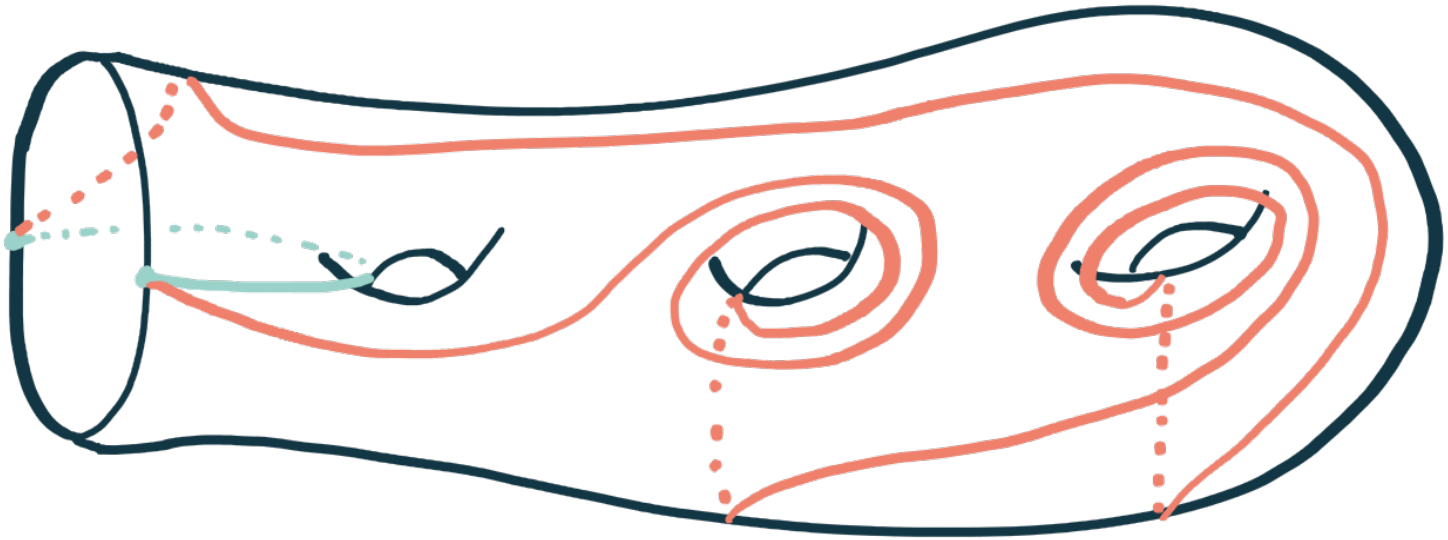
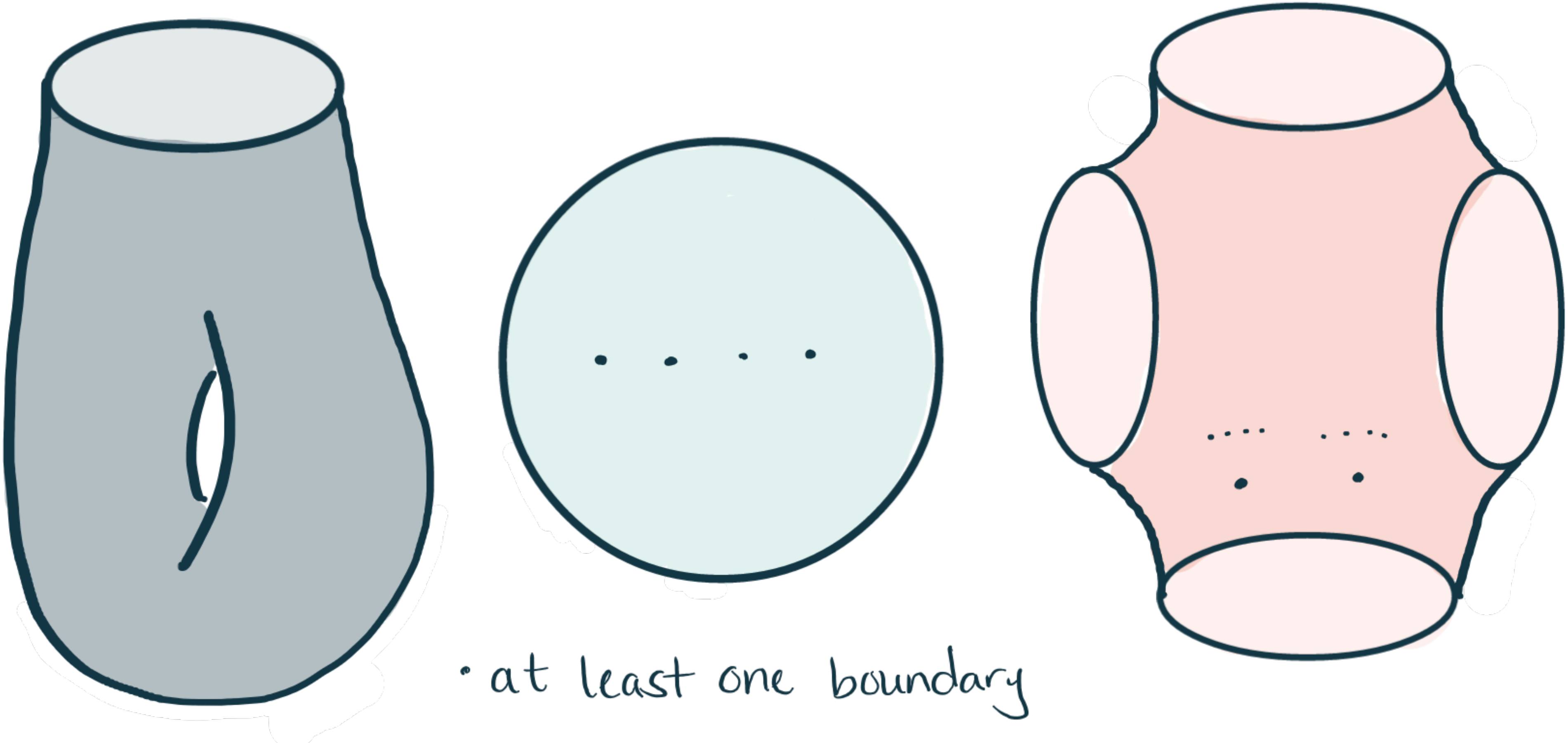
FRACTIONAL DEHN TWISTS & LEFT-ORDERS

joint with Diana Hubbard

(work in progress)

