Computations of ECH capacities and infinite staircases of 4D symplectic embeddings

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Summary

We provide an almost complete classification of the Hirzebruch surfaces whose embedding function has an infinite staircase.

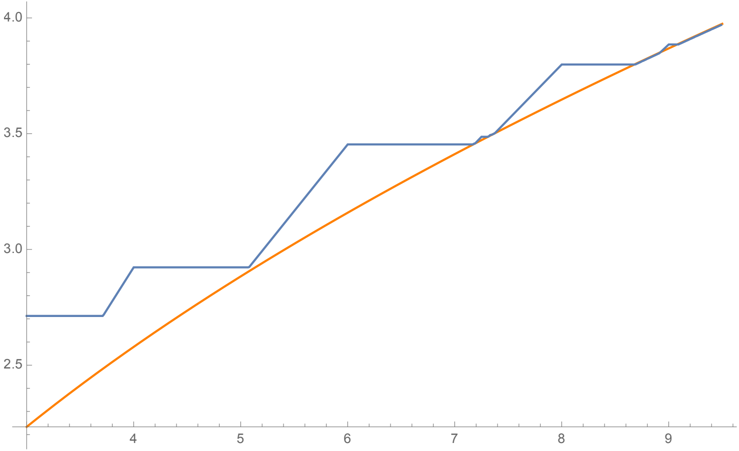

Figure: A new infinite staircase.

Symplectic embeddings

The **standard symplectic form** on $\mathbb{C}^2$ is the 2-form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = r_1\, dr_1 \wedge d\theta_1 + r_2\, dr_2 \wedge d\theta_2.$$

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Definition

A **symplectic embedding** from $X \subset \mathbb{C}^2$ to $X' \subset \mathbb{C}^2$ is a smooth embedding $\varphi : X \to X'$ with $\varphi^*\omega = \omega$.

Notation: $X \overset{s}{\hookrightarrow} X'$.

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The **volume** of X is $\mathrm{vol}(X) := \int_X \omega \wedge \omega$.

$X \overset{s}{\hookrightarrow} X'$ implies $\mathrm{vol}(X) \leq \mathrm{vol}(X')$.

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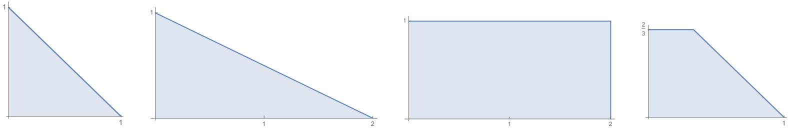
$X \stackrel{s}{\hookrightarrow} X'$ implies $\operatorname{vol}(X) \leq \operatorname{vol}(X')$. **Q:** when are embeddings

- "flexible" – $\operatorname{vol}(X) = \operatorname{vol}(X')$, or
- "rigid" – $\operatorname{vol}(X) < \operatorname{vol}(X')$?

Convex toric domains

A **toric domain** X_Ω in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$.



(a) Ball $B(1)$ (b) Ellipsoid $E(1,2)$ (c) Polydisk $P(1,2)$ (d) $X_{\Omega_{\frac{1}{3}}}$

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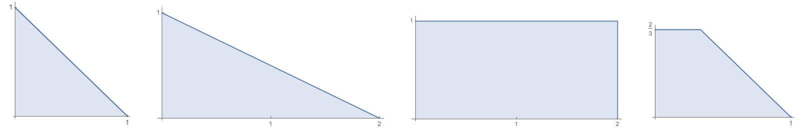


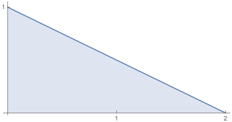
(e) Ball $B(1)$ (f) Ellipsoid $E(1,2)$ (g) Polydisk $P(1,2)$ (h) $X_{\Omega_{\frac{1}{3}}}$

A toric domain is **convex** if Ω is convex in $\mathbb{R}^2$.

$\mathrm{vol}(X_\Omega, \omega) \sim \mathrm{Area}(\Omega)$.

These pictures also enable explicit constructions of embeddings.

Ellipsoid embedding functions



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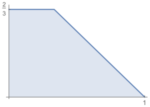

(j) $X_{\Omega_{\frac{1}{3}}}$

We study $X = E(1,z)$ and $X' = X_{\Omega_b}$, where Ω_b is the trapezoid with corner $(b, 1-b)$.

