An Unknotting Number for Transverse Knots

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What are some projects for undergraduates in contact/symplectic topology?
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Today discuss a student project on transverse knots.
Braids $\Rightarrow$ Transverse Knots

Braids have close connections to tranverse knots!

Today: Results from Senior Honors Thesis work of Blossom Jeong, BMC 2020

Comparative Literature + Math Double Major
Given a smooth knot $K$, the **unknotting number** $u(K)$ measures the minimal number of times that a knot must cross through itself in order to become an unknot.
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Unknotting Number for Smooth Knots

Given a smooth knot $K$, the **unknotting number** $u(K)$ measures the minimal number of times that a knot must cross through itself in order to become an unknot.

\[
u(K) = n \implies \exists \text{ projection of } K \text{ such that changing } n \text{ crossings turns projection into projection of unknot}
\]
Standard Contact Structure:

\[ \xi_{std} = \ker \alpha, \quad \alpha = dz - ydx \]

\[ = \langle \vec{j}, \vec{i} + y\vec{k} \rangle \]
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\[ \xi_{std} = \ker \alpha, \quad \alpha = dz - ydx \]

\[ = \langle \vec{j}, \vec{i} + y\vec{k} \rangle \]

Natural Curves:

- **Legendrian Curves**: \( \Lambda(t) \) s.t. \( \alpha \left( \frac{d}{dt} \Lambda(t) \right) = 0 \)
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\[ \xi_{std} = \ker \alpha, \quad \alpha = dz - ydx \]

\[ = \langle j, i + yk \rangle \]

Natural Curves:

- **Legendrian Curves**: \( \Lambda(t) \) s.t. \( \alpha \left( \frac{d}{dt} \Lambda(t) \right) = 0 \)
- **Transverse Curves**: \( T(t) \) s.t. \( \alpha \left( \frac{d}{dt} T(t) \right) > 0 \)
There are many smooth curves in 3-dimensional space that project to this curve:

\[ z = f(x) \]
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But, there is a **special** curve where the missing 3\(^{rd}\) coordinate is given by the slope! This makes the curve **Legendrian**.
Constructing Legendrian Curves

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But, there is a **special** curve where the missing 3\(^{rd}\) coordinate is given by the slope! This makes the curve **Legendrian**.
Cusped curve in the plane (without vertical tangents) can be lifted to 3-space using slope as the third coordinate: 

\[ dz - y \, dx = 0 \quad \Rightarrow \quad y = \frac{dz}{dx}. \]
Cusped curve in the plane (without vertical tangents) can be lifted to 3-space using slope as the third coordinate: $dz - ydx = 0 \implies y = dz/dx$.

Projection crossing resolved by slope:
**Constructing Transverse Curves**

**Transverse curves** are more flexible:

\[ T(t) = (x(t), y(t), z(t)) \]

\[ z'(t) - y(t)x'(t) > 0 \implies z'(t) > y(t)x'(t) \]
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Get “slope” bounds on \( y(t) \):

- When \( x'(t) > 0 \), \( y(t) < \frac{dz}{dx} \).
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- When \( x'(t) < 0 \), \( y(t) > \frac{dz}{dx} \).
Transverse Forbidden Shapes

In the $xz$-diagram of a transverse curve:

- There are no downward vertical tangencies:
Transverse Forbidden Shapes

In the \( xz \)-diagram of a transverse curve:

- There are no downward vertical tangencies:

- There are no \( +\text{-down-down-down} \) crossings:
Forbidding Downward Vertical Tangencies

- When $x'(t) > 0$, $y(t) < \frac{dz}{dx}$.
- When $x'(t) < 0$, $y(t) > \frac{dz}{dx}$.
Forbidding Downward Vertical Tangencies

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Forbidding + -Down-Down Crossings

- When $x'(t) > 0$, $y(t) < \frac{dz}{dx}$.
- When $x'(t) < 0$, $y(t) > \frac{dz}{dx}$.
Every smooth knot has a transverse representative.
Constructing Transverse Knots

Every smooth knot has a transverse representative.

- Start with smooth projection.
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- Remove forbidden downward vertical tangencies:
Constructing Transverse Knots

Every smooth knot has a transverse representative.

- Start with smooth projection.
- Remove forbidden downward vertical tangencies:

  ![Diagram of forbidden downward vertical tangencies]

- Remove forbidden +-down-down crossings:

  ![Diagram of forbidden +-down-down crossings]
Moving Transverse Knots

Transverse Reidemeister Moves: no + - down-down-crossings:

![Diagram of transverse Reidemeister moves]

**Figure 24.** Transverse Reidemeister II moves.

**Figure 25.** Transverse Reidemeister III moves.
Stabilization

Transverse Modification:

Stabilization Operation
Stabilization

Transverse Modification:

Classical Transverse Invariant: **Self-Linking Number**

\[ sl(T) = \text{writhe of } xz\text{-projection} \]

Stabilization lowers \( sl \) by 2.
Every smooth knot has an infinite number of different transverse representatives.

\[
\text{Bennequin} \quad \text{sl} \left( T \right) \leq 2g \left( K \right) - 1
\]

Example: Ray of transverse unknots

\[-1 \quad -3 \quad -5 \]

Unknot is transversely simple: transverse representatives classified by sl #.

