Computers, complex curves, and Khovanov homology

Kyle Hayden (joint with Isaac Sundberg and Alan Du)

April 28, 2022
What is Khovanov homology?

\[ L \leadsto \text{Kh}(L) \quad \text{(bigraded } \mathbb{Z}\text{-module)} \]

\[ \Sigma \subset S^3 \times [0, 1] \]

\[ \leadsto \text{Kh}(\Sigma) : \text{Kh}(L_0) \to \text{Kh}(L_1) \]
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Categorifies the Kauffman bracket and the Jones polynomial.
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Related tools and invariants:

- **Skein modules for 3-manifolds**  
  (Turaev ’88, Przytycki ’91)

- **Khovanov homology for links in thickened surfaces**  
  (Asaeda-Przytycki-Sikora ’04)

- **Khovanov-Rozansky invariants for 4-manifolds**  
  (Morrison-Walker ’12, Morrison-Walker-Wedrich ’19,  
  Manolescu-Neithalath ’20)

- **Slicing obstructions in $B^4$ and other 4-manifolds**  
  (Rasmussen ’04, Manolescu-Marengon-Sarkar-Willis ’19)
Q1: How can we use the maps $\text{Kh}(\Sigma)$ to study surfaces $\Sigma \subset B^4$?
Q2: Can we use computers to study these cobordism maps?
Q3: Can we understand these cobordism maps for complex curves?
**Theorem** (H-Sundberg, 2021)

Khovanov homology can distinguish smooth surfaces in $B^4$ that are isotopic via homeos but not diffeos of $B^4$ (rel $\partial$).

The examples will be holomorphically embedded in $B^4 \subset \mathbb{C}^2$, and we’ll use a braid-theoretic perspective to calculate these maps.
Khovanov homology
Khovanov homology is (relatively) computable.

**Example.** \( J = 17nh_{73} \) (the “positron knot”)

Dirk Schuetz’s *KnotJob* calculates \( \text{Kh}(J) \) in <1 sec (on 2013 MacBook).
R.I.P.

August 2013 - April 25, 2022

Performed > 1.6 million KnotJob calculations.
Khovanov homology is (relatively) computable.

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How can we use Khovanov homology in dimension 4?

Most 4D applications use Rasmussen’s invariant $s(K) \in 2\mathbb{Z}$

$$\Sigma$$

$K$

$B^4$

$g(\Sigma) \geq |s(K)|/2$
$\Sigma \subset S^3 \times [0, 1]$

$\Rightarrow$ \hspace{2cm} $\text{Kh}(\Sigma) : \text{Kh}^{h,q}(L_0) \to \text{Kh}^{h,q+\chi(\Sigma)}(L_1)$
**Theorem** (Jacobsson, Bar-Natan, Khovanov, Morrison-Walker-Wedrich)

$\text{Kh}(\Sigma)$ is invariant (up to sign) under smooth isotopy rel $\partial \Sigma$.

Can these maps distinguish non-isotopic embeddings?

Rasmussen, Tanaka ('05-06): Not if $\Sigma$ is closed (i.e., $\partial \Sigma = \emptyset$).
Sundberg-Swann ('21): $\text{Kh}(\cdot)$ can distinguish certain disks in $B^4$.

- $D, D' : \emptyset \rightarrow K \implies \text{Kh}(D), \text{Kh}(D') : \mathbb{Z} \rightarrow \text{Kh}(K)$

- Showed $\text{Kh}(D)(1) \neq \text{Kh}(D')(1)$ in $\text{Kh}(K)$. 
(For code that checks nontriviality of Khovanov homology classes, check out imsundberg.github.io)
In fact, Khovanov homology distinguishes many pairs of disks.

$9_{46}$

$6_1$ (stevedore knot)

$15n_{103488}$

$17nh_{73}$ (positron knot)
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Note: Several of these embed holomorphically in $B^4 \subset \mathbb{C}^2$. 

946 

61 (stevedore knot) 

15n_{103488} 

17nh_{73} (positron knot)
Note: Some already distinguished by $\pi_1(S^3 \setminus K) \to \pi_1(B^4 \setminus D)$!

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**Theorem** (H-Sundberg, 2021)

Khovanov homology can distinguish smooth surfaces (of all genera) in $B^4$ that are isotopic via homeos but not diffeos of $B^4$ (rel $\partial$).

Further work:

- **Lipshitz-Sarkar ’21**: Mixed Kh-invariant for non-orientable surfaces

- **Lipshitz-Sarkar ’22**: Distinguish surfaces via spectral sequence relating Kh of equivariant knot and its quotients
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The equivariant perspective works for all of these disks.