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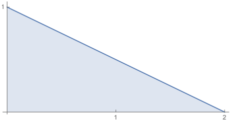
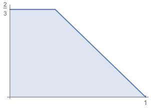


(k) Ellipsoid $E(1, 2)$



(l) $X_{\Omega_{\frac{1}{3}}}$

We study $X = E(1, z)$ and $X' = X_{\Omega_b}$, where Ω_b is the trapezoid with corner $(b, 1 - b)$.

X_{Ω_b} above the top and right sides of Ω_b are T^2-fibered; collapsing this boundary via a Hopf subfibration determined by the slope of $\partial\Omega_b$ gives a Hirzebruch surface (a blown up $\mathbb{C}P^2$).

Ellipsoid embedding functions

Define the **ellipsoid embedding function** of X_Ω by

$$C_\Omega(z) := \inf \left\{ C > 0 \;\middle|\; E(1,z) \overset{s}{\hookrightarrow} X_{C\Omega} \right\}.$$

We say C_Ω has an **infinite staircase** if it is nonsmooth at infinitely many points.

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Immediate facts:

- C_Ω is increasing and sublinear.
- Set $C_b = C_{\Omega_b}$. Lower bound:

$$\operatorname{vol}(E(1,z)) \leq \operatorname{vol}(X_{C_b(z)\Omega_b}) \Rightarrow C_b(z) \geq \sqrt{\frac{z}{1-b^2}}.$$

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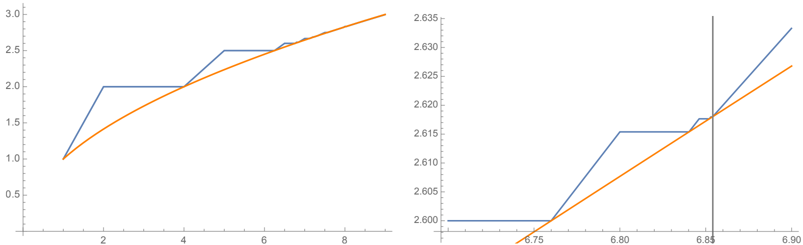
Goal: classify those C_b with infinite staircases.

Theorem (McDuff-Schlenk '12)

C_0 has an infinite staircase accumulating from below to (τ^4, τ^2), where $\tau = \frac{1+\sqrt{5}}{2}$.



(m) C_0. (n) Zoomed in. $z = \tau^4 = \left(\frac{1+\sqrt{5}}{2}\right)^4$.

Figure: C_0, volume bound $\sqrt{a}$.

Outer corners show rigidity while inner corners show flexibility.

What is known about C_b

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b irrational:

- BHMMMP-Weiler '21 found infinitely many infinite staircases for *b* irrational, including descending
- extended by Magill-McDuff '21 using symmetries, see also Usher '19

Main theorem and ingredients

Accumulation points

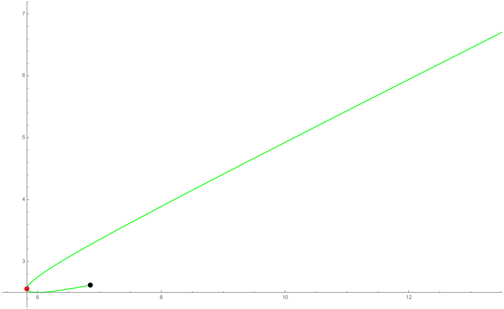
Theorem (C-G–H–M–P '20)

If C_b has an infinite staircase, the larger solution to

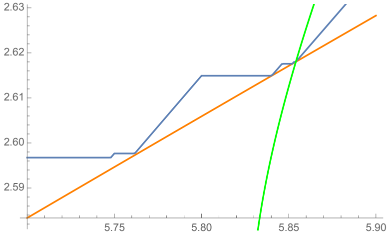
$$z^2 - \left(\frac{(3-b)^2}{1-b^2} - 2\right) z + 1 = 0$$

is the z-coordinate acc(b) of the accumulation point. For its y-coordinate: the accumulation point is on the volume obstruction.

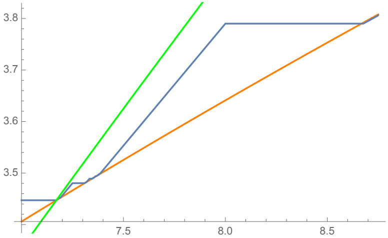
Note: if $b \in \mathbb{Q}$ then acc(b) is rational or quadratic irrational.



