An algorithm to distinguish Legendrian knots

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Braids in Low-Dimensional Topology
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Legendrian knots are *equivalent*, if they are isotopic within the class of Legendrian knots.

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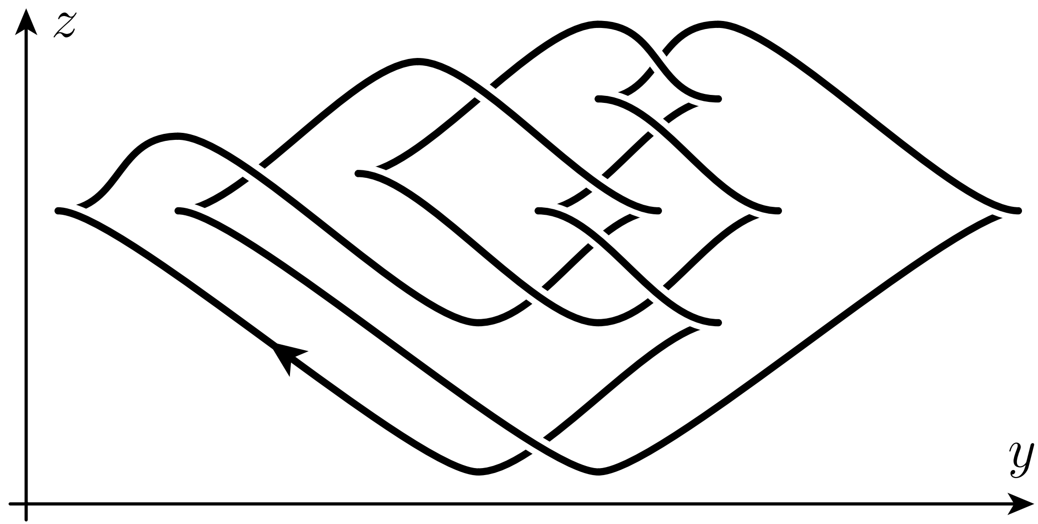
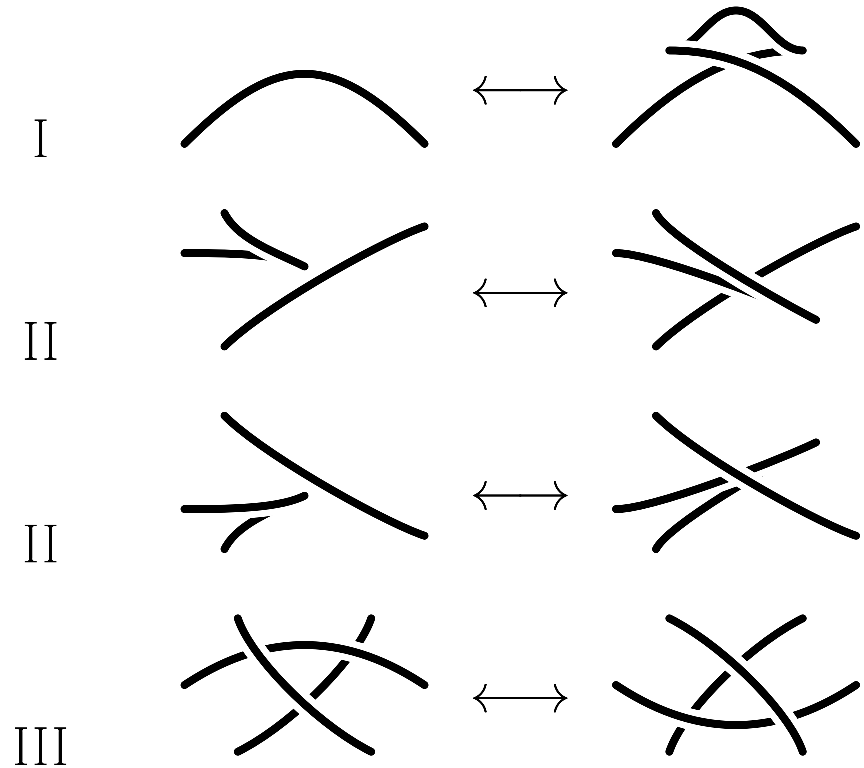


Reidemeister moves for front projections (J.Świątkowski, 1992)



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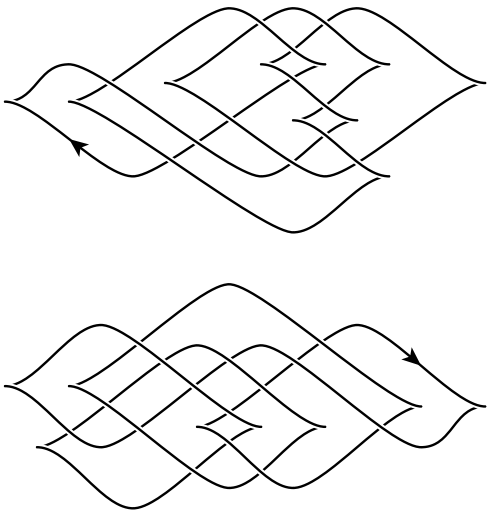
Q. Can it be used in practice?

A. Yes, for knots of low complexity and, sometimes, for specific knots of high complexity.

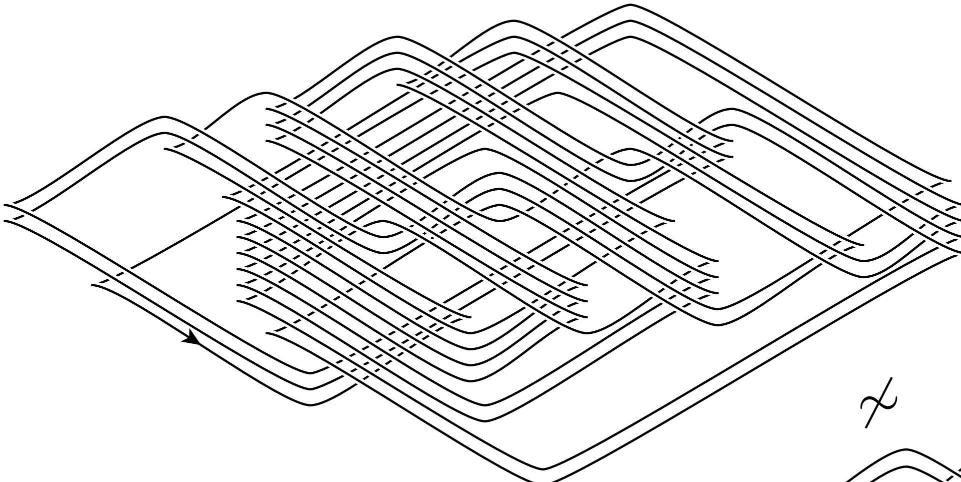
Example

The following Legendrian knots are pairwise non-equivalent:

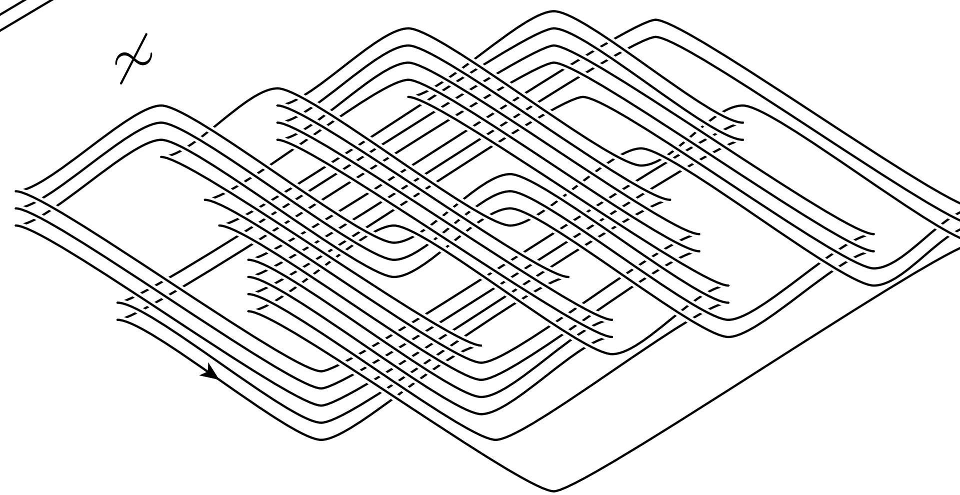


(topological type: 10_{160}).

Another example



$\not\approx$



Classical invariants of Legendrian knots

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Thurston–Bennequin number $\operatorname{tb}(K)$ of a Legendrian knot K is defined as

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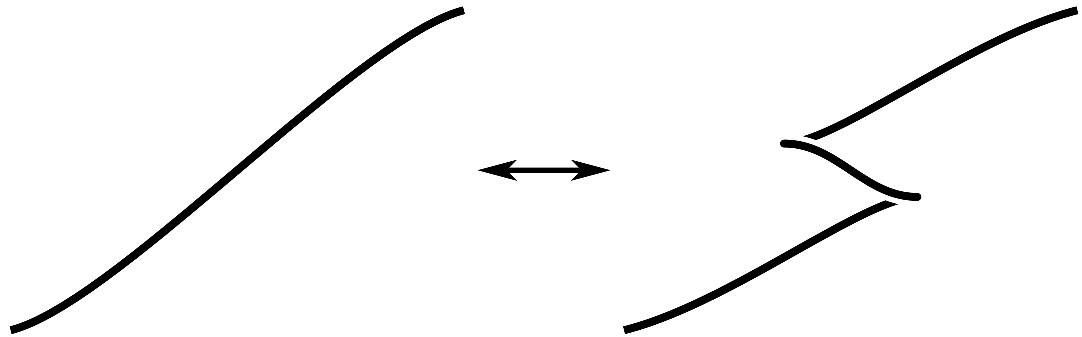
Rotation number $r(K)$ of an oriented Legendrian knot K is

$$\frac{1}{2}(c_- - c_+),$$

where c_+ (respectively, c_-) is the number of cusps oriented down (repsectively, oriented up).

Stabilizations and *destabilizations* of Legendrian knots



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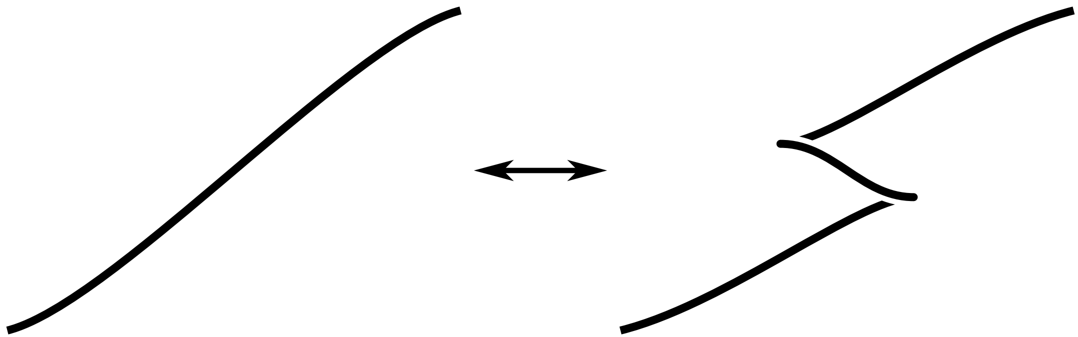

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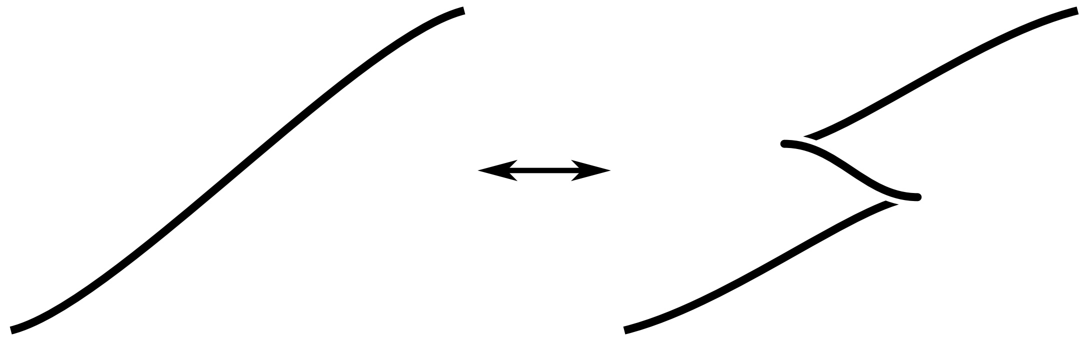

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Two topologically equivalent Legendrian knots become equivalent after some number of stabilizations (D.Fuchs–S.Tabachnikov, 1997).