9_{46}  

6_1 (stevedore knot)  

15n_{103488}  

17n_{h73} (positron knot)
Instead, we’ll consider braids and braided surfaces.
Braided surfaces
Rudolph ’84: Ribbon surfaces in $B^4$ are isotopic to braided surfaces.
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Informal 3D definition

\textit{braided surface} = a finite collection of parallel disks joined by half-twisted bands (with “ribbon” intersections)

Formal 4D definition

A smooth surface $\Sigma$ in $B^4 \approx D^2 \times D^2$ is braided if the first-coordinate projection $D^2 \times D^2 \to D^2$ restricts to a simple branched covering $\Sigma \to D^2$. 
Braided surfaces are encoded by factorizations $\beta = \prod_{k=1}^{\ell} w_k \sigma_{i_k}^{\pm 1} w_k^{-1}$. 
Rudolph ‘83, Boileau-Orevkov ‘01:

\[
\{ \text{Bounded complex curves in } B^4 \subset \mathbb{C}^2 \} \leftrightarrow \{ \text{pos. braided surfaces} \} \leftrightarrow \left\{ \prod_k w_k \sigma_i^{+1} w_k^{-1} \right\}
\]
Inequivalent factorizations can yield inequivalent surfaces.

Example (Auroux): \[(\sigma_2^{-2}\sigma_1\sigma_2^2)\sigma_1^2 = (\sigma_1^{-3}\sigma_1\sigma_3)\sigma_1\sigma_2 \in B_3\]
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Computers, complex curves, and Khovanov homology
Braided surfaces and Khovanov homology
Khovanov homology sees a surface as a sequence of link diagrams.
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Khovanov homology has elementary maps associated to

• births/deaths
• saddles
• Reidemeister moves (R1, R2, R3)

Braided surfaces do not require R3.

⇒ Easier to work with in theory, practice, and on computer.

[For code that computes cobordism maps for braided surfaces (in the near future), check out Alan Du’s github.]
Combine these to get other cobordisms. For example:

Note: The induced map is the “identity” on \( \bowtie \) but kills \( \bowtie \).
How does Khovanov homology relate to braids / braid factorizations?
**Plamenevskaya’06:** Khovanov homology contains a braid invariant.

**Recipe:** Oriented (braid-like) resolution at every crossing, assign every circle the element $x \in \mathcal{A} = \mathbb{Z}[x]/(x^2)$. 

$$\psi(\beta) \in \text{Kh}$$
Plamenevskaya: If $\Sigma$ is positively braided, then $\psi(\partial \Sigma) \xrightarrow{\text{Kh}(\Sigma)} \pm 1$.

✓ Useful for showing that $\text{Kh}(\Sigma)$ is nonvanishing.

✗ Not useful for distinguishing positively braided surfaces.
Does Khovanov homology contain a more sensitive invariant associated to a (quasipositive) braid factorization?

**Example.** The knot $10_{148}$ bounds $\geq 2$ positively braided surfaces:

Note: No obvious reason that equivariant considerations could apply here; $10_{148}$ has no symmetries.
**Candidate?** Oriented resolutions at core $\sigma_i$ of each band $w\sigma_iw^{-1}$ and *disoriented* resolutions elsewhere. [And add labels 1, $x$ appropriately...]
E.g., this element gets sent to $\pm 1$ by $\text{Kh}(\Sigma)$ and to 0 by $\text{Kh}(\Sigma')$. ✓
So can we apply this to our desired examples?
Sadly, such elements $\phi_1 \in C\text{Kh}(\beta)$ aren’t usually cycles. But can often be “completed” to a cycle by adding some $\phi_0 \in \ker (C\text{Kh}(\Sigma))$. 
Computers, complex curves, and Khovanov homology
Lastly, some motivational bad news.
Gompf ’91: The knot $12n_{121}$ bounds infinitely many knotted tori whose branched covers are distinguished by Seiberg-Witten invariants.

**Theorem (Du-H, 2022)**

Infinitely many of these induce the same map $\text{Kh}(12n_{121}) \to \mathbb{Z}$.

Can we enrich Khovanov homology to distinguish surfaces like this?

**strong parallel:**

\[
\begin{align*}
\{ \text{invariants of surfaces} & \} \quad \leftrightarrow \quad \{ \text{invariants of} \quad \text{4-manifolds} & \}
\end{align*}
\]
Thank you!