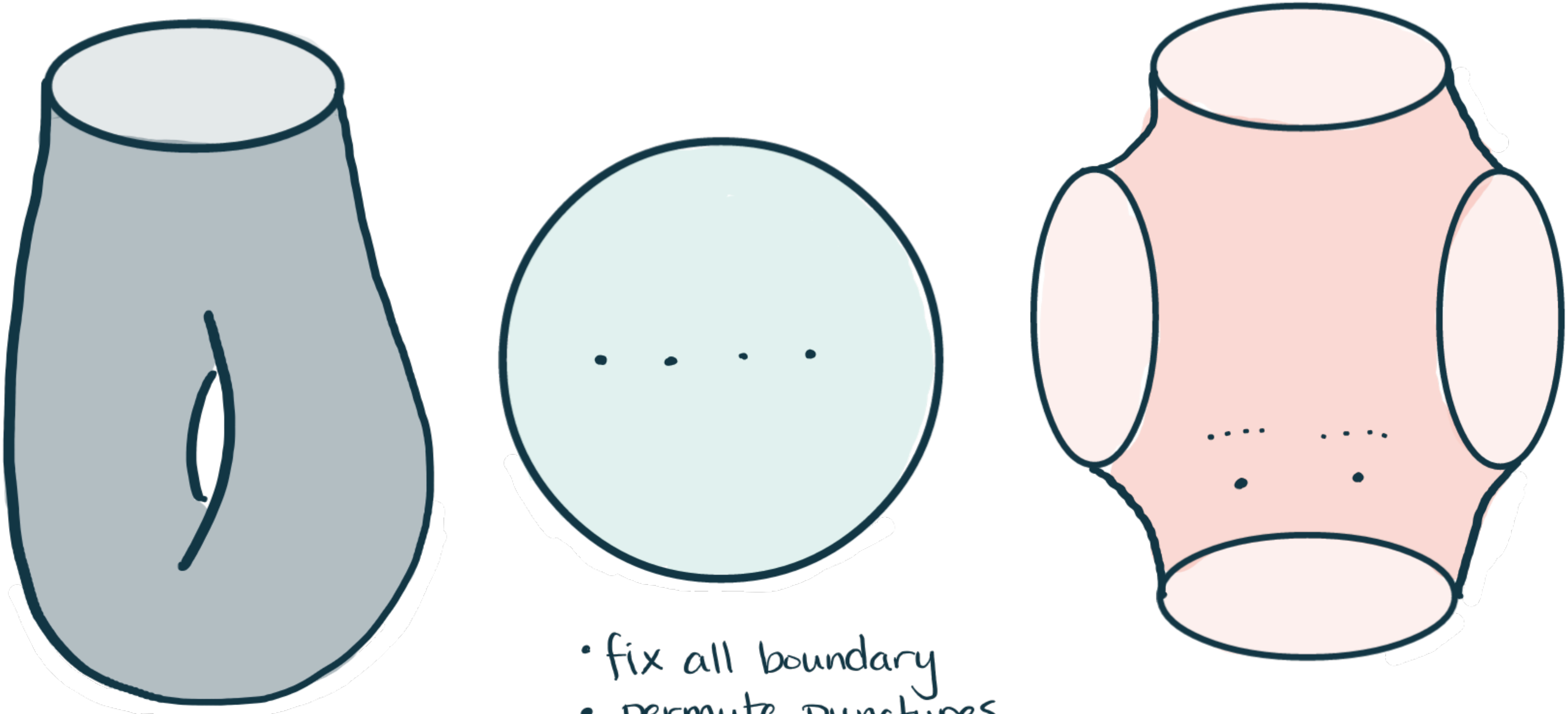


Good surfaces



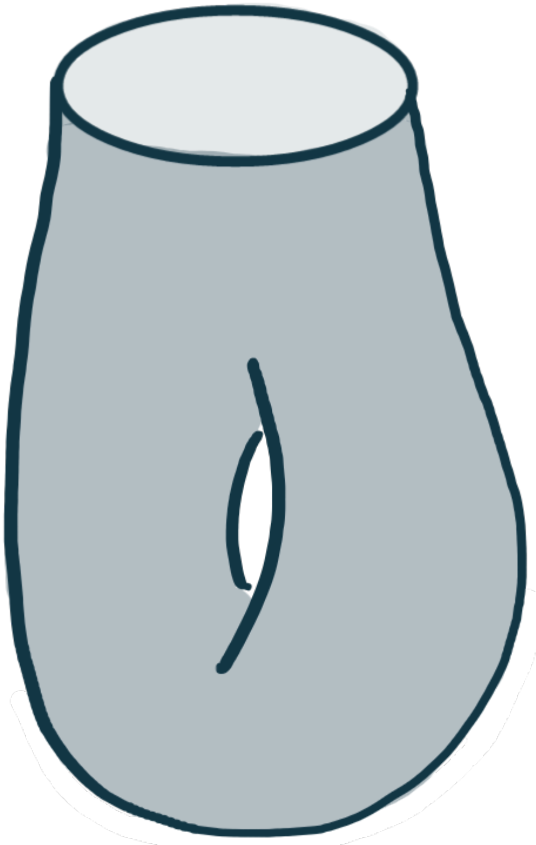
- at least one boundary

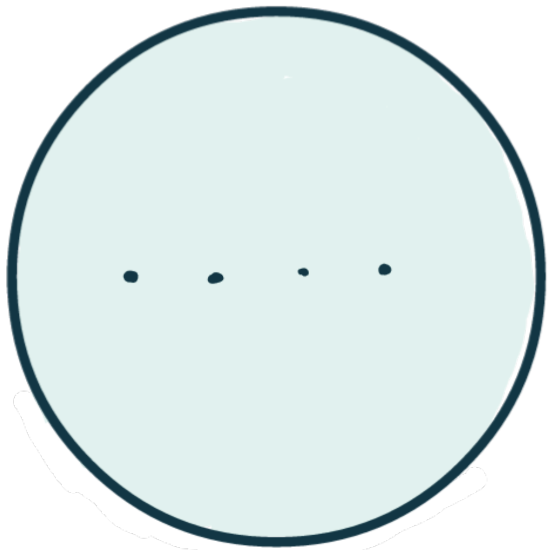
Good homeomorphisms

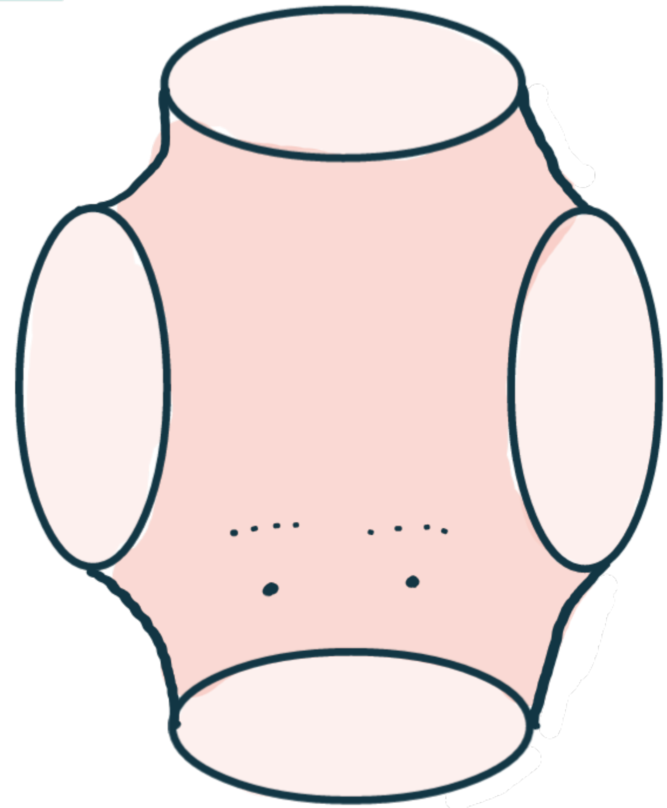


- fix all boundary
- permute punctures

Good homeomorphisms

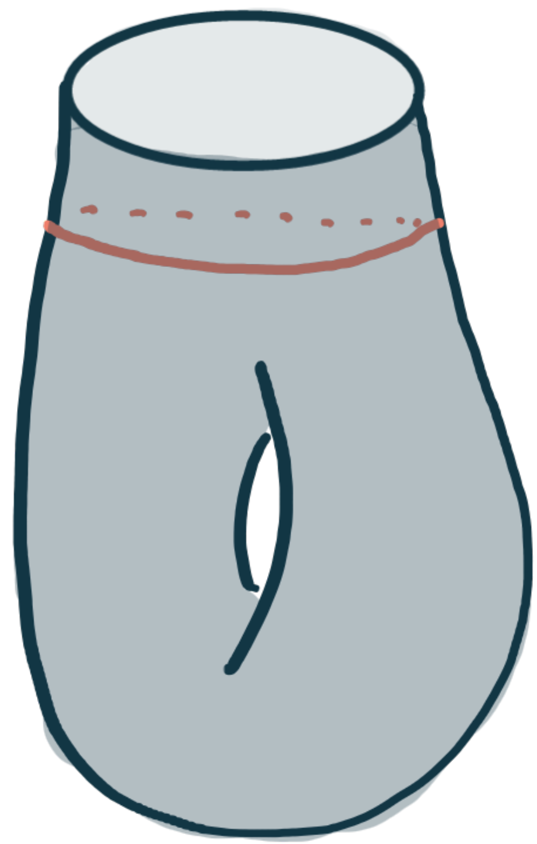






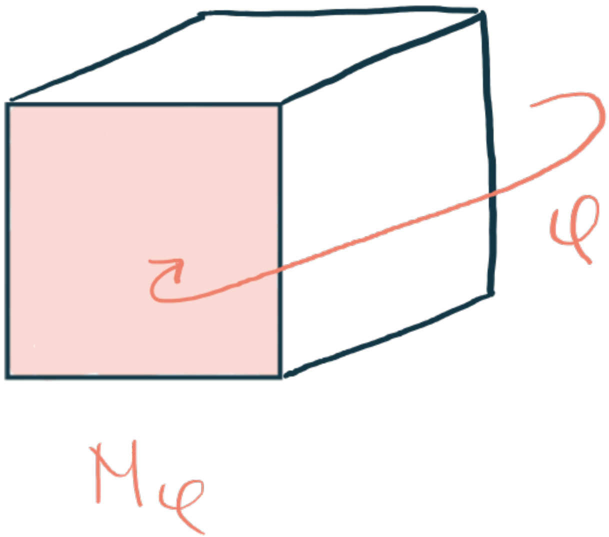
- up to isotopy fixing boundaries and punctures

Heuristic: the fractional Dehn twist coefficient $\mathrm{FDTC}(\varphi, C)$ measures the amount of twisting φ affects near C.



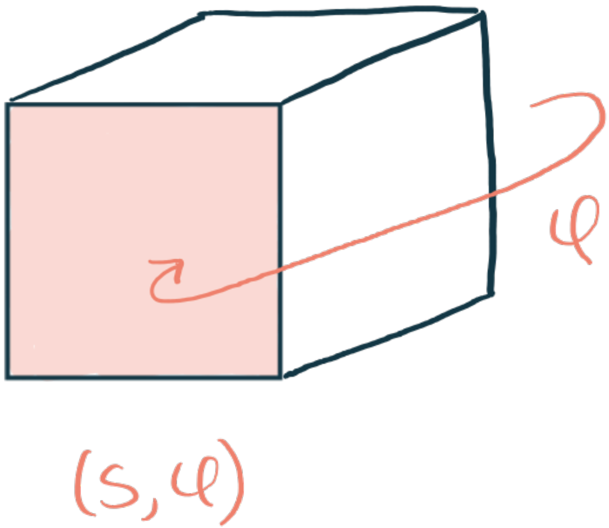
Ex: A (positive) Dehn twist along a curve parallel to C has $\mathrm{FDTC} = 1$.

Why study the FDTC?



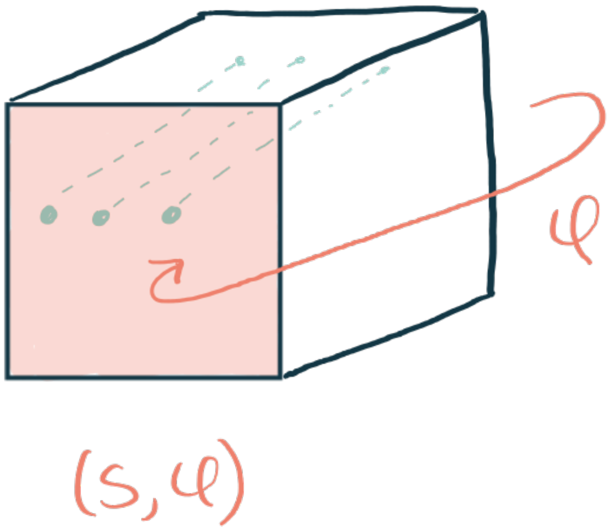
- M_φ is a 3-mfd with torus boundary components

Why study the FDTC?



- M_φ is a 3-mfd with torus boundary components