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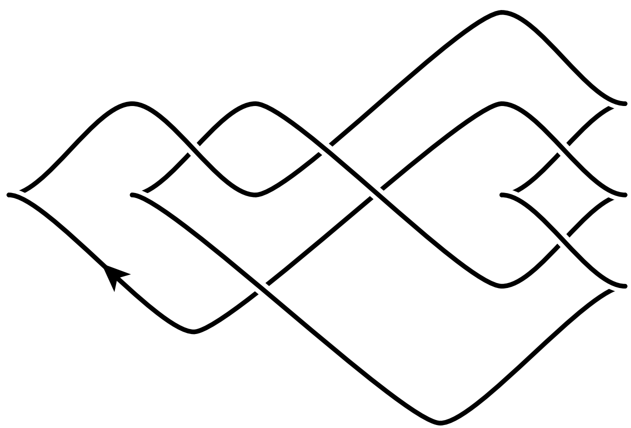
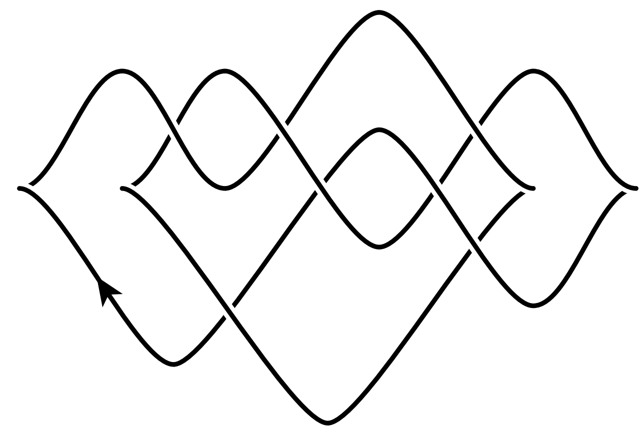
Example (knot type 5_2):



$\not\sim$



Methods to distinguish Legendrian knots

Algebraic invariants (Yu. Chekanov, Ya. Eliashberg, D. Fuchs, L. Ng, P. Pushkar', P. Ozsváth, Z. Szabó, D. Thurston)

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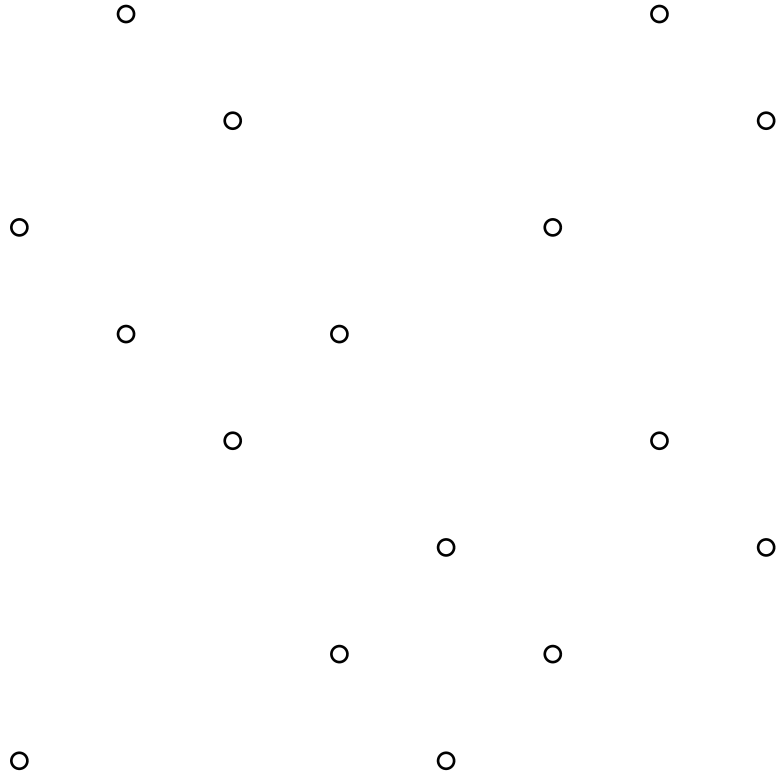
Our approach: Giroux's convex surface + rectangular diagrams

Rectangular diagram of a knot (or link)



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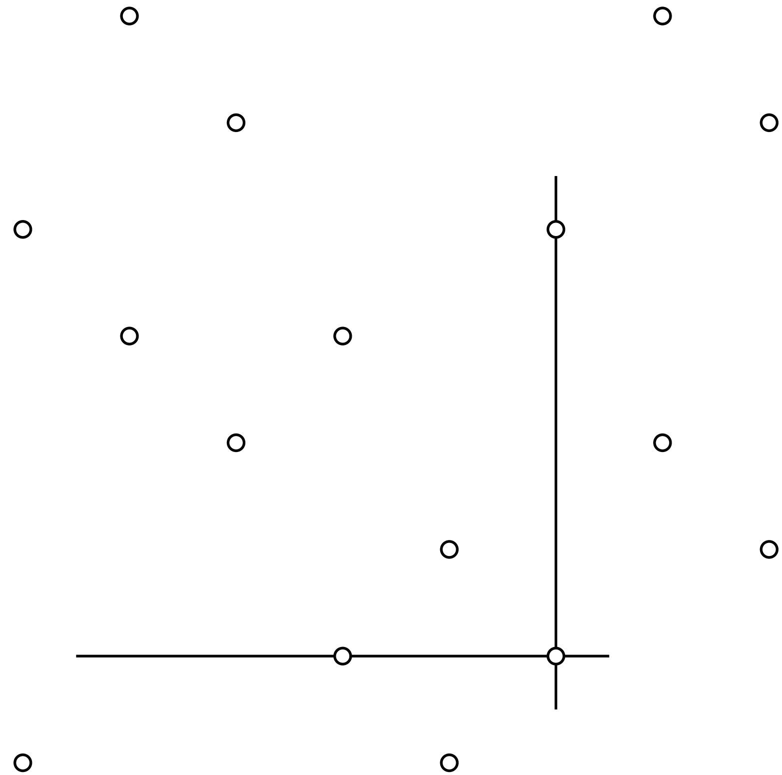
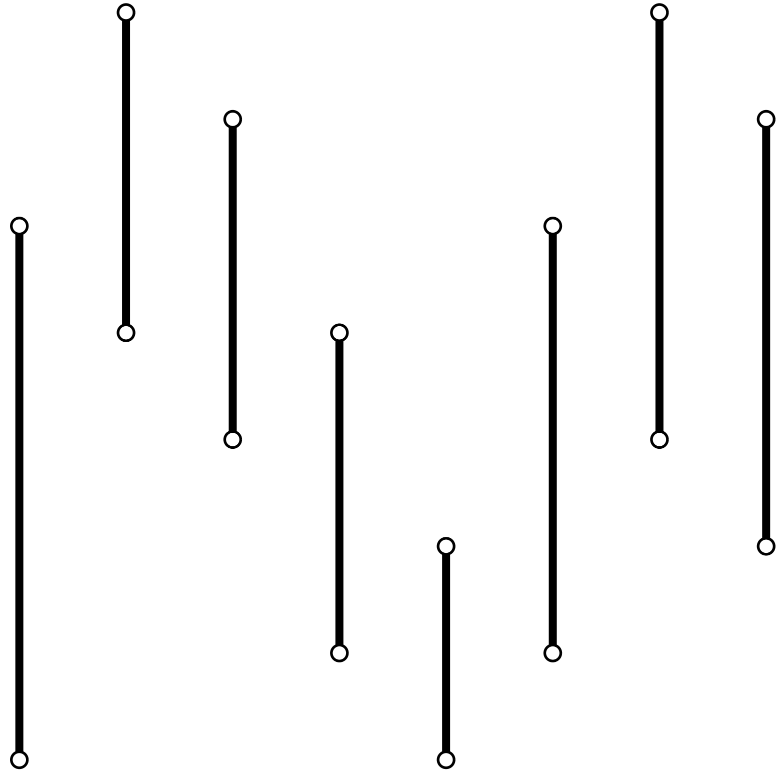


Rectangular diagram of a knot (or link)



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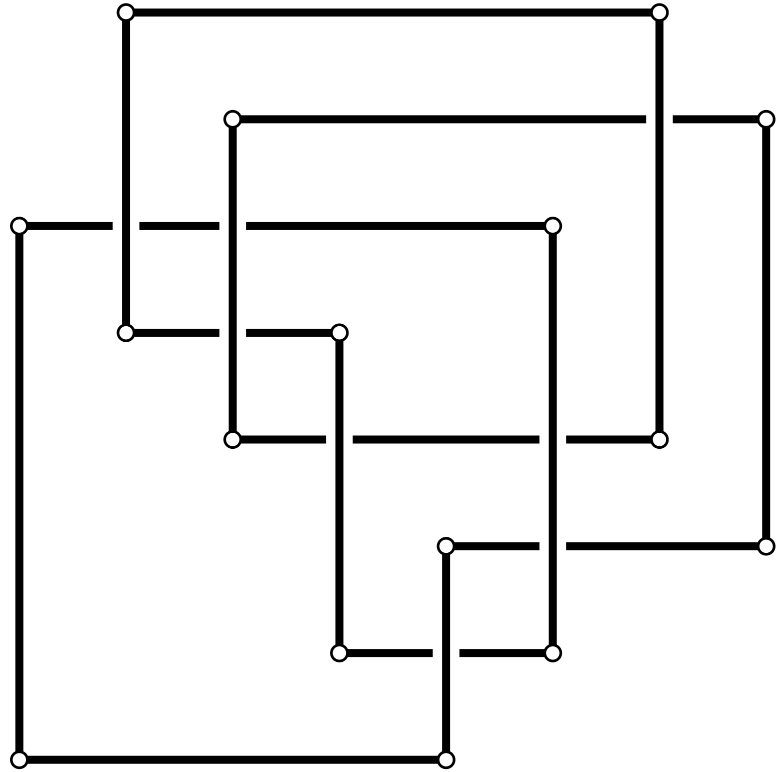