∃ non-transversely simple, e.g. \((2, 3)\)-cable of \((2, 3)\)-torus knot [Etnyre-Honda]
Every smooth knot has an infinite number of different transverse representatives.

Organize all transverse representatives of a fixed knot type with a ray: 
(Bennequin) $sl(T) \leq 2g(K_T) - 1$
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**Example:** Ray of transverse unknots

Unknot is **transversely simple:**
transverse representatives classified by \(\text{sl} \) #.

\[\exists \text{ non-transversely simple, e.g. } (2,3)\text{-cable of } (2,3)\text{-torus knot [Etnyre-Honda]}\]
Legendrian Unknotting: ??

exists

never exists
Unknotting in the Contact World

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Transverse Unknotting: Use the following crossing changes

Transverse Crossing Changes
Unknotting in the Contact World

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exists never exists

Transverse Unknotting: Use the following crossing changes

Transverse Crossing Changes

Rigid Crossing: --down-down crossing
Transverse Wall Crossing

Transverse crossing changes gives us a path in the space of immersed transverse curves.
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If $T'$ is obtained from $T$ by a transverse crossing change, then

- $T'$ is a transverse knot, and
Transverse Wall Crossing

Transverse crossing changes gives us a path in the space of immersed transverse curves.

If $T'$ is obtained from $T$ by a transverse crossing change, then

- $T'$ is a transverse knot, and
- $|sl(T') - sl(T)| = 2$. 
Q: Can every transverse knot be “unknotted” by allowable crossing changes?
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All rigid crossings!
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All rigid crossings!

After transverse isotopy, can we perform transverse crossing changes to convert to a transverse unknot?
A start to answering this:

**Claim:** Every smooth knot type has a transverse representative that can be converted to a transverse unknot by passing through self intersections.
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- Start with any transverse representative.
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- Start with any transverse representative.
- Get another transverse representative after removing rigid crossings:
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- Start with any transverse representative.
- Get another transverse representative after removing rigid crossings:

![Diagram](image)

- Standard topological argument shows that crossing changes can be performed to get to an unknot.
Smooth World: there is a unique unknot.

Contact World: there are an infinite number of transverse unknots.
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Contact World: there are an infinite number of transverse unknots.

Q: Can we always get to the “simplest” transverse unknot?
Smooth World: there is a unique unknot.

Contact World: there are an infinite number of transverse unknots.

Q: Can we always get to the “simplest” transverse unknot?

A: Yes!
Claim: We can move between any two transverse unknots by transverse crossing changes.
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- All transverse unknots are related by stabilization.
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- All transverse unknots are related by stabilization.
- Stabilization/destabilization can be done by allowable crossing changes:
Theorem: If $T, T'$ are transversal representatives of the same knot type $\mathcal{K}$, then can move between $T$ and $T'$ by transversal crossing changes in such a way that all transversal knots along the way are in $\mathcal{K}$.
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- After stabilization $T$ and $T'$ are transversely isotopic. [Fuchs-Tabachnikov]
Moving between transverse representatives of fixed knot type

**Theorem:** If $T, T'$ are transversal representatives of the same knot type $\mathcal{K}$, then can move between $T$ and $T'$ by transversal crossing changes in such a way that all transversal knots along the way are in $\mathcal{K}$.

- After stabilization $T$ and $T'$ are transversely isotopic.
  [Fuchs-Tabachnikov]

- Stabilize via transverse crossing changes, transverse isotopy, destabilize via transverse crossing changes.
Corollary: For all knot types $\mathcal{K}$, any transverse representative $T$ of $\mathcal{K}$ can be converted to a (and thus any) transverse unknot by transverse crossing changes.
All Transverse Knots can be Unknotted

**Corollary:** For all knot types $\mathcal{K}$, any transverse representative $T$ of $\mathcal{K}$ can be converted to a (and thus any) transverse unknot by transverse crossing changes.

- Earlier saw that there exists a representation $T'$ of $\mathcal{K}$ with a projection that can be unknotted.
**Corollary:** For all knot types $\mathcal{K}$, any transverse representative $T$ of $\mathcal{K}$ can be converted to a (and thus any) transverse unknot by transverse crossing changes.