- canonical way to fill in to get closed 3-mfd (S, φ) (open book)

Why study the FDTC?



Keeping track of punctures traces a braid in (S,φ)

Why study the FDTC?

Q:

- how could one open book give us any info on $M^3 = (S, \varphi)$?

- how can info on one braid say anything about its closure?

Why study the FDTC?

- braids / open books have FDTCs

- essential laminations of (S, φ) (Gabai–Oertel)
- taut foliations of (S, φ) (Roberts)
- contact structures on (S, φ) (Honda–Kazez–Matic, Colin–Honda)
- geometry of (S, φ) (Ito–Kawamuro)

Why study the FDTC ?

braids have a FDTC

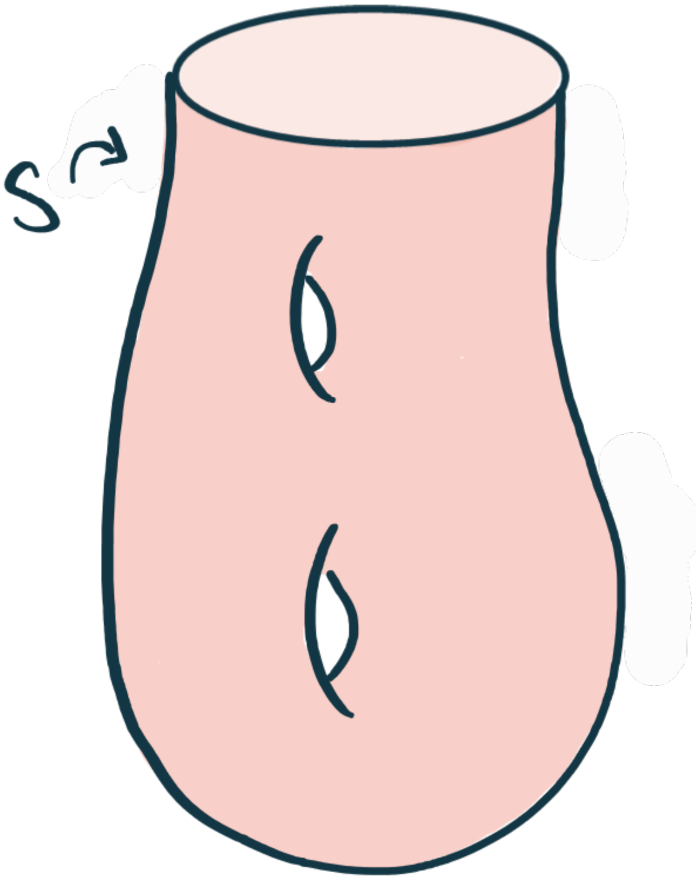
- primeness of $\hat{\beta}$ (Malyutin)
- braid index of $\hat{\beta}$ (Feller-Hubbard)
- genus of surfaces bounded by $\hat{\beta}$ (Ito, Feller)
- geometry of $S^3 - \hat{\beta}$ (Ito-Kawamuro)

Thm (Hubbard-T) The FDTC for $\varphi: S \to S$ can be computed via any Thurston type left-order on $MCG(S)$.

