FRACTIONAL DEHN TWISTS & LEFT-ORDERS

Joint with Diana Hubbard
(work in progress)
Good surfaces

- at least one boundary
Good homeomorphisms

- fix all boundary
- permute punctures
Good homeomorphisms

- up to isotopy fixing boundaries and punctures
Heuristic: the fractional Dehn twist coefficient $\text{FDTC}(\gamma,C)$ measures the amount of twisting $\gamma$ affects near $C$.

Ex: A (positive) Dehn twist along a curve parallel to $C$ has $\text{FDTC} = 1$. 
Why Study the FDTC?

- $M_{\varphi}$ is a 3-mfld with torus boundary components.
Why Study the FDTC?

- $M^4$ is a 3-mfd with torus boundary components
- Canonical way to fill in to get closed 3-mfd $(S, \mathcal{U})$
  (open book)
Why Study the FDTU?

Keeping track of punctures traces a braid in \((S, \mathcal{U})\)
Why Study the FDTC?

Q:

- How could one open book give us any info on \( M^3 = (S, \ell) \)?

- How can info on one braid say anything about its closure?
Why Study the FDTC?

- braids/open books have FDTCs
- essential laminations of \((S, \mathcal{F})\) (Gabai-Oertel)
- taut foliations of \((S, \mathcal{F})\) (Roberts)
- contact structures on \((S, \mathcal{F})\) (Honda-Kazez-Matic, Colin-Honda)
- geometry of \((S, \mathcal{F})\) (Itô-Kawamura)
Why Study the FDTC?

- primeness of $\hat{\beta}$ (Malyutin)
- braid index of $\hat{\beta}$ (Feller-Hubbard)
- genus of surfaces bounded by $\hat{\beta}$ (Ito, Feller)
- geometry of $S^3 - \hat{\beta}$ (Ito-Kawamura)
Thm (Hubbard-T) The FRTC for $\phi: S \to S$ can be computed via any Thurston type left-order on $\text{MCG}(S)$.

- Kazet-Roberts
- Malyutin
- Hto-Kawamura
Thurston's left-orders on MCG

(Short - Wiest)
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Say \( \varphi >_\alpha \text{id} \) if \( \tilde{\varphi}(\tilde{\alpha}) \) to the right of \( \tilde{\alpha} \)
Thurston's left-orders on MCG

• This relationship determines a left-order: i.e. \( f < g \Rightarrow hf < hg \)

• these LOSs might be partial
Thurston's left-orders on MCG

What if $\varphi(\alpha) = \alpha$?
Even if $LO$ is partial, we have:

1. \[ T_c^{-3} <_x T_c^{-2} <_x T_c^{-1} <_x \text{id} <_x T_c <_x T_c^2 <_x T_c^3 \ldots \]

2. $\varphi \in \text{MCG}(S)$ is incomparable to at most one $T_c^k$

$\varphi \in [T_c^k, T_c^{k+1})$  

Def: the floor $\lfloor \varphi \rfloor_x = k$
\[ T_c^{-3} \prec \alpha T_c^{-2} \prec \alpha T_c^{-1} \prec \alpha \text{id} \prec \alpha T_c \prec \alpha T_c^2 \prec \alpha T_c^3 \]

**Def:** \( \psi \in [T_c^k, T_c^{k+1}) \) the floor \( \lfloor \psi \rfloor_\alpha = k \)

**Thm (Hubbard - T):** \( \text{FDTC}(\psi, C) = \lim_{n \to \infty} \frac{\lfloor \psi \rfloor_\alpha^n}{n} \)
Defining the FDTC
Defining the FDTC

$\cdot \text{MCG} \sim \mathbb{R}$
Defining the FDTC (Ito-Kawamuro)

- $\text{MCG} \sim \mathbb{R}$
- Can identify with $\mathbb{R}$ so that $\tilde{T}_c$ is translation by 1.
Defining the FDTC

Def the translation number for $f \in \text{Homeo}^+(S^1)$ is:

$$\lim_{n \to \infty} \frac{f^n(x) - x}{n}$$

and FDTC($\lambda$) = translation $\#(\lambda)$
Does this help compute FDTC?

For braids, there is an algebraic algorithm to determine $L_{\beta_1 \alpha}$.

Q: Does this generalize?

A: Probably not. But yes, in some cases:

- symmetric mapping classes
- braids in $S^3$
- braids in OBD with page
- braids in OBD with symmetric monodromy
Existing algorithms are geometric:
- need to separate into Nielsen-Thurston type
  - reducible
  - periodic
  - Pseudo Anosov
- produce invariant laminations of CP
- understand when arcs are moved to the right
For braids:

\( \beta \) moves \( c_i \) to the right
iff there is a rep \( \omega \) which is \( \sigma_i \) positive
i.e. \( \sigma_i \) only occurs with positive exponent

\[ \omega_i = \sigma_i \sigma_2 \sigma_i^{-1} = \sigma_2^{-1} \sigma_i \sigma_2 =: \omega_2 \]

\( \sigma_i^{-1} \)-positive (or negative)
For braids:

Dehornoy gives algorithm to produce \( \sigma_i \)-positive/ negative/empty word.

\[ \Rightarrow \text{algorithm to compute } L_{\mathcal{P}_1 \mathcal{C}_i}. \]
For general mapping classes:

If $\mathcal{P}$ is rep by a $c_1$-positive word then $\mathcal{P}$ moves $\alpha$ to the right.

- Not clear that every $\mathcal{P}$ moving $\alpha$ to the right has a $c_1$-positive word.
Algebra of Thurston’s Lo’s

For general mapping classes:

If $\phi$ is rep by a $c_i$-positive word then $\phi$ moves $c_i$ to the right

- not clear that every $\phi$ moving $c_i$ to the right has a $c_i$-positive word
How would this algorithm work?

**Facts:**
- $\text{FDTC}(\varphi) = \frac{K}{n}$ with $n$ constrained by $S$ (Ito-Kawamuro)
- If $L\varphi J_\alpha = K$ then $K \leq \text{FDTC}(\varphi) \leq K + 1$

**Idea:**
- There is some $N(S)$ so that $\text{FDTC}(\varphi^{N(S)})$ is an integer.
- Then $L\varphi^{2N(S)}$ determines $\text{FDTC}(\varphi^{2N(S)}) = 2N(S) \cdot \text{FDTC}(\varphi)$
Thank you!