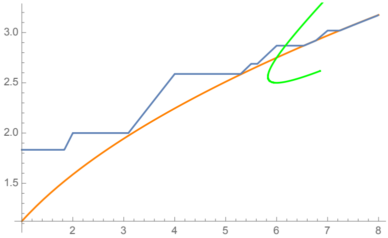
Four possibilities for C_b



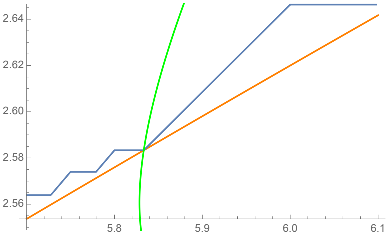
(a) Ascending staircase



(b) Descending staircase



(c) **Blocked**



(d) Unblocked, no staircases

New results

Theorem (Magill-McDuff-Weiler '22)

(1) *The set of unblocked b with* $acc(b) \in [6, 8]$ *is homeomorphic to the Cantor set.*

(2) *Assume* $acc(b) \in [6, 8]$ *and b is not blocked.*

- *If b is an endpoint of a blocked interval,* C_b *has either an ascending or a descending infinite staircase.*
- *Otherwise,* C_b *has both.*

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- *If b is an endpoint of a blocked interval, C_b has either an ascending or a descending infinite staircase.*
- *Otherwise, C_b has both.*

(3) *All C_b are equivalent to one with $acc(b) \in [6, 8]$, except for a countable set with C_b like (d). These have no ascending infinite staircases.*

(4) *Weak C-G–H–M–P conjecture holds for Hirzebruch surfaces.*

Staircase steps are ECH capacity ratios

Each step of C_b represents a Floer-theoretic obstruction to embedding.

[1] can think "length"

[2] these are Reeb orbits

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A metric space's **systole** is the length of its shortest noncontractible curve.

Embedded contact homology provides a $\mathbb{Z}_{\geq 0}$-family of "systoles" for X_Ω called **ECH capacities**, which are integer linear combinations of the actions[1] of certain torus knots[2] in ∂X_Ω.

$$0 = c_0(X_\Omega) < c_1(X_\Omega) \leq c_2(X_\Omega) \leq \cdots \leq \infty.$$

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Theorem (McDuff '09, Cristofaro-Gardiner '19)

If X_Ω is a convex toric domain,

$$\mathrm{int}(E(1,z)) \overset{s}{\hookrightarrow} \mathrm{int}(X_\Omega) \Leftrightarrow c_k(E(1,z)) \leq c_k(X_\Omega)\ \forall k.$$

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Matching capacities to steps

Using the theorem and fact that $c_k(X_{C\Omega}) = Cc_k(X_\Omega)$:

$$C_b(z) = \inf\left\{ C > 0 \,\middle|\, E(1,z) \overset{s}{\hookrightarrow} X_{C\Omega_b} \right\}$$
$$= \sup_k \left\{ \frac{c_k(E(1,z))}{c_k(X_{\Omega_b})} \right\}.$$

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We can upgrade the sup to a max using a Weyl law for ECH capacities (Cristofaro-Gardiner–Hutchings–Ramos '15).

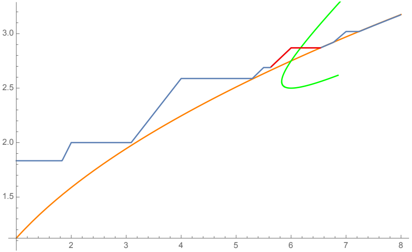

Figure: $C_{5/11}$, volume obstruction, accumulation point curve. The obstruction $c_6(E(1,z))/c_6(X_{\Omega_{5/11}})$ blocks an infinite staircase for $C_{5/11}$.

From steps to staircases

Each c_k ratio has an outer corner at $p/q \geq 1 \in \mathbb{Q}$. To identify p/q:

- $\partial X_{\Omega_b} = T_1 \cup T_2$, where T_1 is foliated by $(1,0)$-knots $K_{(1,0)}$ (action $1-b$) and T_2 by $(1,-1)$-knots $K_{(1,-1)}$ (action 1)

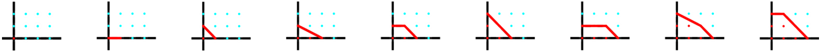

Figure: The carriers of the capacities $c_0(X_{1/3}), \ldots, c_8(X_{1/3})$ in $H_1(T^2)$.