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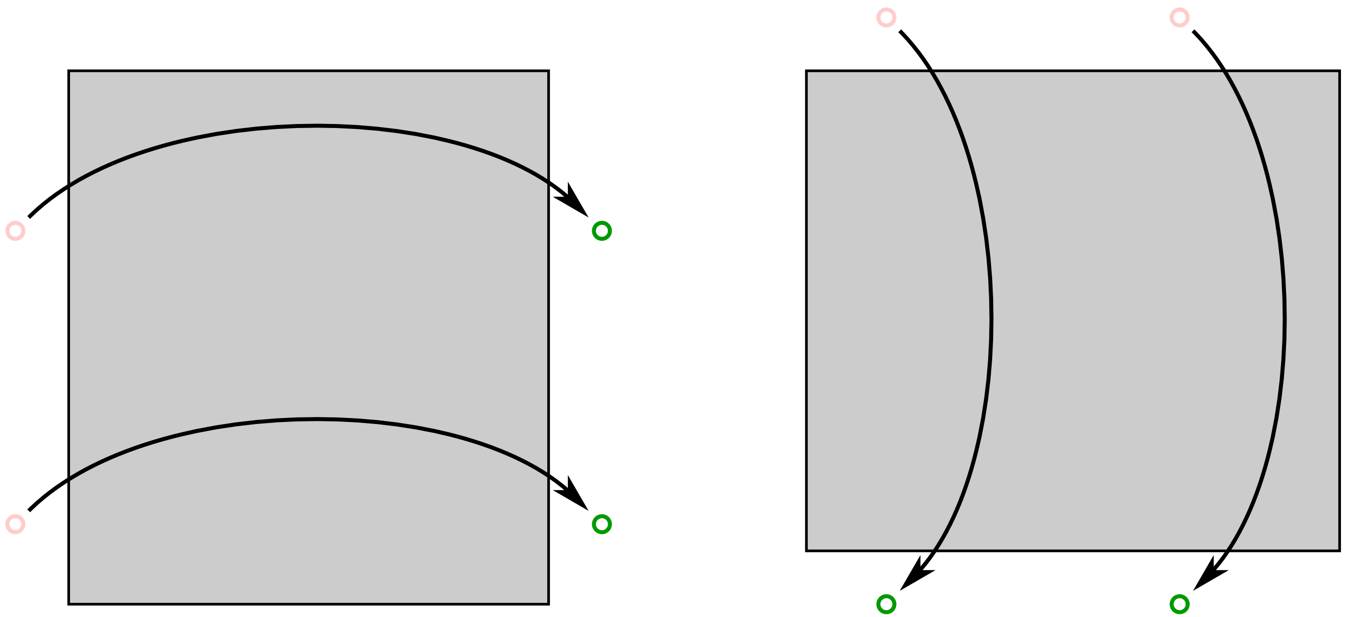
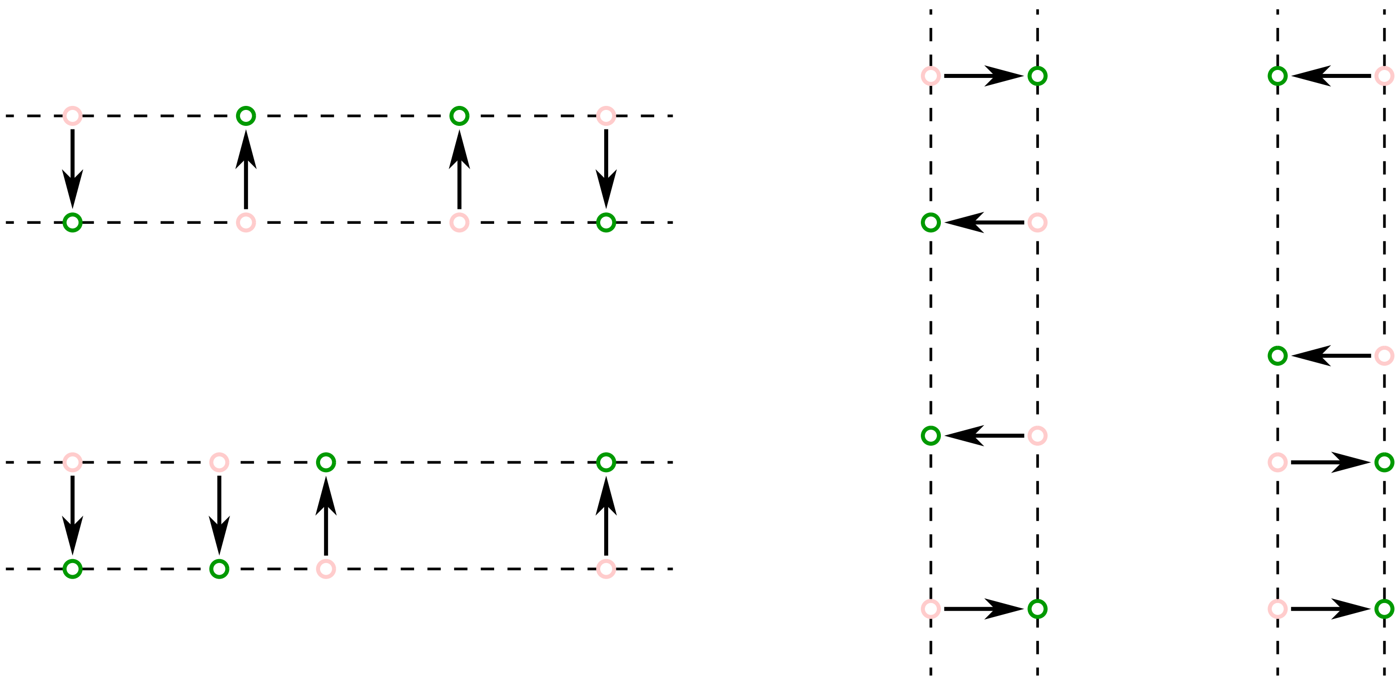
- stabilizations and destabilizations (alter the number of vertices).

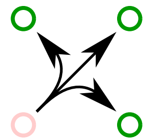
Cyclic shifts:



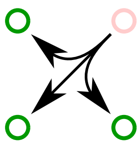
Exchange moves:

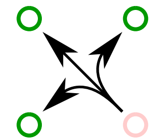


Stabilizations:

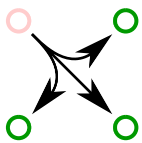


Type I





Type II



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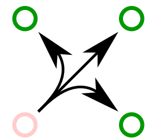

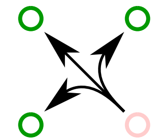

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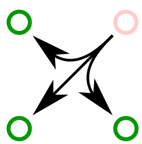

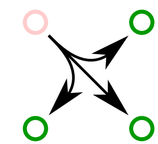

The inverse of a stabilization is *a destabilization*.
For oriented rectangular diagrams, stabilizations and destabilizations are further subdivided into four subtypes $\overrightarrow{\mathrm{I}}$, $\overrightarrow{\mathrm{II}}$, $\overleftarrow{\mathrm{I}}$, $\overleftarrow{\mathrm{II}}$.

Morphisms between knots

A morphism from a knot K_1 to a knot K_2 is an orientation preserving diffeomorphism $(\mathbb{S}^3, K_1) \to (\mathbb{S}^3, K_2)$ viewed up to isotopy (within the class of such diffeomorphisms).

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Thm. *Every morphism of rectangular diagrams can be represented by a sequence of elementary moves.*

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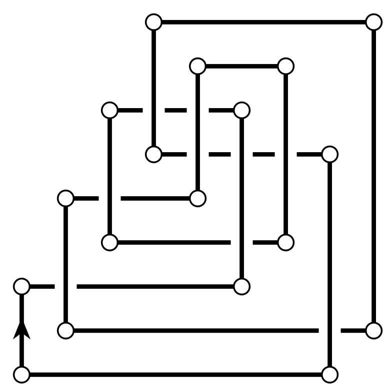

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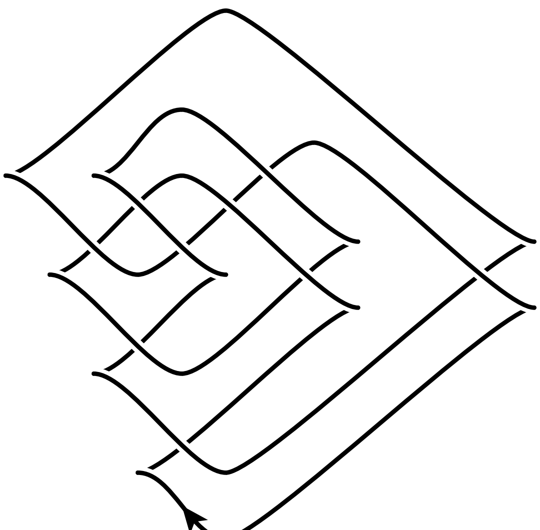
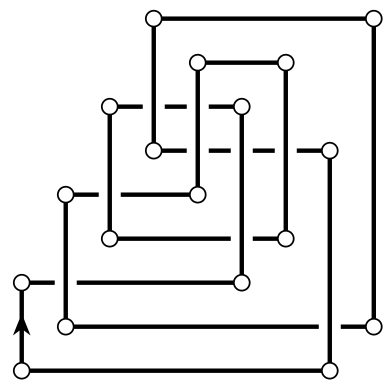


$\mathscr{L}_+(R)$



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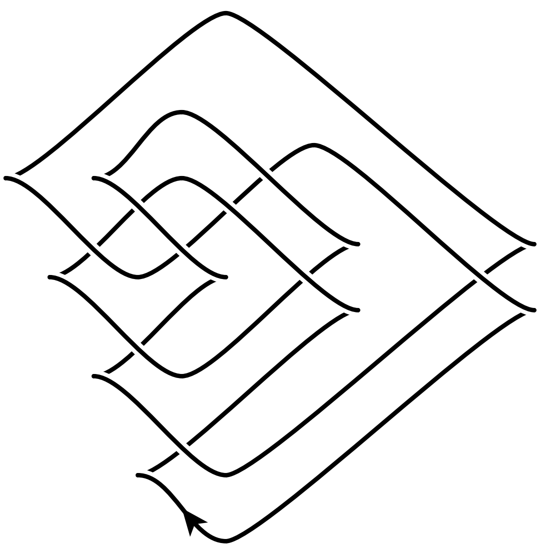

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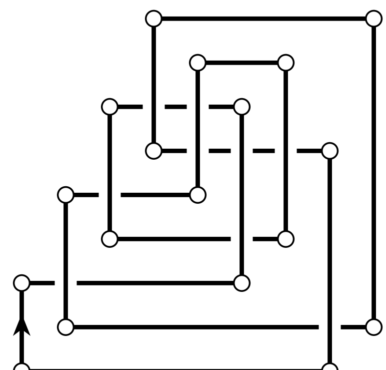
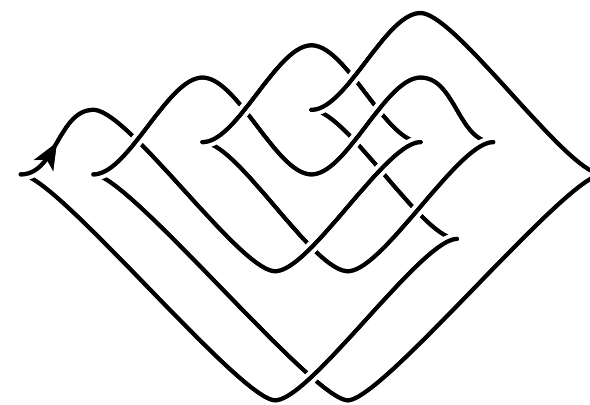


R



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If $\mathscr{L}_+(R_1) = \mathscr{L}_+(R_2)$ (respectively, $\mathscr{L}_-(R_1) = \mathscr{L}_-(R_2)$), then R_1 and R_2 are related by a sequence of elementary moves not including type II (respectively, type I) (de)stabilizations.

$$\frac{\text{Rectangular diagrams}}{\text{exchange moves, type I (de)stabilizations}}$$

$$\updownarrow \mathscr{L}_+\text{-construction}$$

Legendrian knot types

$$\updownarrow \mathscr{L}_-\text{-construction}$$

$$\frac{\text{Rectangular diagrams}}{\text{exchange moves, type II (de)stabilizations}}$$

Thm. (I.D., M.P., 2013) *The Legendrian types* $\mathscr{L}_+(R)$ *and* $\mathscr{L}_-(R)$ *are 'independent': if* $\mathscr{L}_+(R)$ *admits a destabilization, then this destabilization can be presented by a sequence of elementary moves without distorting* $\mathscr{L}_-(R)$.