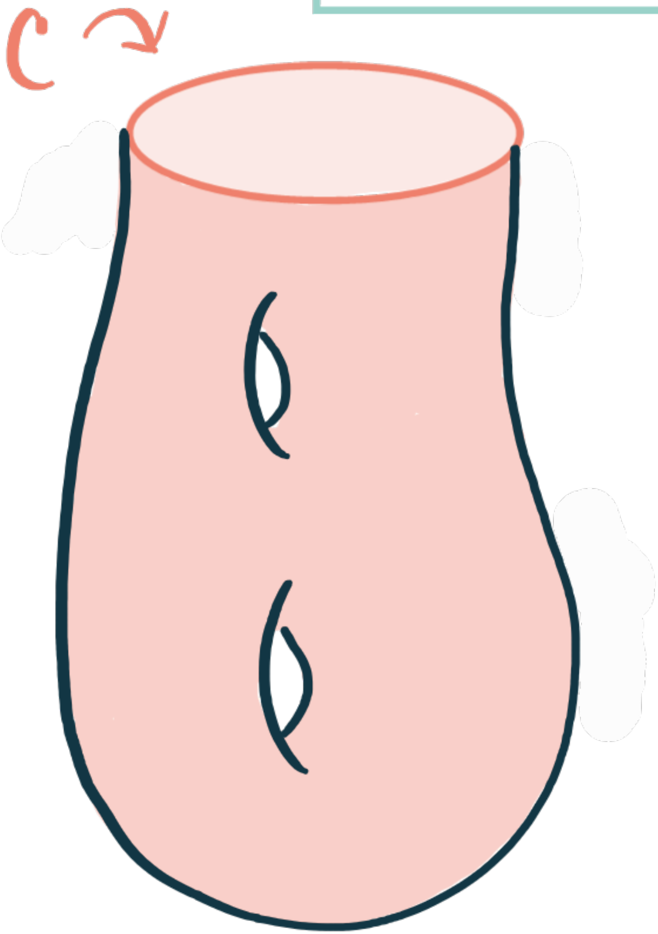
- Kazez-Roberts
- Malyutin
- Ito-Kawamuro

Thurston's left-orders on MCG

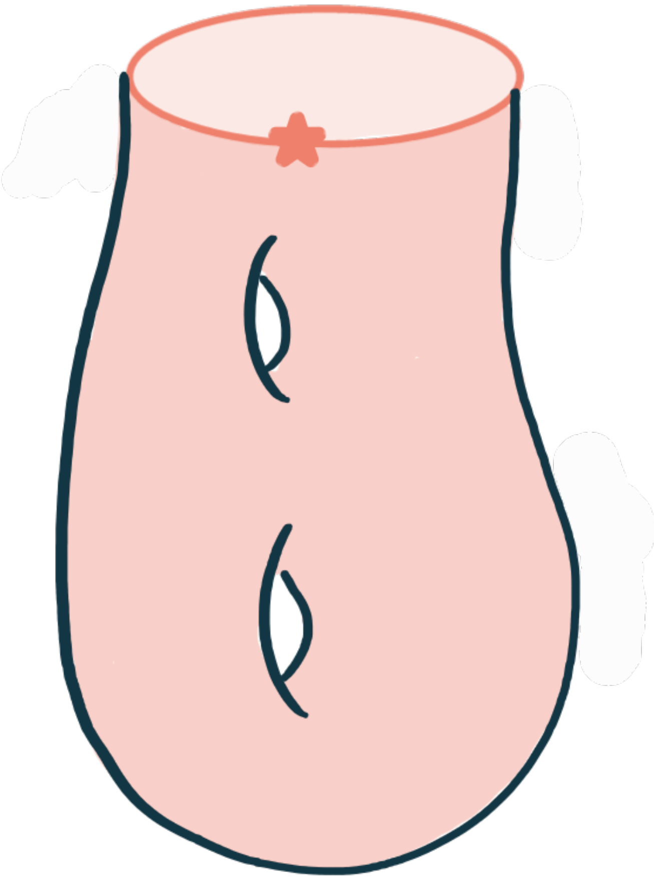
(Short-Wiest)



Thurston's left-orders on MCG



Thurston's left-orders on MCG



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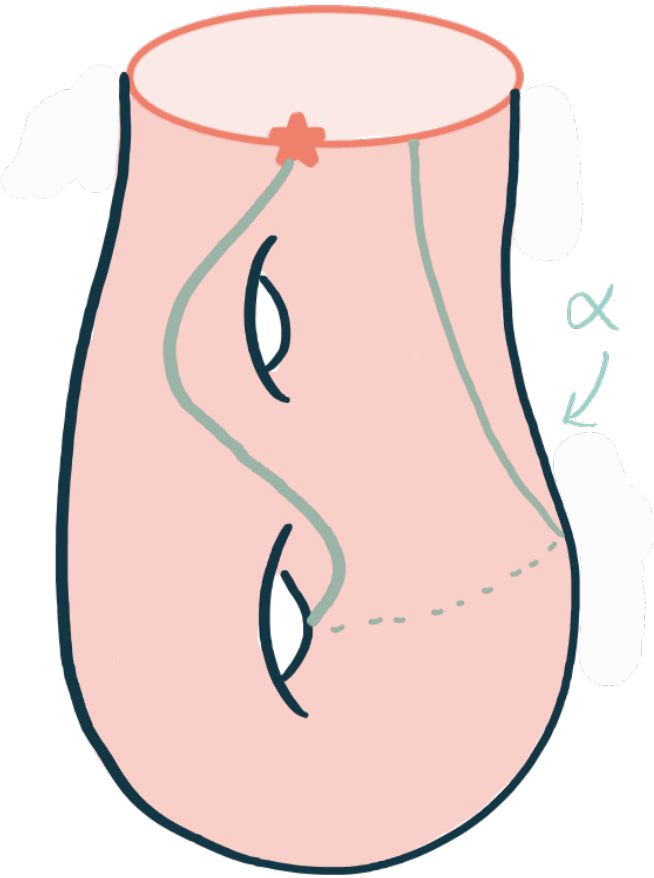

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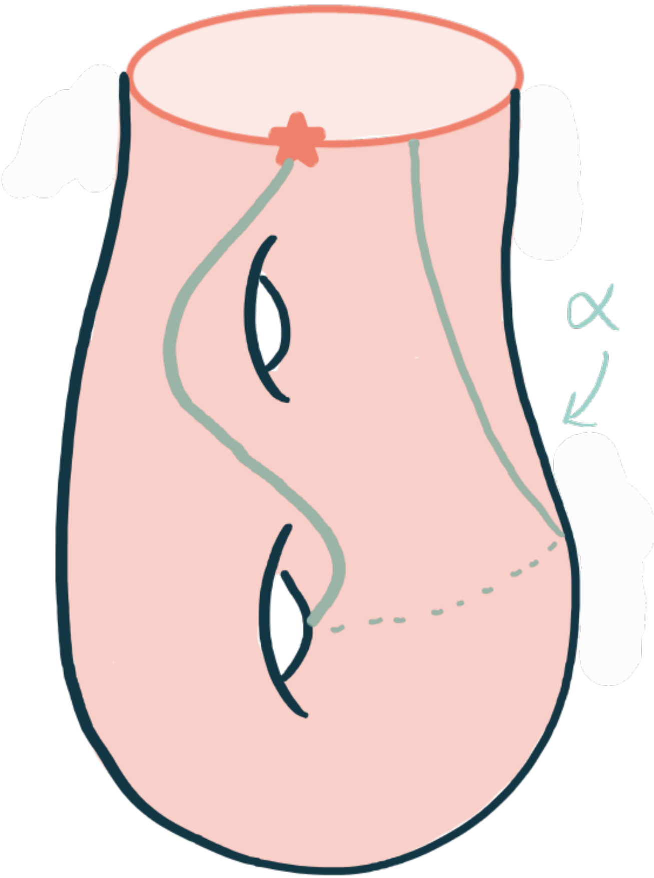