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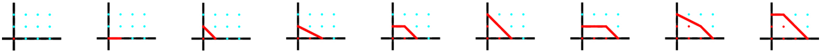
$$(p+1)(q+1)/2 = k+1$$

lattice points, with $p \geq q$



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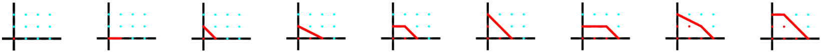

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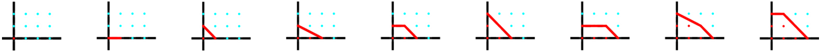
- $c_k(X_{\Omega_b}) =$ action of $mK_{(1,0)} + (d-m)K_{(1,-1)} = d - mb$



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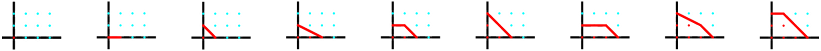

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k	1	2	3	4	5	6	7	8
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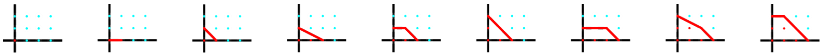

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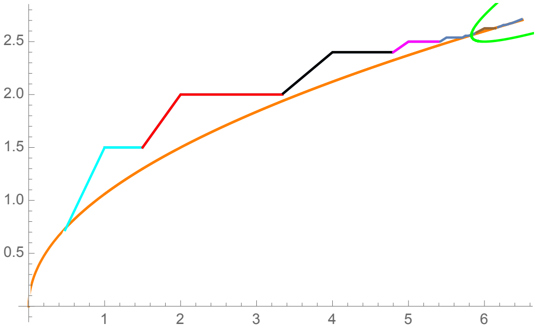

Figure: The $k = 1, 2, 4, 5$, and 8 steps of $C_{1/3}$. $k = 8$ capacity is unusual and supercedes $k = 6$ at $z = 6$ for $b = 1/3$.

Main Theorem #1 and #2

Theorem (Magill-McDuff-Weiler '22)

(1) *The set of unblocked b with* $acc(b) \in [6, 8]$ *is homeomorphic to the Cantor set.*

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When c_k blocks an infinite staircase, near $\mathrm{acc}(b) = z = p/q$

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an open condition on b. Thus c_k blocks an open interval containing b with $\mathrm{acc}(b) = z = p/q$. □

The fundamental staircases for the Cantor set

Continued fractions crash course: $[a, b, c, d] = a + \frac{1}{b + \frac{1}{c + 1/d}}$, and $[a, \{b, c\}^2] = [a, b, c, b, c]$, etc.

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Unproven claims from BHMMMPW '21:

- $k = 6$ blocks an infinite staircase for C_b if

$$\mathrm{acc}(b) \in ([\{5,1\}^\infty], [7, \{5,1\}^\infty]) \approx (5.851, 7.171)$$

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Unproven claims from BHMMMPW '21:

- $k=6$ blocks an infinite staircase for C_b if

$$\mathrm{acc}(b) \in ([\{5,1\}^\infty],[7,\{5,1\}^\infty]) \approx (5.851, 7.171)$$

- $C_{\mathrm{acc}^{-1}([7,\{5,1\}^\infty])}$ has a descending infinite staircase with steps

$$[8],[7,4],[7,5,2],[7,5,1,4],[7,5,1,5,2],[7,5,1,5,1,4],\ldots$$

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Proof of Main Theorem #1 and #2.

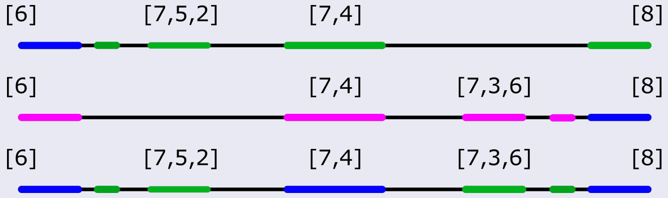

Figure: Horizontal direction: z variable. Colored intervals are acc of the blocked intervals including p/q as labeled by its continued fraction.

Proof of Main Theorem #1 and #2.

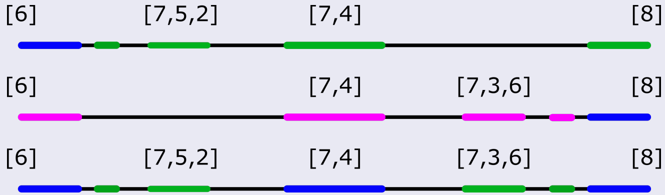

Figure: Horizontal direction: z variable. Colored intervals are acc of the blocked intervals including p/q as labeled by its continued fraction.

$$p/q = [7,4] = 29/4 \Rightarrow k = \frac{(p+1)(q+1)}{2} - 1 = 74$$

The pattern repeats with the ECH capacities c_6 and c_{74} taking the place of c_8, and again with the ECH capacities c_{74} taking the place of c_6 and c_8.