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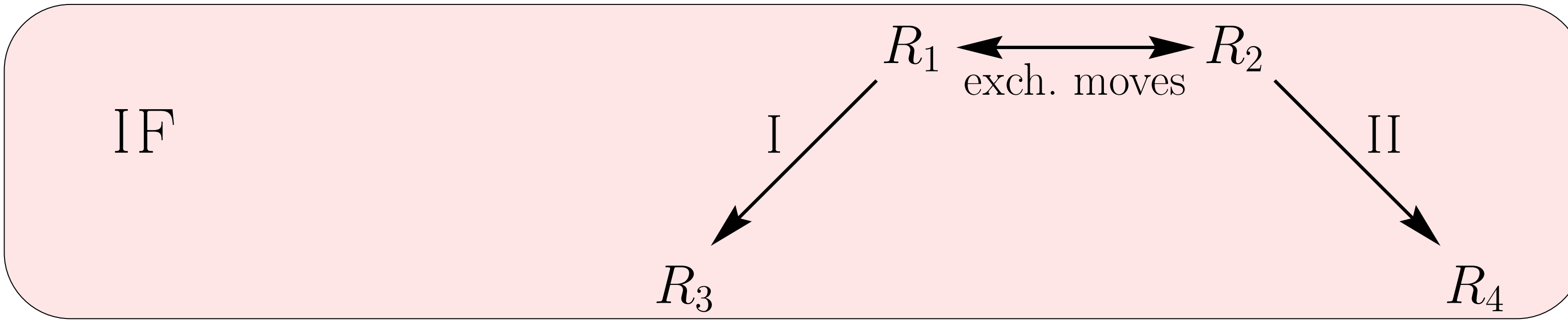

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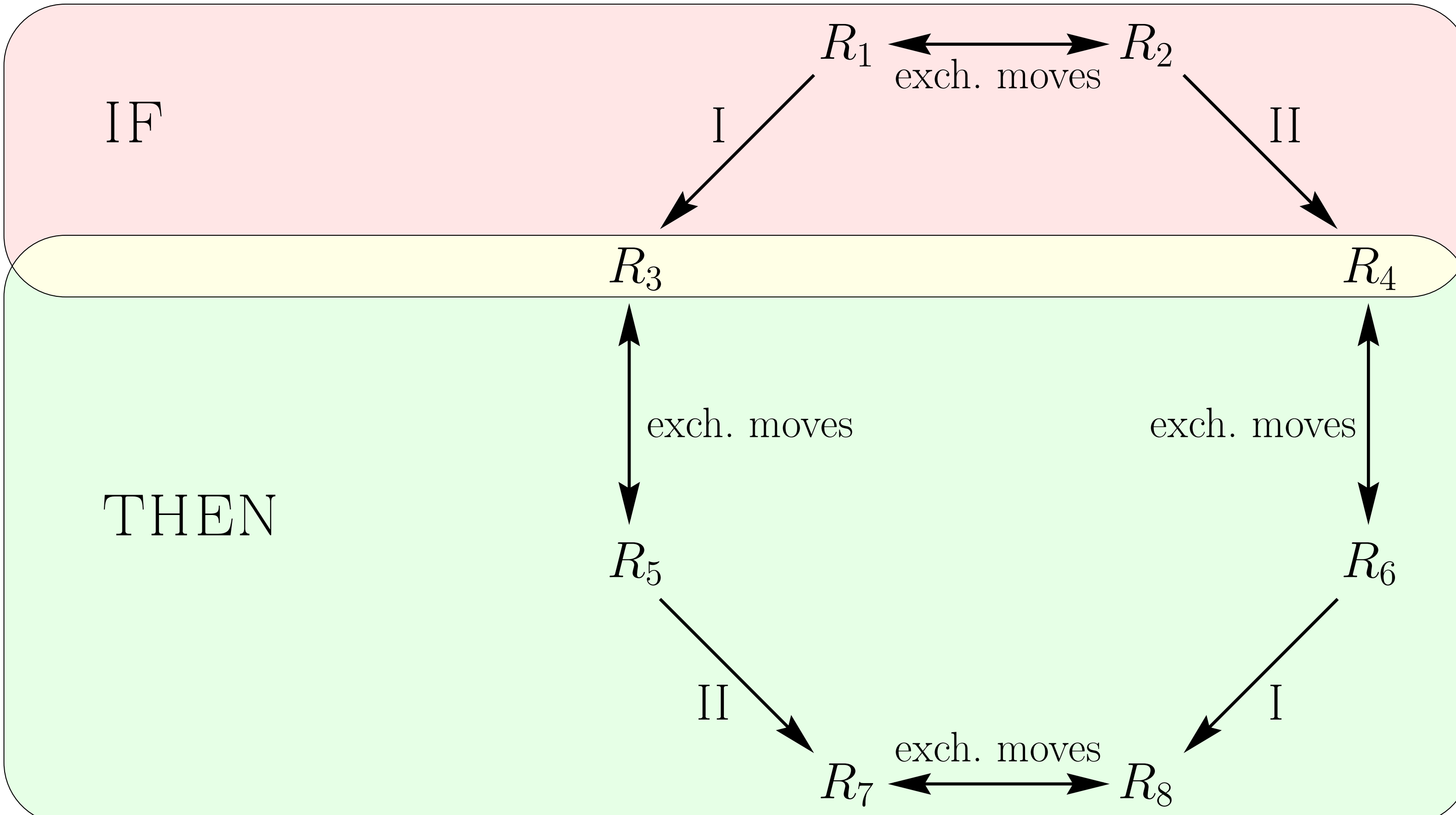
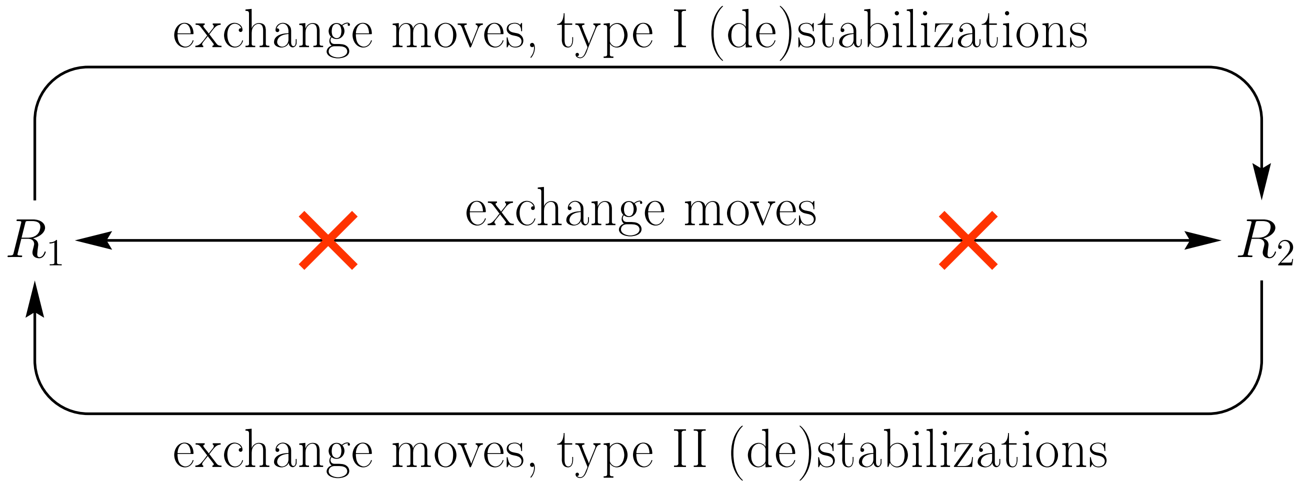


Thm. *If $\mathscr{L}_\pm(R_1) = \mathscr{L}_\pm(R_2)$, but R_1 and R_2 are not exchange-equivalent, that is*



then the round trip yields a non-trivial element of the symmetry group.

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Lemma. Suppose that $\{g_1, \ldots, g_m\}$ is a generating set for $\mathrm{Sym}(R)$, and for all $i = 1, \ldots, m$, the element g_i can be realized by a sequence of elementary moves including k or less stabilizations of each subtype ($\overrightarrow{\mathrm{I}}$, $\overrightarrow{\mathrm{II}}$, $\overleftarrow{\mathrm{I}}$, and $\overleftarrow{\mathrm{II}}$). Then any element of $\mathrm{Sym}(R)$ can be realized by a sequence of elementary moves with at most k stabilizations of each subtype.

The algorithm

Let R_1 and R_2 be two given rectangular diagrams representing isotopic knots. We have to decide whether or not $\mathscr{L}_+(R_1) = \mathscr{L}_+(R_2)$.

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3. Apply k stabilizations of each of the subtypes $\overrightarrow{\mathrm{I}}$ and $\overleftarrow{\mathrm{I}}$ to R_1' and R_2 to obtain R_3 and R_4, respectively.