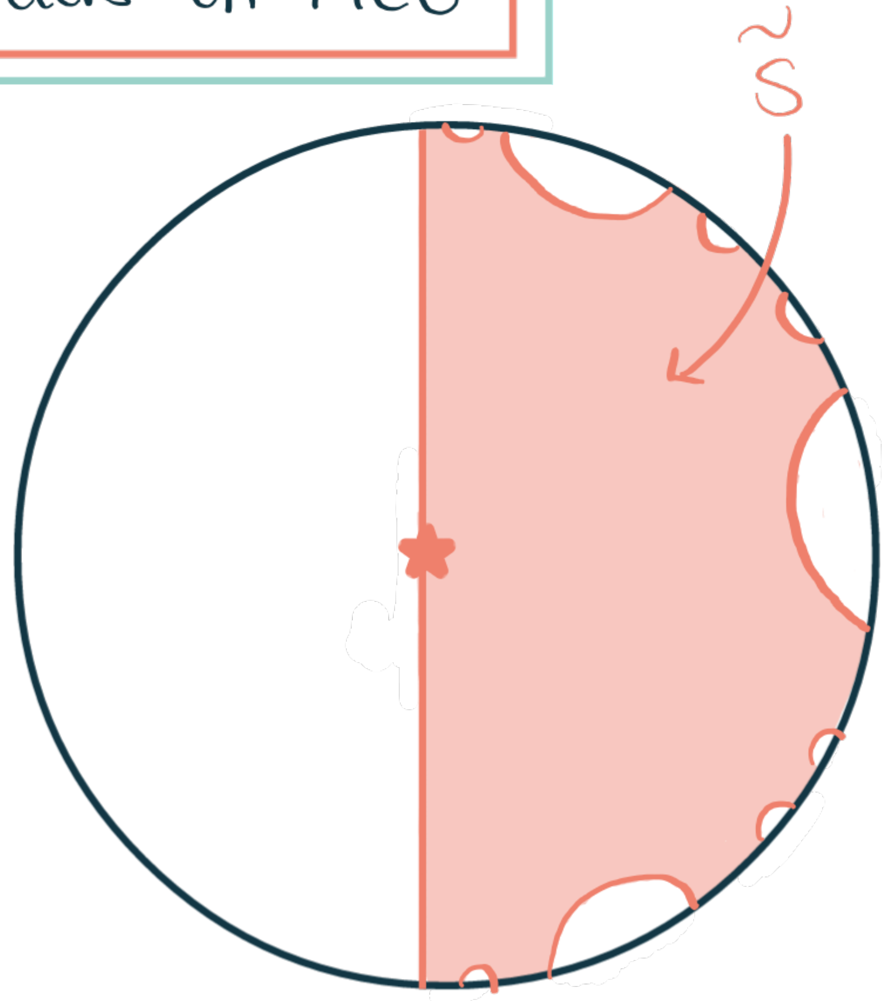

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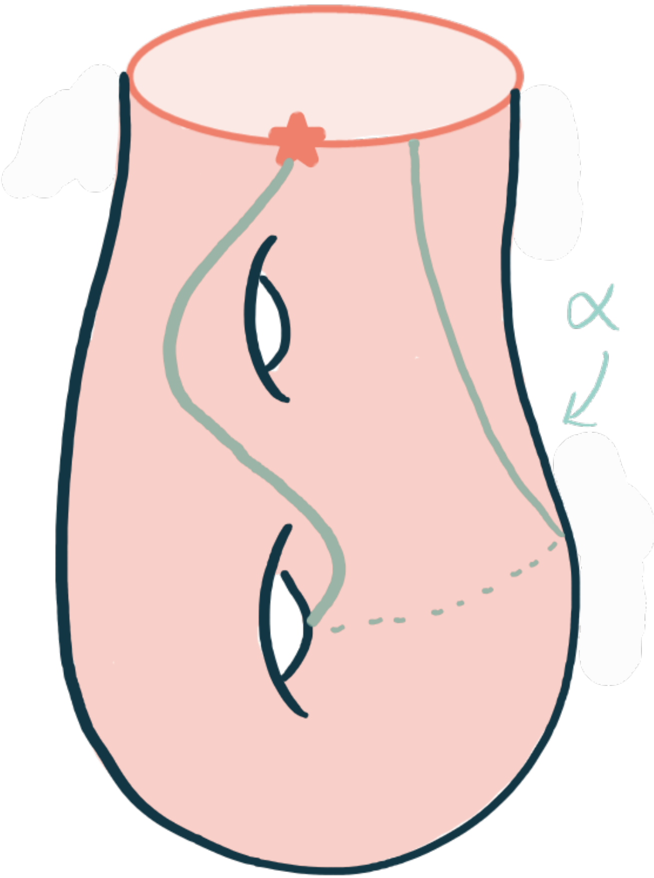