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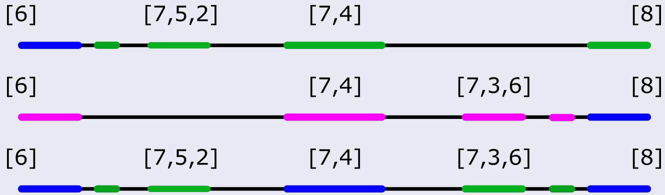

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The pattern repeats with the ECH capacities c_6 and c_{74} taking the place of c_8, and again with the ECH capacities c_{74} taking the place of c_6 and c_8.

Repeating forever produces a Cantor set, with infinite staircases for b at the end of each removed interval.

□

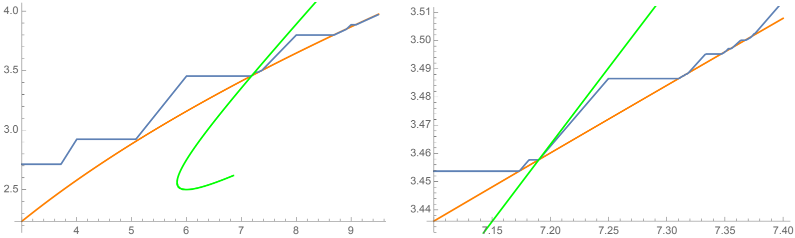
First slide again

This is the new ascending infinite staircase for the lower endpoint of the interval blocked by c_{74}, with steps having corners

$$[6], [7,5,2], [7,5,3,1,6], [7,5,3,1,7,5,2], \ldots$$

and

$$k = 6, 479, 73072, 12113009, \ldots$$



(a) First slide, with acc pt curve

(b) Zoomed in

C_b with two staircases

Theorem (Magill-McDuff-Weiler '22)

(2) *Assume* $\mathrm{acc}(b) \in [6,8]$ *and* b *is not blocked.*

- *If* b *is not an endpoint of a blocked interval,* C_b *has both an ascending and a descending infinite staircase.*

Proof.

If $\mathrm{acc}(b) \in [6,8]$ is not the endpoint of a blocked interval, infinitely many of the steps obtained via the fractal procedure will still be visible on either side of $\mathrm{acc}(b)$. □

Weak C-G–H–M–P conjecture: Main theorem #4

C-G–H–M–P conjectured: if $b \in \mathbb{Q}, b \neq 0, 1/3$, then C_b does not have an infinite staircase.

We prove: if $\mathrm{acc}(b) \in [6, 8]$ with finite continued fraction, then b is blocked. (Finite continued fractions $\leftrightarrow$ rationals.)

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Future goal: if $\mathrm{acc}(b) \in [6, 8]$ is quadratic irrational (periodic CF), then b is blocked.

Unblocked special rational b: Main theorem #3

Magill-McDuff '21: every $b \in [0,1)$ besides $b_i = \text{acc}^{-1}(y_i/y_{i-1})$

$$y_1 = 1, y_2 = 6, y_i = 6y_{i-1} - y_{i-2}; \quad b_2 = 1/5, b_3 = 11/31, b_4 = 59/179, \ldots$$

has C_b equivalent to some $C_{b'}$ with $\text{acc}(b') \in [6,8]$.

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The C_{b_i} do not have ascending infinite staircases.

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Theorem (Magill-McDuff-Weiler '22)

The C_{b_i} do not have ascending infinite staircases.

In progress: The C_{b_i} do not have descending infinite staircases.

$C_{\frac{11}{31}}$

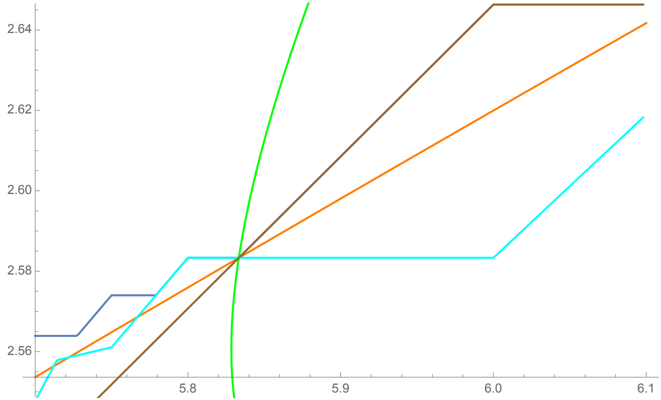

Figure: $C_{\frac{11}{31}}$ near $\text{acc}(11/31) = 35/6 = y_3/y_2$. Volume obstruction, accumulation point curve. They all intersect at $35/6$.
Obstruction from c_{89}.
Obstruction from c_8.

Thank you!