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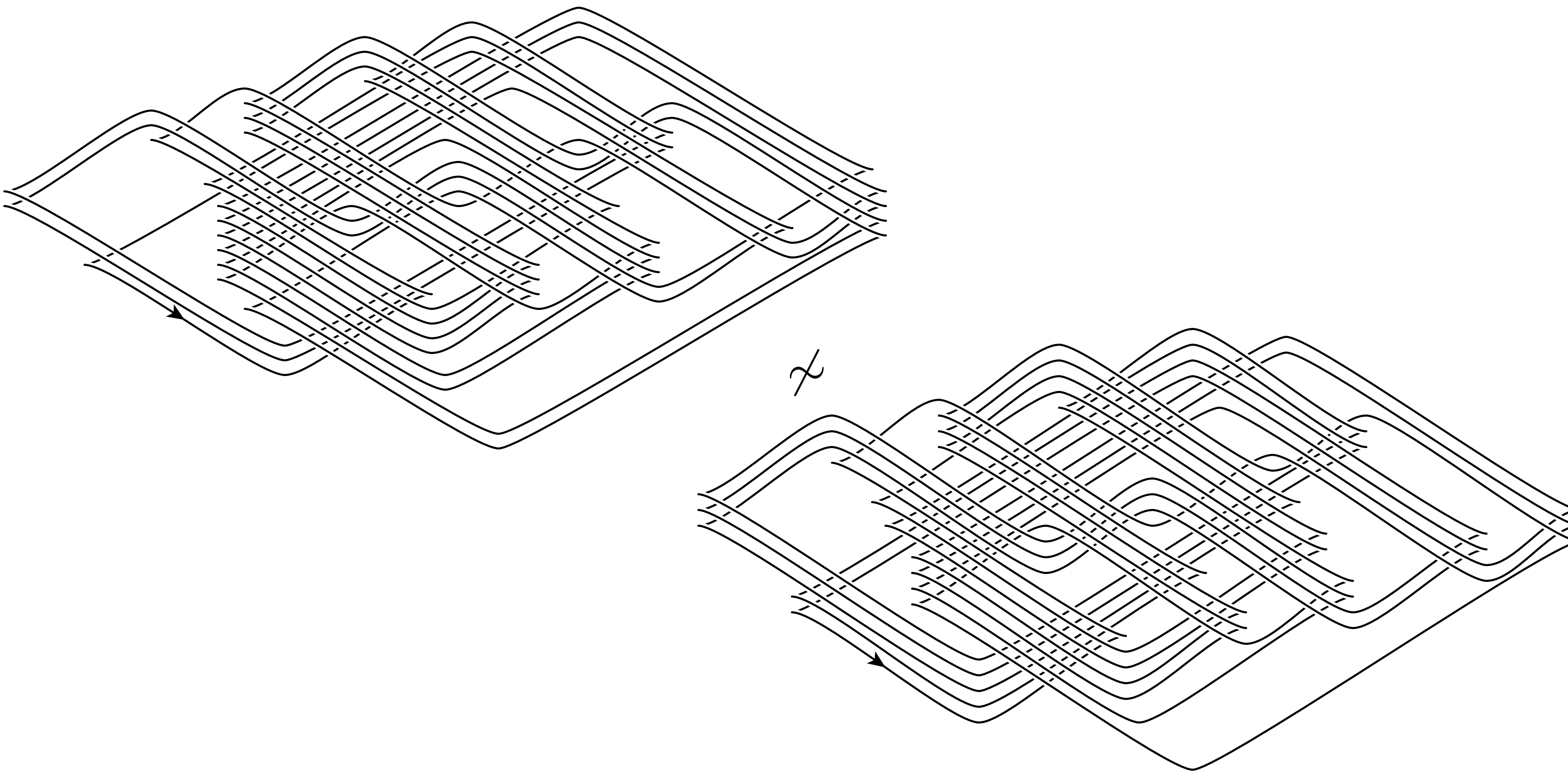
3. Apply k stabilizations of each of the subtypes $\overrightarrow{\mathrm{I}}$ and $\overrightarrow{\mathrm{II}}$ to R_1' and R_2 to obtain R_3 and R_4, respectively.

4. Check whether or not R_3 and R_4 are related by a sequence of exchange moves.

These two knots cobound an annulus in $\mathbb{S}^3$ tangent to ξ_+ along the entire boundary.



Let $r_0, r_1, r_2, \ldots, r_{2n} = r_0$ be rectangles in $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with sides parallel to those of the fundamental domain (which is a square) such that:

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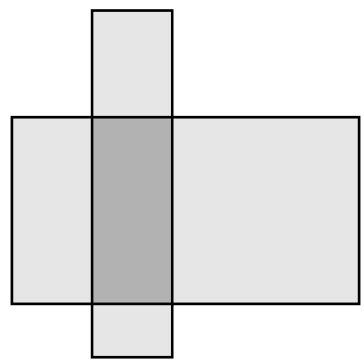
- for all $i = 0, 1, \ldots, 2n-1$, the intersection $r_i \cap r_{i+1}$ is the top right vertex of r_i and the bottom left vertex of r_{i+1};
- if $j > i+1$ then the intersection $r_i \cap r_j$ (if not empty) is disjoint from the vertices of r_i and r_j:



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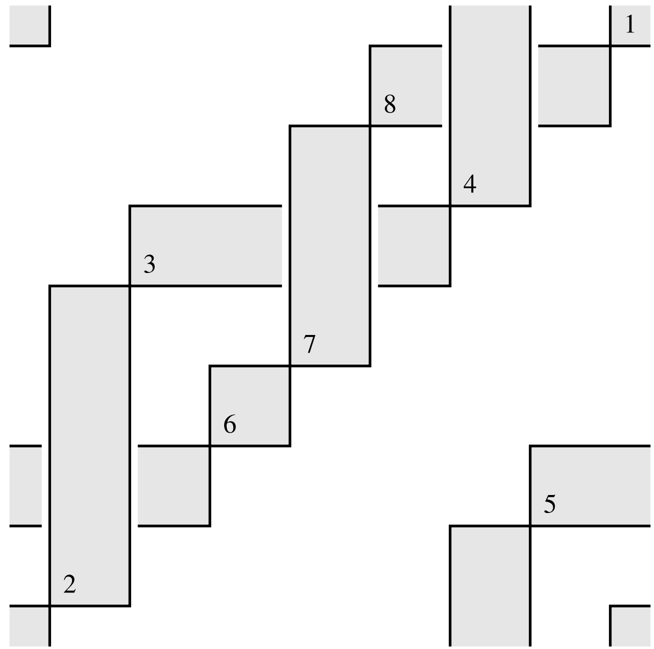

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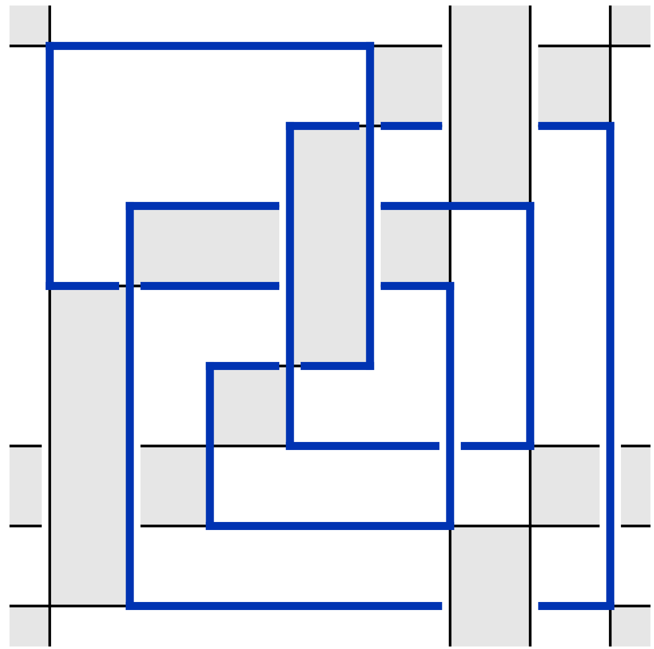

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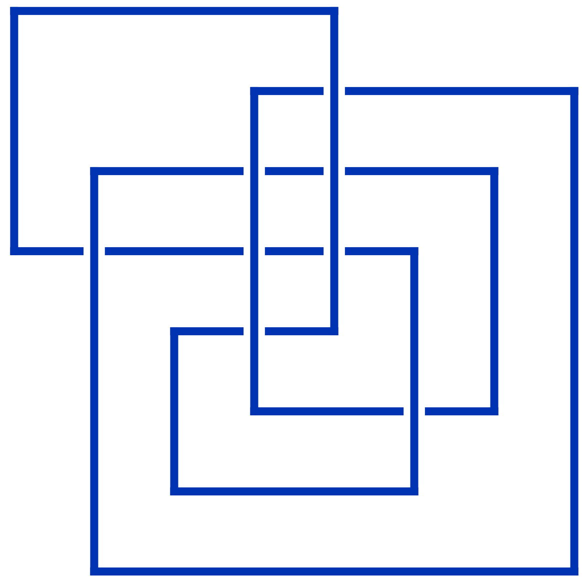

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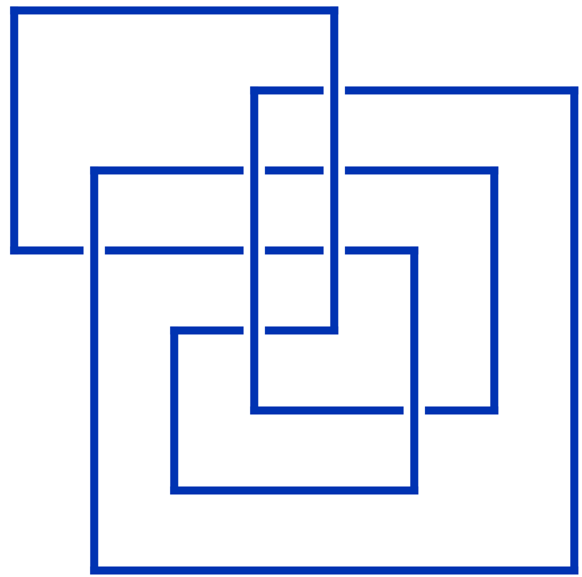

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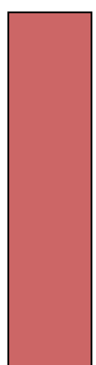