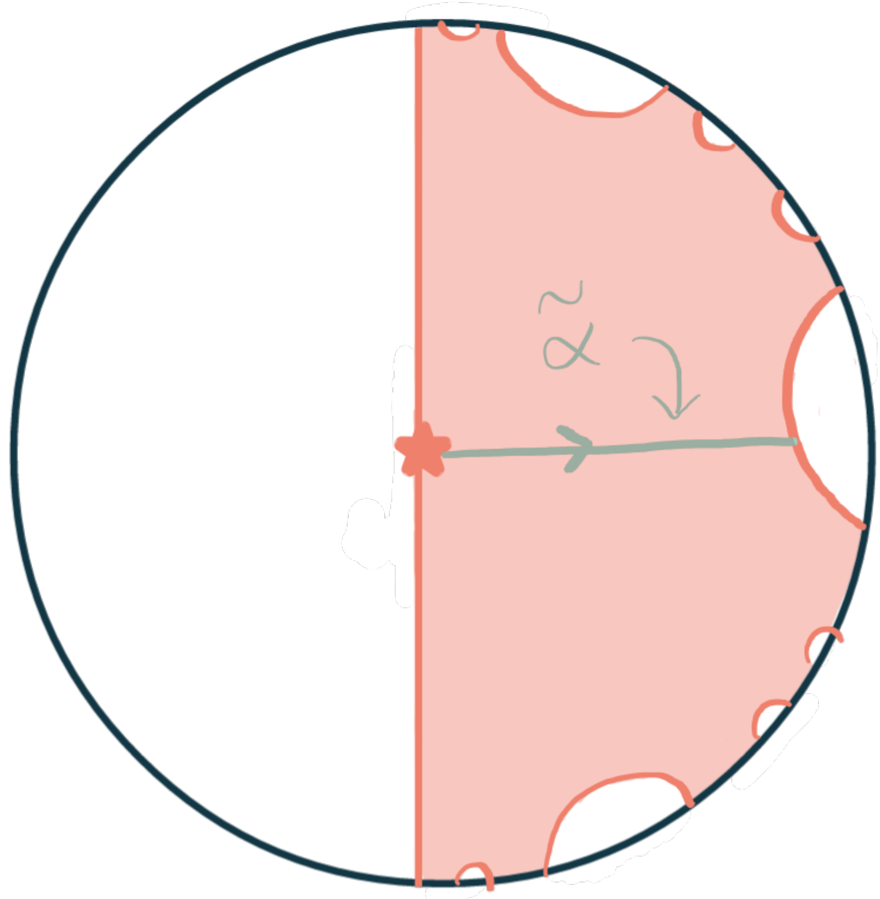

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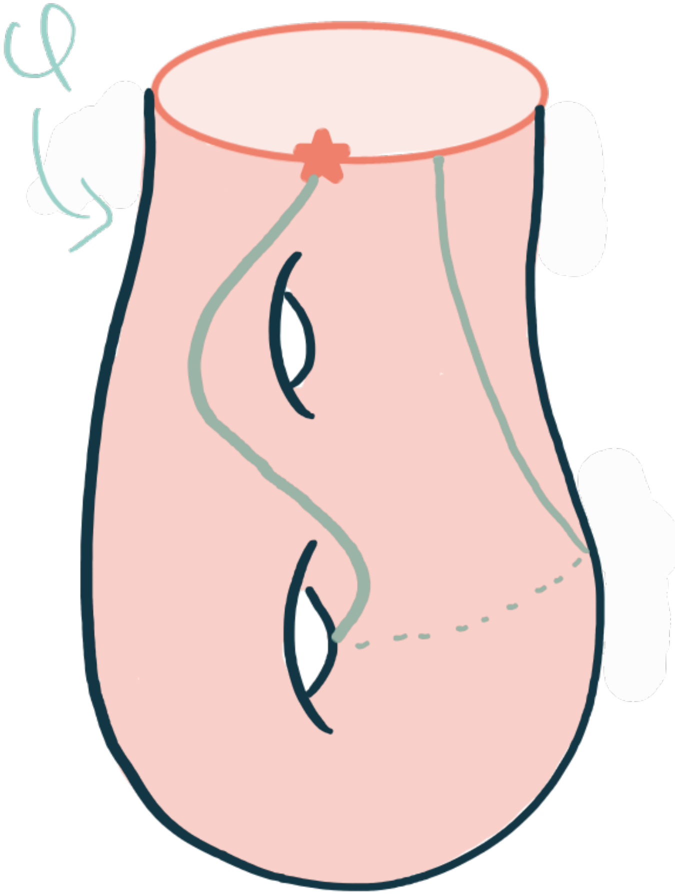

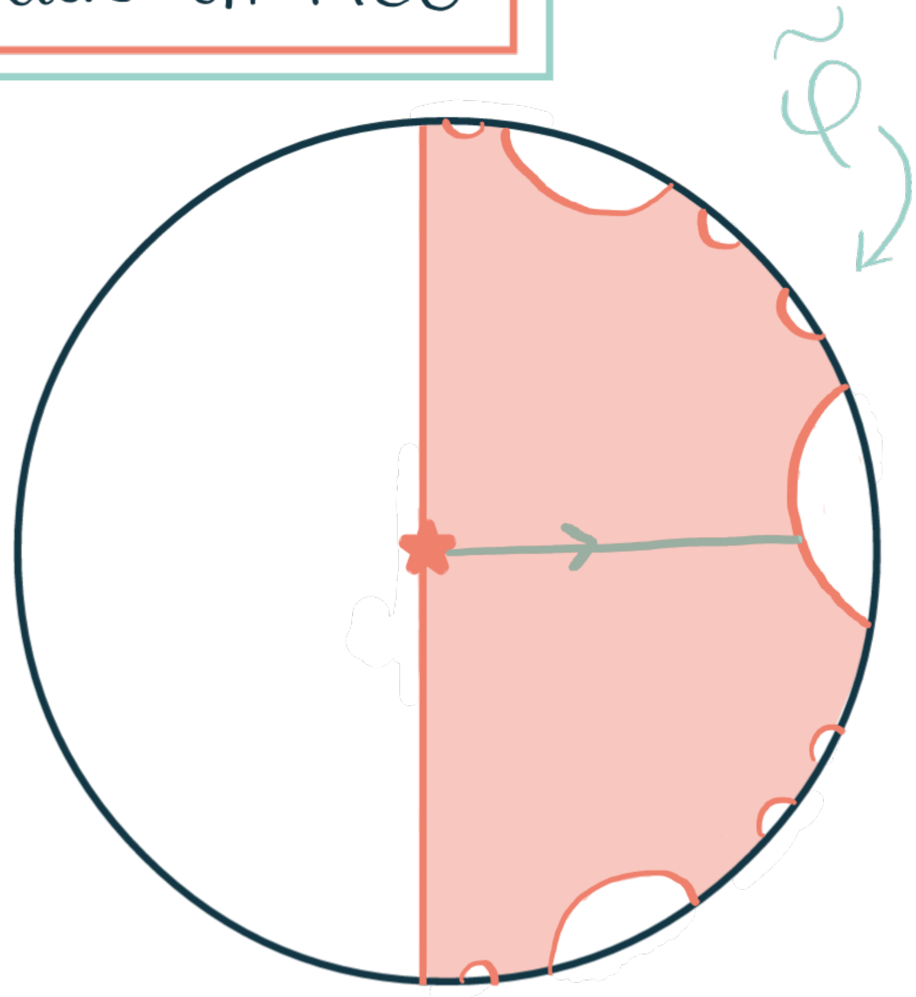

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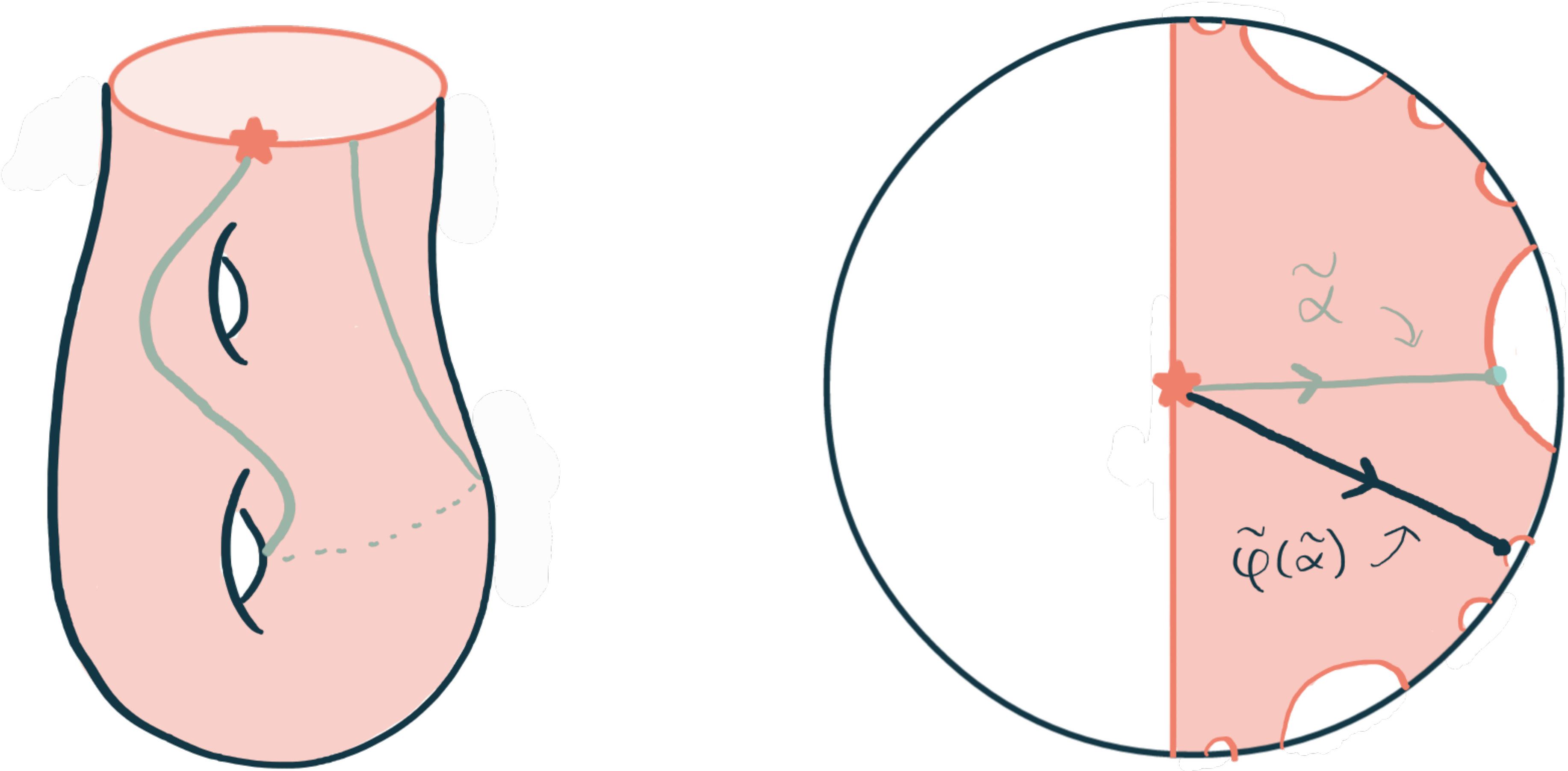