- Earlier saw that there exists a representation $T'$ of $\mathcal{K}$ with a projection that can be unknotted.
- By previous result, we can move from $T$ to $T'$ via transverse crossing changes.
**Definition:** A transverse knot $T$ has **transverse unknotting number** $n$ if there exists a front diagram of $T$ such that transversely changing $n$ crossings in the diagram turns $T$ into the transverse unknot $U$ with $sl(U) = -1$, and there is no diagram of $T$ such that fewer crossing changes would have produced the transverse unknot with self-linking number -1.
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Notation: $U_{-1}^h(T)$:
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**Notation:** $U_{-1}^h(T)$:

World of immersed transverse knots: the transverse unknotting number is the minimum number of times the knot must pass through itself in order to become the transverse unknot with self-linking number -1.
Get upper bounds to $U_{-1}(T)$ by constructions.
Get upper bounds to $U_{-1}^h(T)$ by constructions.

**Lower bound:**

**Lemma:** Suppose $T$ is a transverse knot in the smooth knot type $\mathcal{K}$. Then

$$ \max \left\{ u(\mathcal{K}), \left| \frac{sl(T) + 1}{2} \right| \right\} \leq U_{-1}^h(T), $$

where $u(\mathcal{K})$ is the smooth unknotting number.
Example: $\mathcal{K} = (2, -5)$-torus knot: transversely simple with $\bar{sl}(\mathcal{K}) = -7$. 
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Max $sl$ representative $T$: 

![Diagram of a (2, -5) torus knot representing the max $sl$ representative $T$.]
**Example:** $\mathcal{K} = (2, -5)$-torus knot: transversely simple with $\overline{sl}(\mathcal{K}) = -7$.

Max $sl$ representative $T$:

$$sl(T) = -7, \ u(K) = 2 \quad \Longrightarrow \quad \max \left\{ u(K), \left| \frac{sl(T) + 1}{2} \right| \right\} = 3 \leq U_{-1}^h(T).$$
**Example:** $\mathcal{K} = (2, -5)$-torus knot: transversely simple with $sl(\mathcal{K}) = -7$.

Max $sl$ representative $T$:

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For any transverse representative $T'$ of $\mathcal{K}$,

\[ u_{-1}^h(T') = \left| \frac{sl(T') + 1}{2} \right|. \]
Further Questions

Have established an unknotting number for transverse knots.

Q: What are some other questions related to unknotting in the transverse world?
Ancestor-Descendant Relationship


Defn: A smooth knot $K_1$ is an ancestor of $K_2$ if a diagram for $K_2$ can be obtained from a minimum crossing diagram of $K_1$ by some number of crossing changes.

A knot with crossing number $n$ is fertile if it is an ancestor of every knot with crossing number less than $n$.

Examples: $11_{a135}$ is an ancestor of the trefoil; $7_6$ is fertile.
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**Examples:**
$11_{a135}$ is an ancestor of the trefoil; $7_6$ is fertile.
If don’t require minimum crossings in diagram of ancestor, all knots are related by ancestor-descendant.
Properties of Ancestor-Descendant Relationship

- If don’t require minimum crossings in diagram of ancestor, all knots are related by ancestor-descendant.

- Every knot is an ancestor of the unknot.
Properties of Ancestor-Descendant Relationship

- If don’t require minimum crossings in diagram of ancestor, all knots are related by ancestor-descendant.

- Every knot is an ancestor of the unknot.

- The unknot is not an ancestor of any non-trivial knot.
1. CHMOPRZ studied family trees of twist and torus knots to answer: What are all possible descendants of a twist/torus knot?
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“Insular Families”: 
- twist descendants are twist knots;
- \((2, p)\)-torus knot descendants are \((2, q)\)-torus knots.
Knot Lineage

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   “Insular Families”:
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2. Hanaki [2020]: There are no alternating fertile knots with crossing number \(> 7\).

3. Ito [2021]: A knot whose minimum crossing number \(c(K)\) is even and greater than 30 is not fertile.
Q: What is the contact world analogue of this Ancestor-Descendant Relationship?
**Defn:** A sequence of transverse knots \( (T_1, T_2, \ldots, T_n) \) is a **transverse family tree** if each \( T_{i+1} \) can be obtained from \( T_i \) by a single transverse crossing change.
Transverse Family Trees

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A transverse family tree is:

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A transverse family tree is:

- **maximal** if each \(T_i\) has maximal self-linking number in its knot type.

- **increasing** (**decreasing**) if the self-linking numbers of \(T_i\) are strictly increasing (decreasing).
Given any smooth knot types $K_1, K_2$, there exists a transverse family tree $(T_1, \ldots, T_2)$ where $T_1$ is in the knot type of $K_1$, $T_2$ is in the knot type of $K_2$. Moreover, we can assume that $T_1$ and $T_2$ have maximum self-linking numbers (but not that all knots in the family tree have maximal self-linking numbers).
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Properties of Transverse Family Trees

Given any smooth knot types $K_1, K_2$, there exists a transverse family tree $(T_1, \ldots, T_2)$ where $T_1$ is in the knot type of $K_1$, $T_2$ is in the knot type of $K_2$.