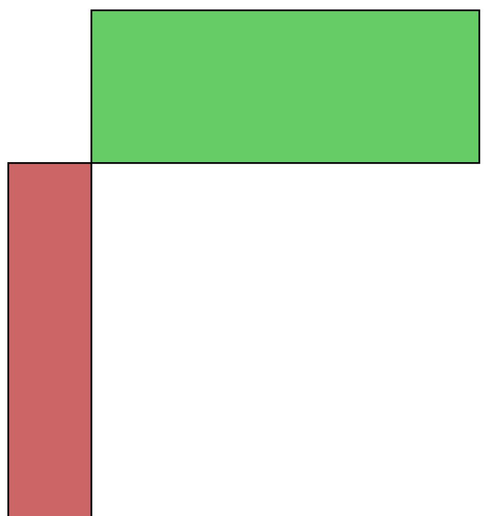

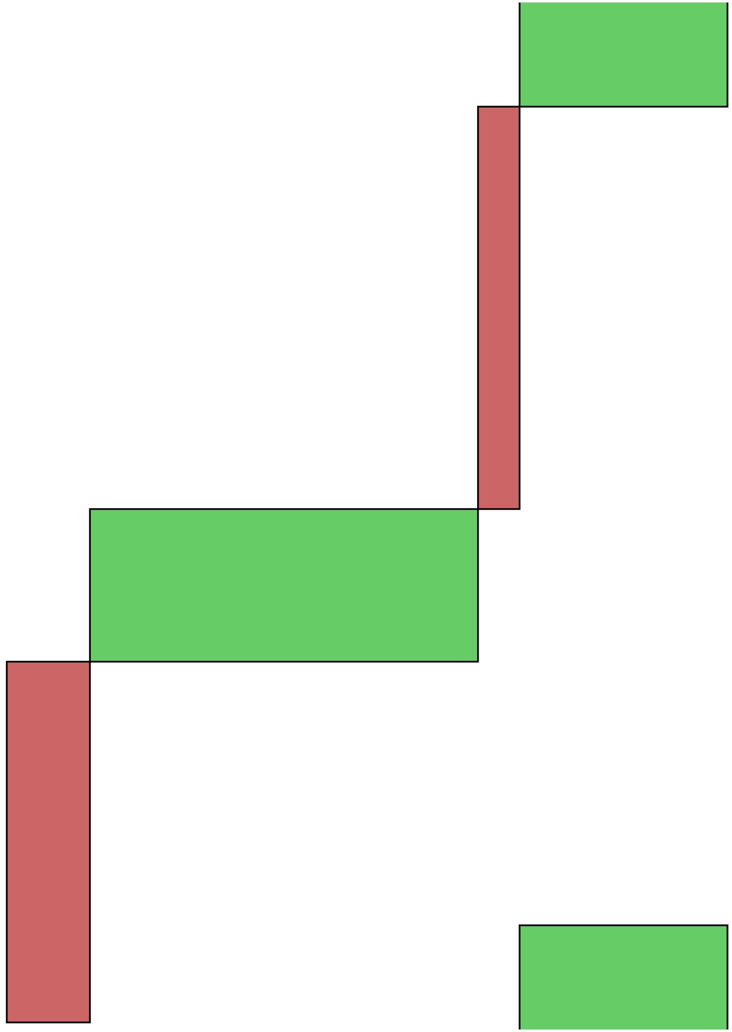

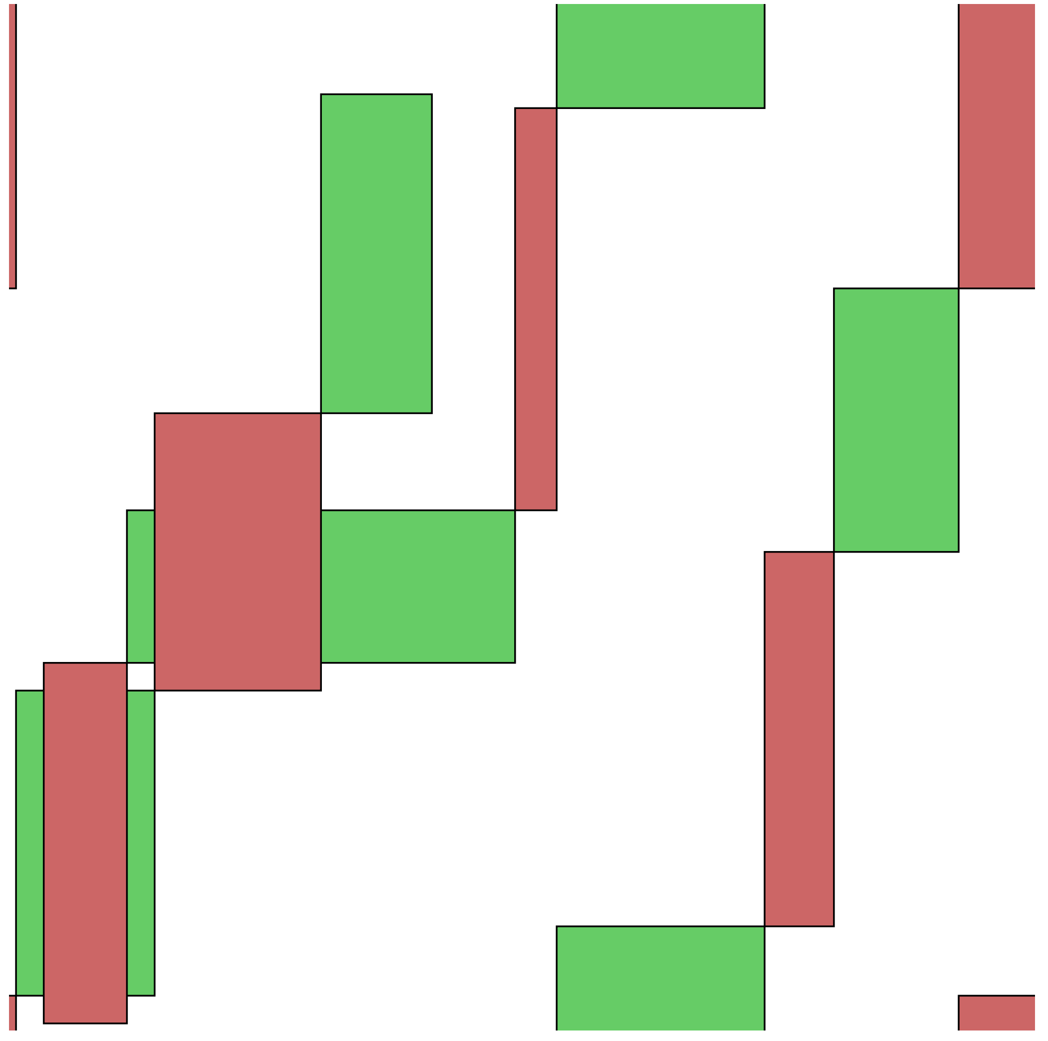

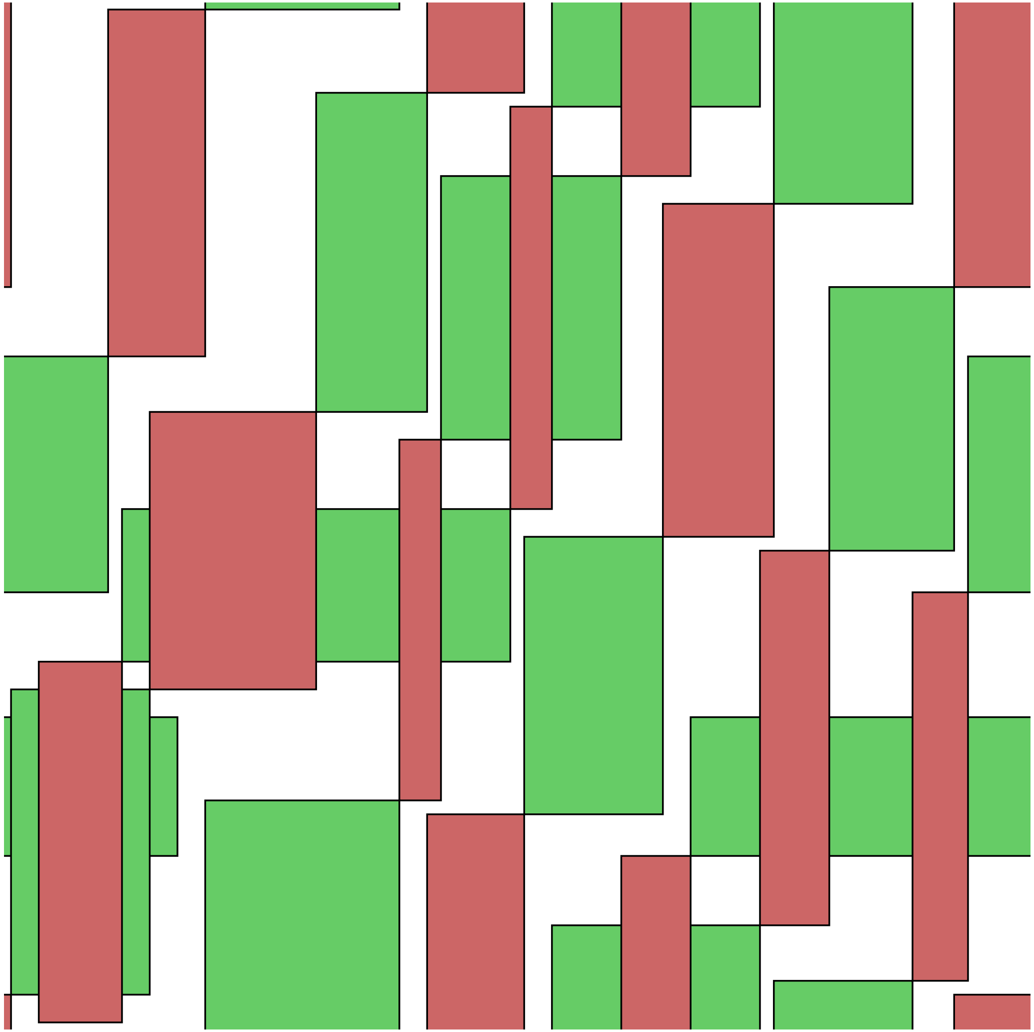

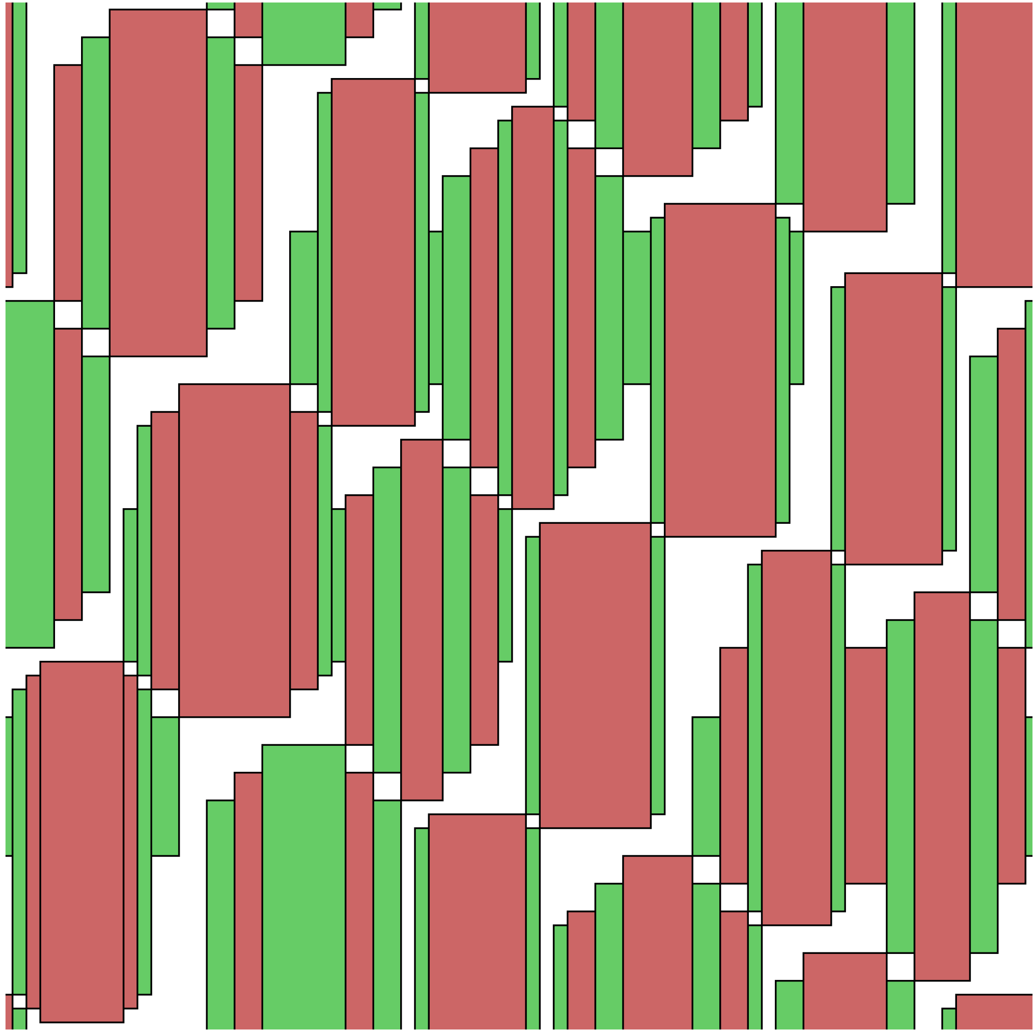

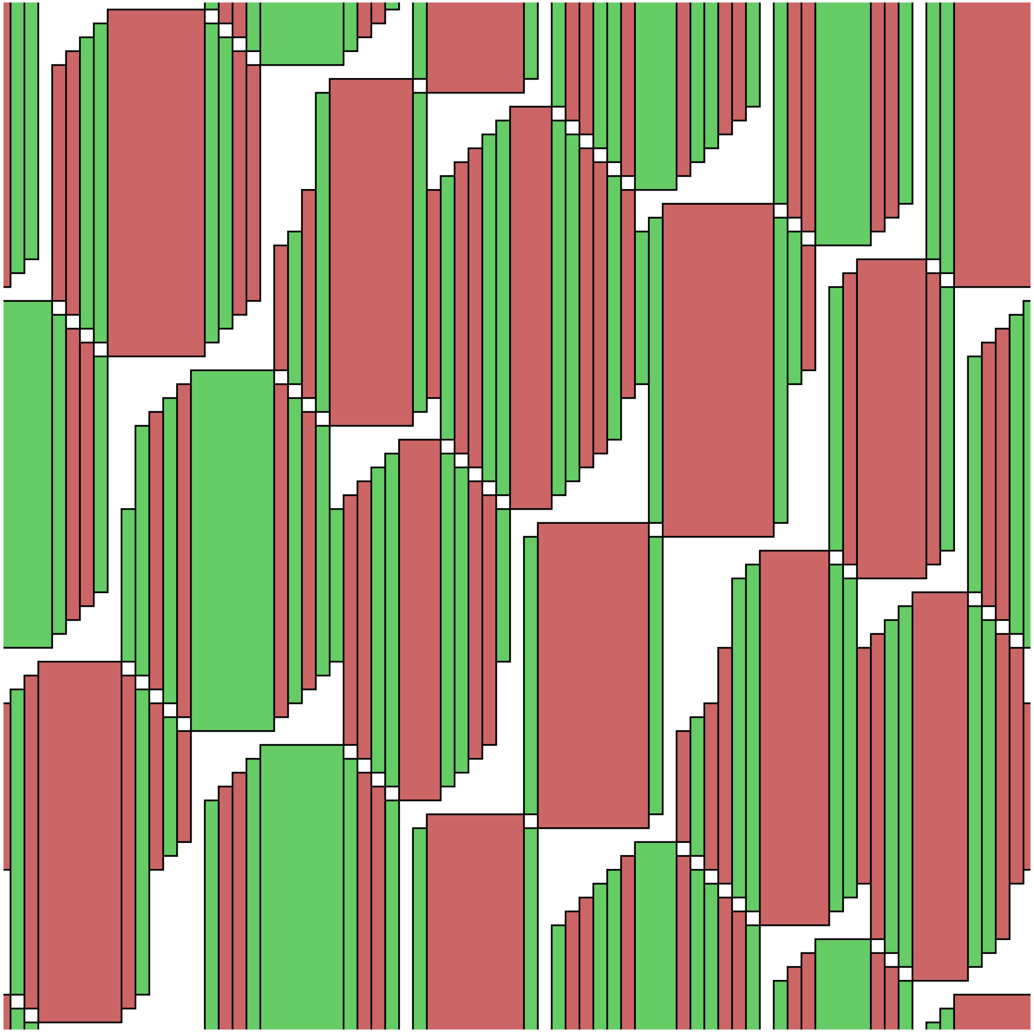

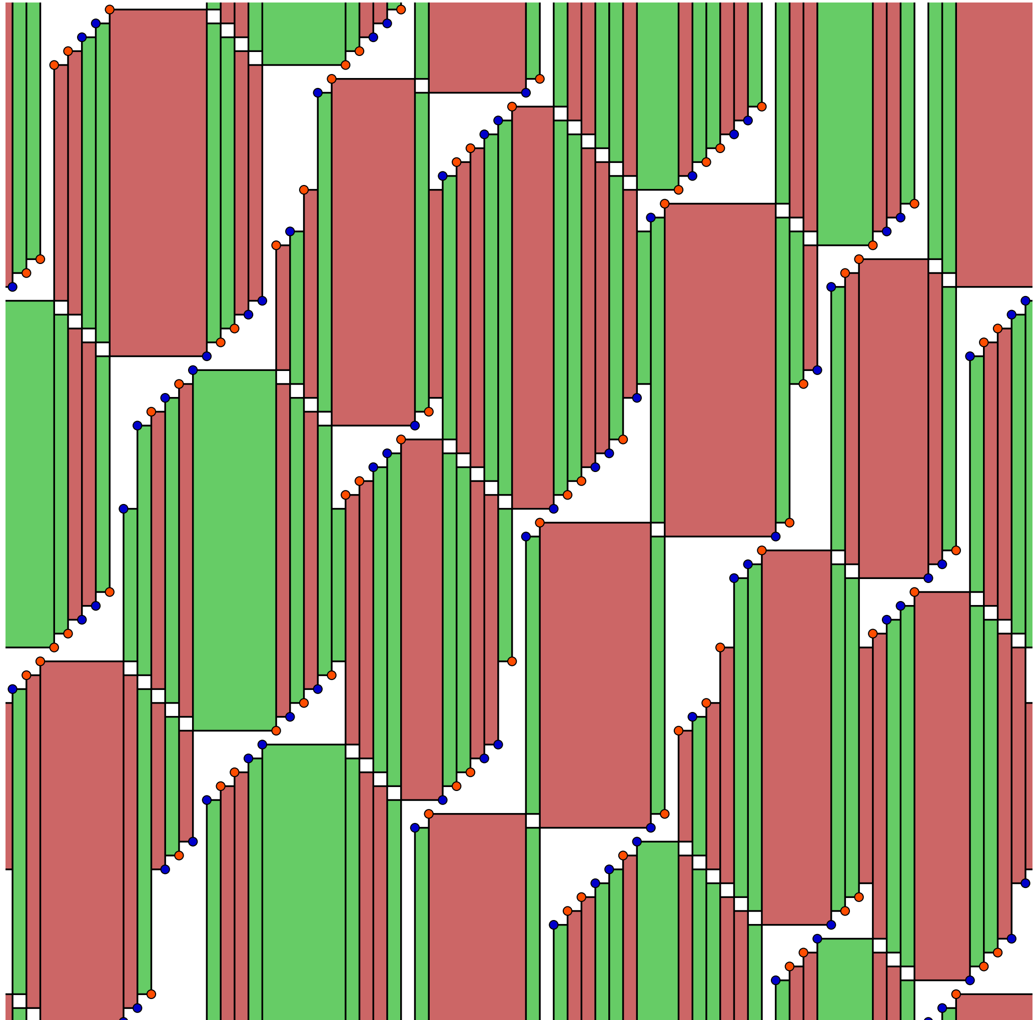