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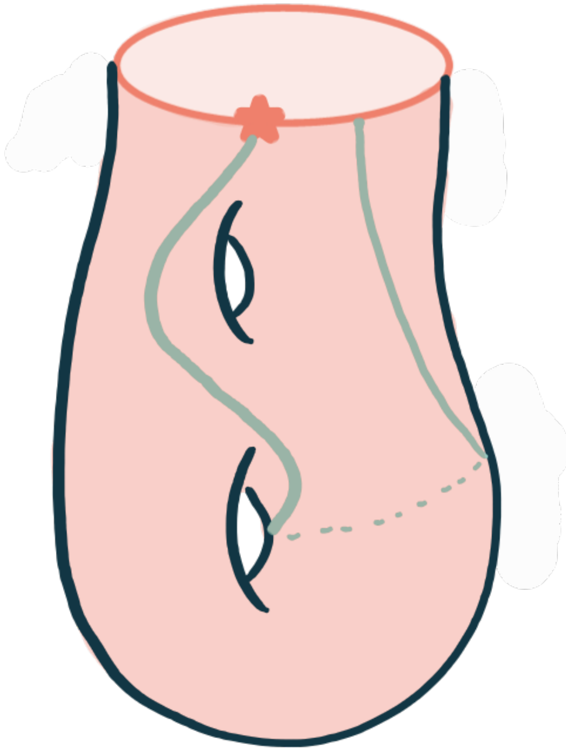
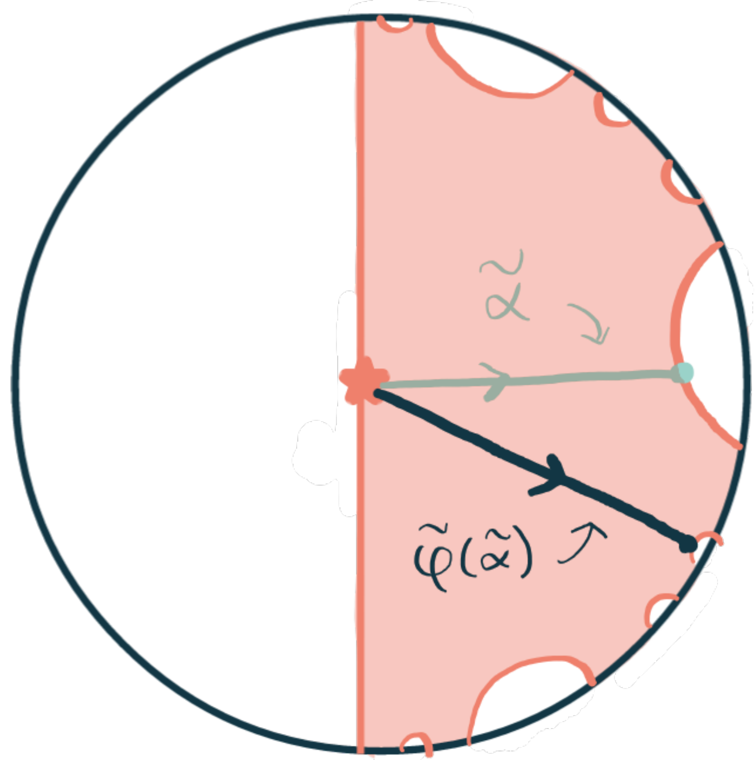


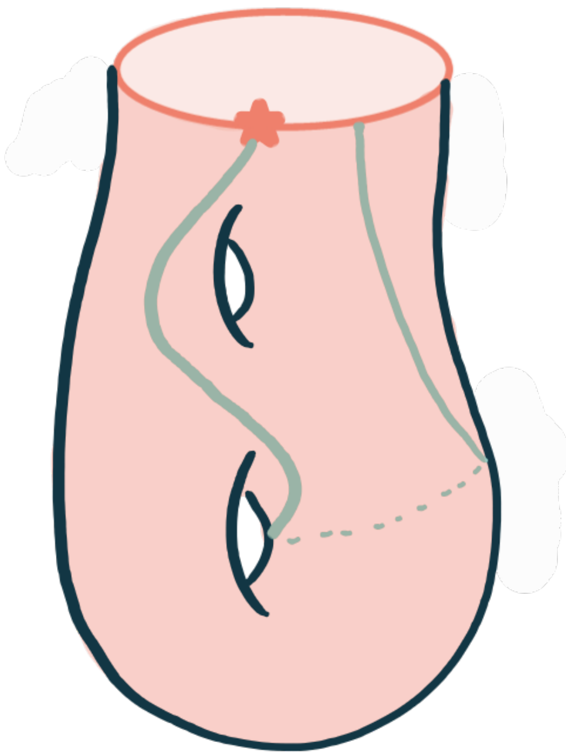


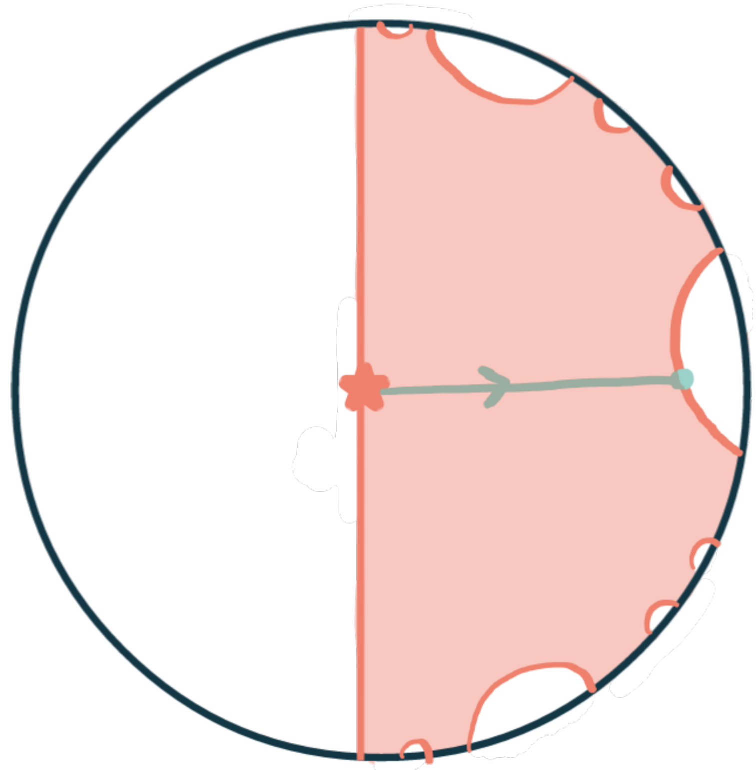
Say $\varphi >_\alpha \mathrm{id}$ if $\tilde{\varphi}(\tilde{\alpha})$ to the right of $\tilde{\alpha}$