Moreover, we can assume that $T_1$ and $T_2$ have maximum self-linking numbers (but not that all knots in the family tree have maximal self-linking numbers).

Q: What are examples of maximal transverse family trees?
Theorem 7.4 (Twist Knot Transverse Family Trees).

1. For any odd $m \geq 1$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_1)$, where $T_j$ is a transverse representative of the twist knot $K_j$.
2. For any even $m \geq 2$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_2)$, where $T_j$ is a transverse representative of the twist knot $K_j$.
3. There exists a transverse family tree $(T_m, U, T_{m-1}, U, \ldots, T_1)$, where $T_j$ is a transverse representative of the twist knot $K_j$ and $U$ is a transverse unknot.
**Theorem 7.6** \((2,p)\)-Torus Knot Transverse Family Trees.

1. For all odd \(p \geq 3\), there exists a maximal decreasing transverse family tree \((T_{2,p}, T_{2,p-2}, \ldots, T_{2,3})\), where \(T_{2,j}\) is a transverse representative of the torus knot \(K_{2,j}\).
2. For all odd \(n \leq -3\), there exists a maximal increasing transverse family tree \((T_{2,n}, T_{2,n+2}, \ldots, T_{2,-3})\), where \(T_{2,j}\) is a transverse representative of torus knot \(K_{2,j}\).

**Figure 42.** A transverse representative of a torus knot \(K_{2,p}\) where \(p\) is odd, (a) when \(p\) is positive and (b) when \(p\) is negative. We will call each knot \(T_+\) and \(T_-\) respectively.
Project Evaluation:

- Very accessible with knot theory background.
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- Nice way to rethink knot theory results.
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- Very accessible with knot theory background.
- Nice way to rethink knot theory results.
- Won MAA-EPaDel Student Math Paper Prize.
Final Comments:

Transverse Knots and Braids Connections
Standard Contact Structure: Symmetric Version

\[ \xi_{sym} = \ker \alpha = \ker (dz + r^2 d\theta) \]
Standard Contact Structure: Symmetric Version

\[ \xi_{\text{sym}} = \ker \alpha = \ker (dz + r^2 d\theta) \]

Any smooth closed braid $B$ can be isotoped (through braids) to be a transverse knot.
**Theorem**: [Alexander, 1925] Any knot is isotopic to a closed braid.
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**Theorem**: [Markov, 1935] Two closed braid representatives $X_-, X_+$ of the same oriented knot type $\chi$ are related by a sequence of closed braid representatives of $\chi$:

$$X_- = X_1 \to X_2 \to \cdots \to X_r = X_+$$

such that, up to braid isotopy, $X_{i+1}$ is obtained from $X_i$ by a single stabilization or destabilization.
Braids = Transverse Knots

**Theorem:** [Bennequin, 1983] Any transverse knot in \((\mathbb{R}^3, \xi_{sym})\) is transversely isotopic to a closed braid.
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**Theorem:** [Bennequin, 1983] Any transverse knot in \((\mathbb{R}^3, \xi_{sym})\) is transversely isotopic to a closed braid.

If \(T = \hat{\beta}\), can easily calculate \(sl\) from braid presentation:

\[
sl(\hat{\beta}) = a(\beta) - n,
\]

where \(n = \) braid index, \(a(\beta) = \) algebraic crossing number of \(\beta\).
**Theorem**: [Bennequin, 1983] Any transverse knot in \((\mathbb{R}^3, \xi_{sym})\) is transversely isotopic to a closed braid.

If \( T = \hat{\beta} \), can easily calculate \( sl \) from braid presentation:

\[
sl(\hat{\beta}) = a(\beta) - n,
\]

where \( n = \) braid index, \( a(\beta) = \) algebraic crossing number of \( \beta \).

**Theorem**: [Orevkov and Shevchishin 2003] Given two transversal, closed braid representatives \( TX_- \), \( TX_+ \) of the same oriented knot type \( \chi \) are related by a sequence of transversal closed braid representatives of \( \chi \)

\[
TX_- = TX_1 \rightarrow TX_2 \rightarrow \ldots \rightarrow TX_r = TX_+
\]

such that, up to braid isotopy, \( TX_{i+1} \) is obtained from \( TX_i \) by a single **positive** stabilization or destabilization.
There are lots of interesting topological problems that can be “contactified” to make interesting projects for undergraduate students.

Examples:

- Transversal Unknotting and lineage (today)
- Legendrian versions of multicrossing knots
- Constructions of “decomposable” Lagrangian fillings and cobordisms
Conclusion

There are lots of interesting topological problems that can be “contactified” to make interesting projects for undergraduate students.

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Thank you!