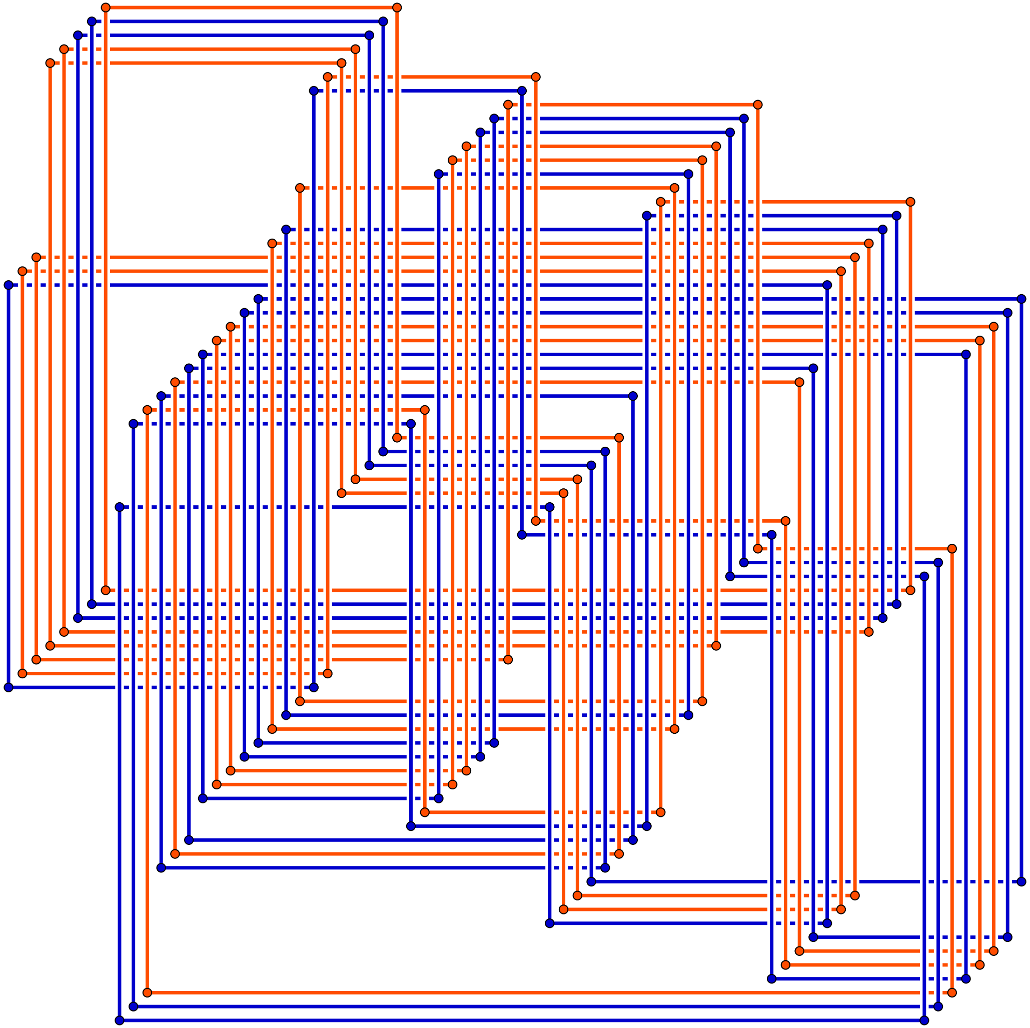

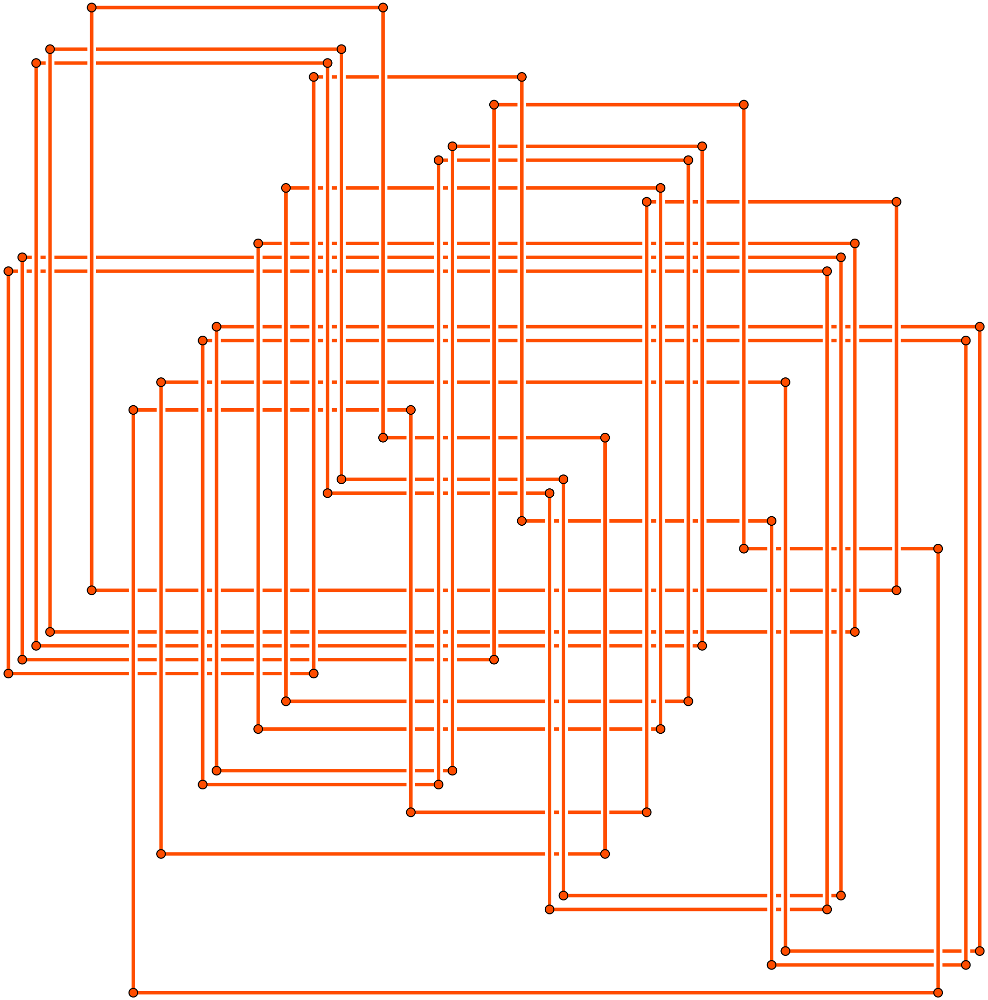

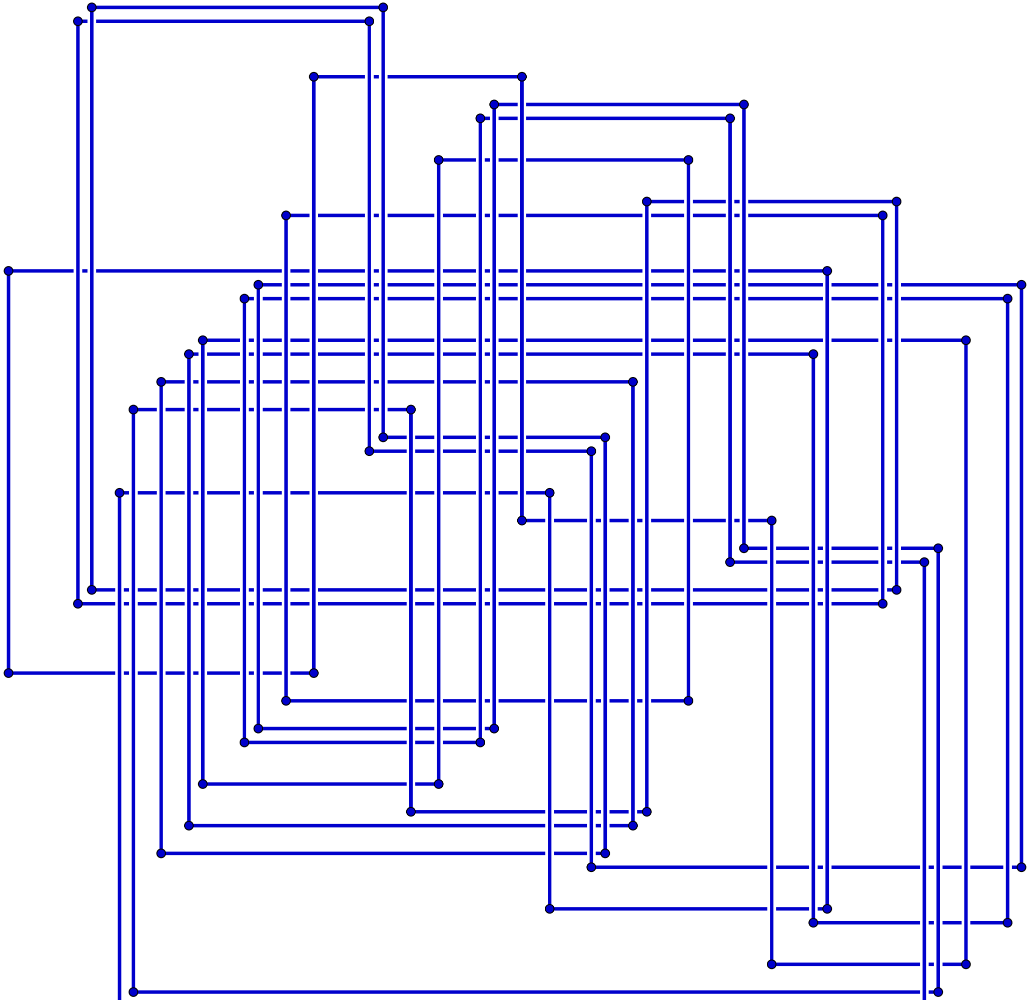

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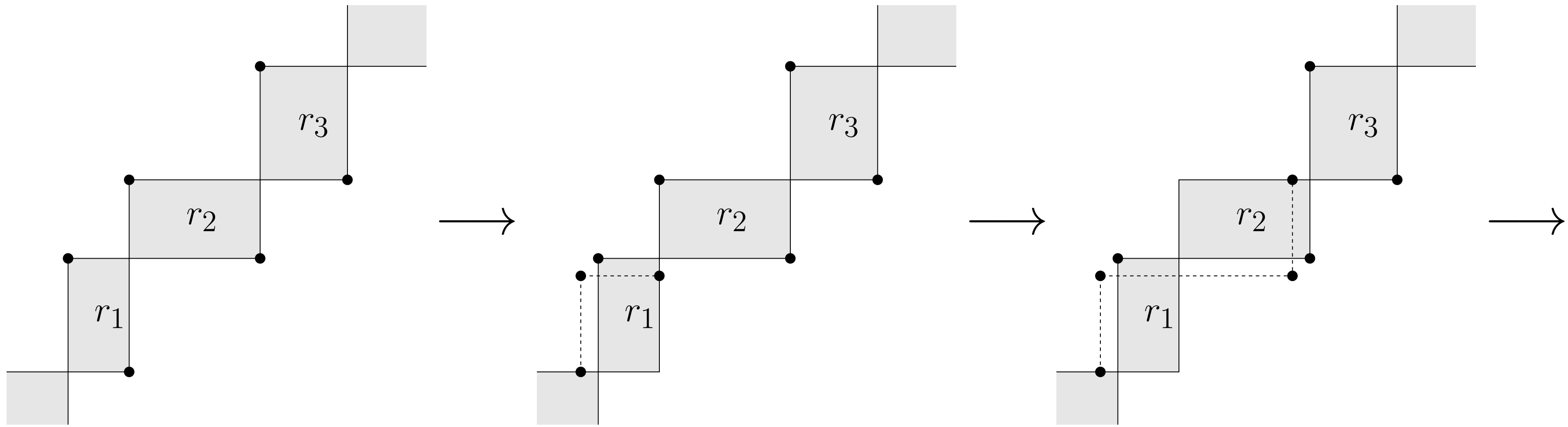

