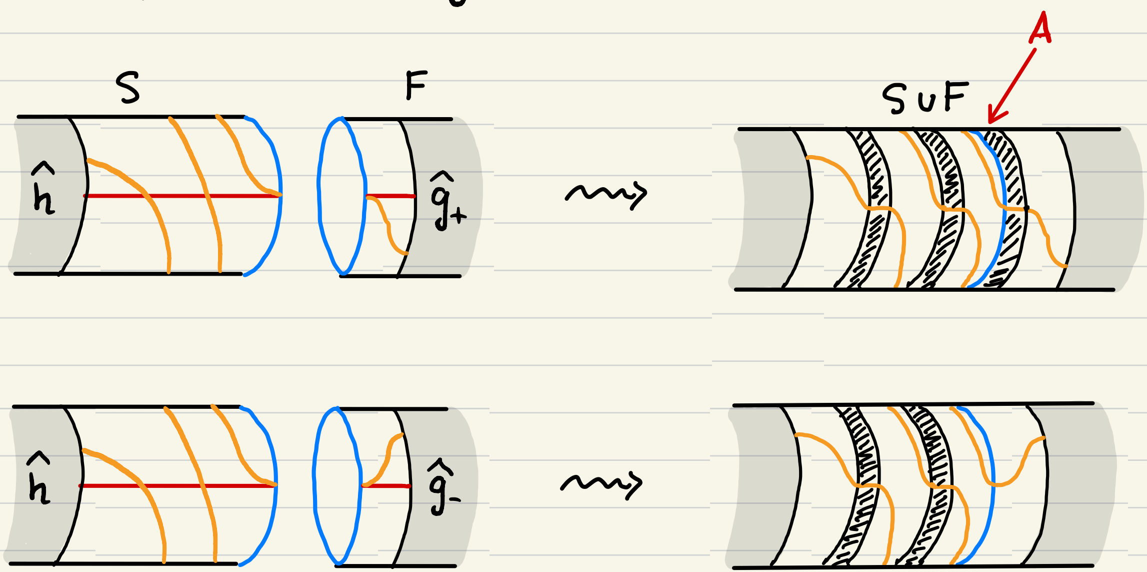
Prop: $\dim HF^{symp}(h \cup g_+) = 2 + \dim HF^{symp}(h \cup g_-)$

Pf: fixed points of $h \cup g_\pm$ differ near $\partial S = \partial F$



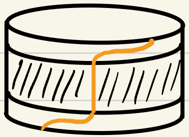
Prop: $\dim HF^{symp}(h \cup g_+) = 2 + \dim HF^{symp}(h \cup g_-)$

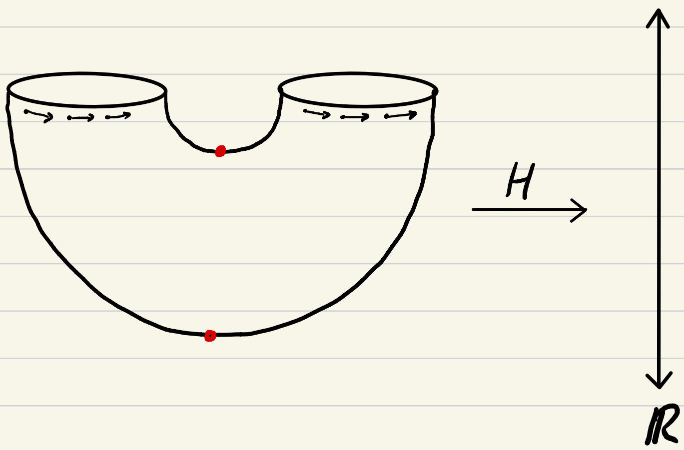
Pf: fixed points of $h \cup g_\pm$ differ near $\partial S = \partial F$



Extra fixed annulus A contributes $H_*(A, \partial A) \cong \mathbb{F}^2$ to $HF^{symp}(h \cup g_+)$.

More details

Perturb on fixed annulus $A \longrightarrow$  by flow of Hamiltonian vector field $\mathcal{X}_H$ associated with function $H: A \to \mathbb{R}$ where ∂A are maxima.



$dH(v) = \omega(\mathcal{X}_H, v)$

$\mathcal{X}_H$ parallel to level sets of H

p a fixed point of flow $\iff \mathcal{X}_H(p) = 0 \iff dH_p = 0 \iff p \in Crit(H)$.