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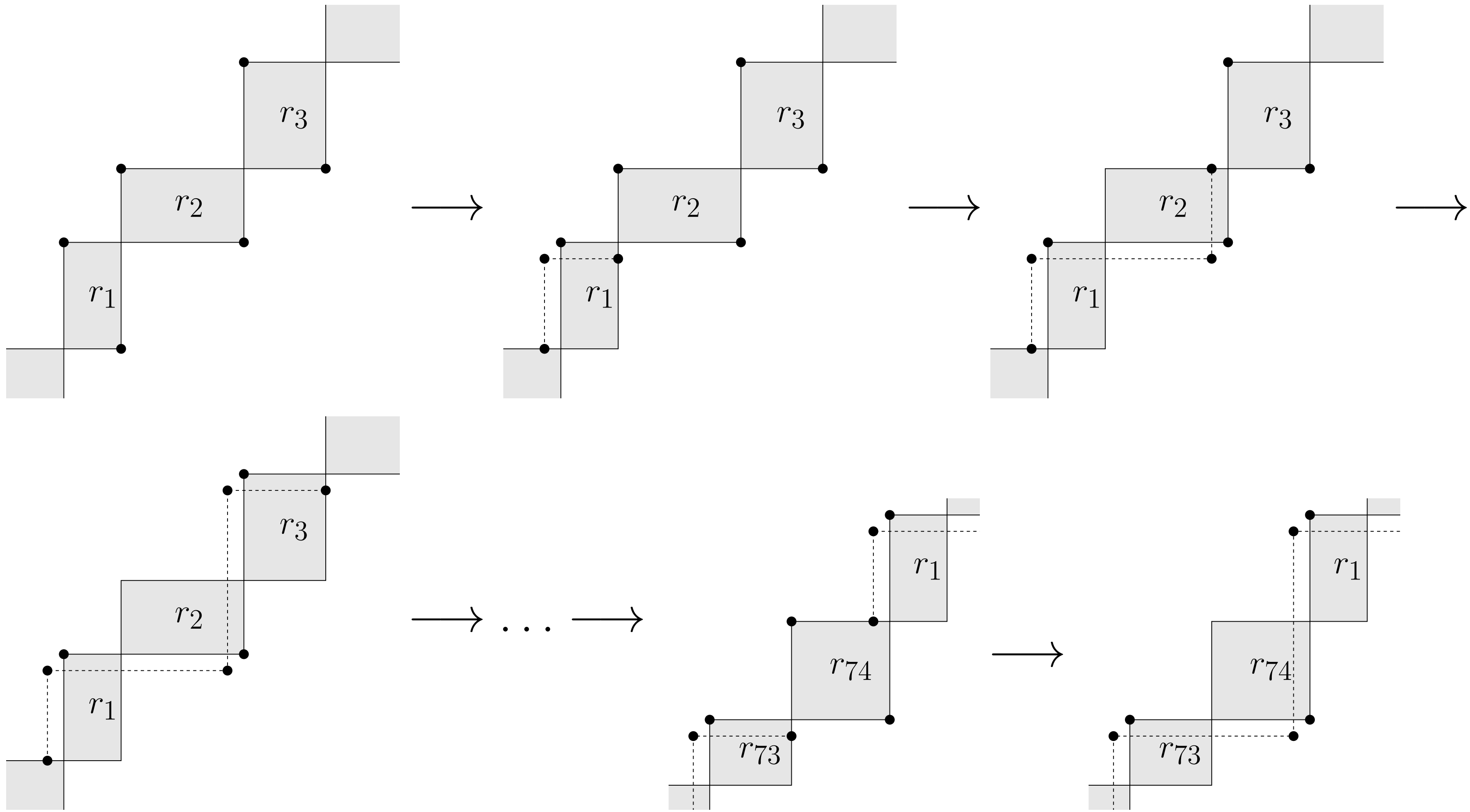

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- for no $p > 1$ the polynomial $\Delta(t^p)$ has a self-reciprocal factor of degree 20 with integer coefficients.

Thank you!

The idea behind the construction