Thurston's left-orders on MCG

- This relationship determines a left-order: i.e. $f < g \Rightarrow hf < hg$
- these LOs might be partial

Thurston's left-orders on MCG





What if $\varphi(\alpha) = \alpha$?

Even if LO is partial, we have:

① $\cdots T_c^{-3} <_\alpha T_c^{-2} <_\alpha T_c^{-1} <_\alpha id <_\alpha T_c <_\alpha T_c^2 <_\alpha T_c^3 \cdots$

② $\varphi \in MCG(S)$ is incomparable to at most one T_c^k

$$\varphi \in [T_c^k, T_c^{k+1})$$

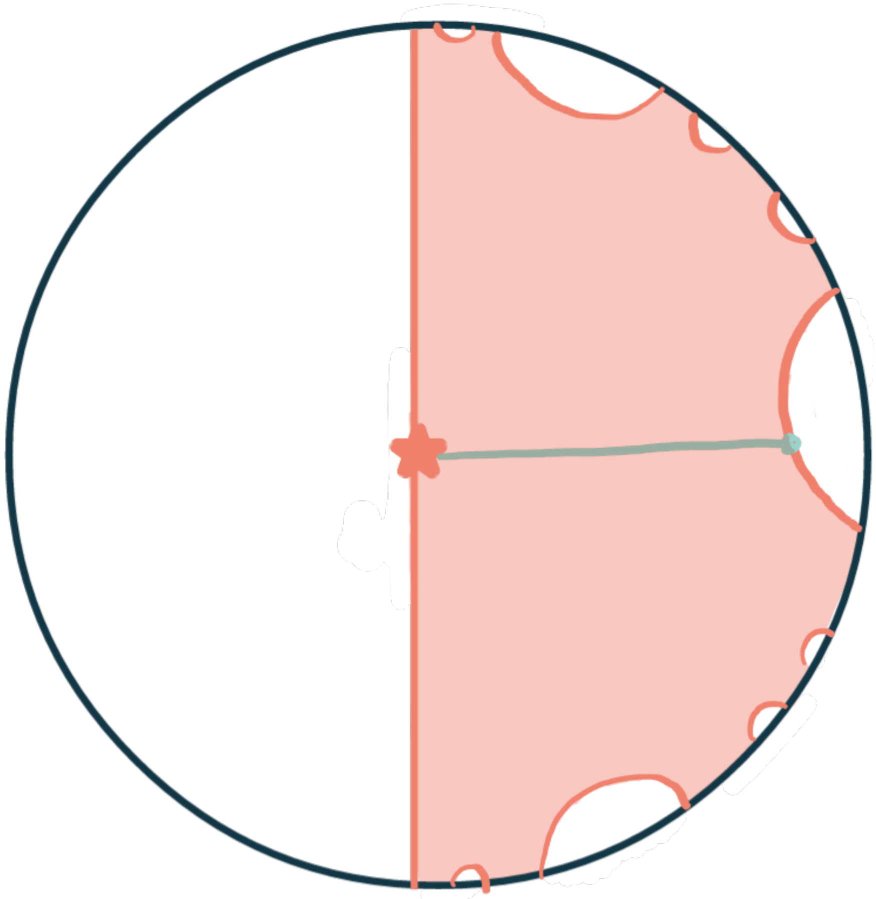
Def: the floor $\lfloor \varphi \rfloor_\alpha = k$

$$\cdots T_c^{-3} <_\alpha T_c^{-2} <_\alpha T_c^{-1} <_\alpha \mathrm{id} <_\alpha T_c <_\alpha T_c^2 <_\alpha T_c^3 \cdots$$

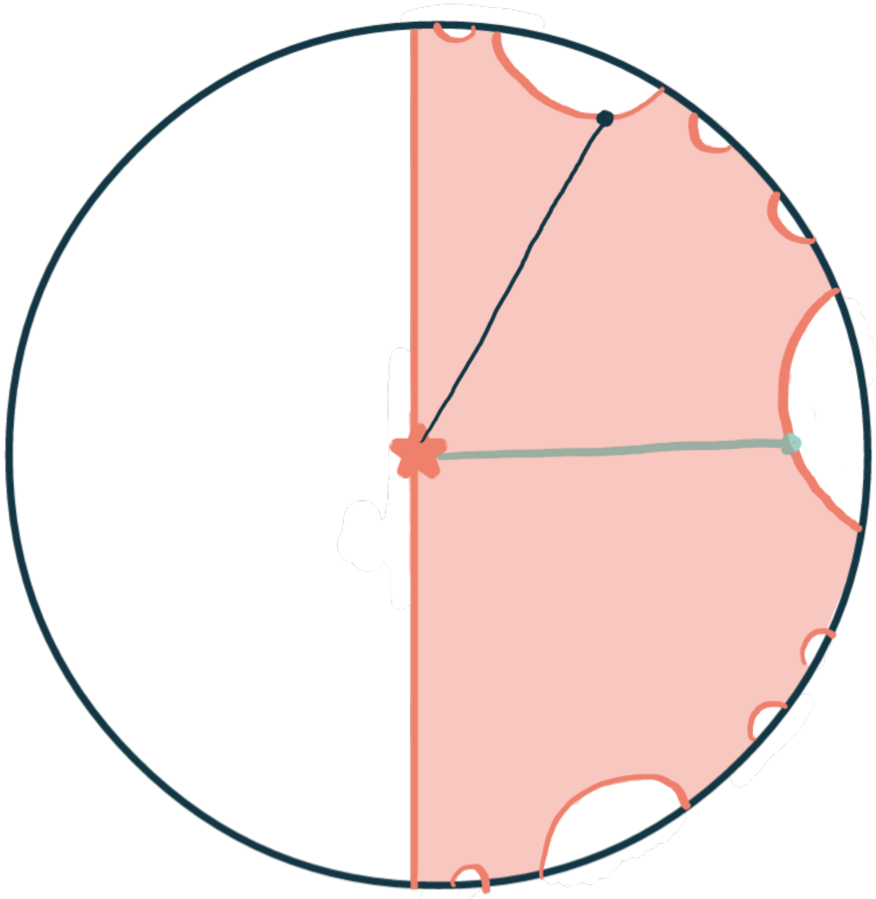
Def: $\varphi \in [T_c^k, T_c^{k+1})$ the floor $\lfloor \varphi \rfloor_\alpha = k$

Thm (Hubbard-T) $\mathrm{FDTC}(\varphi, C) = \lim\limits_{n\to\infty} \dfrac{\lfloor \varphi^n \rfloor_\alpha}{n}$

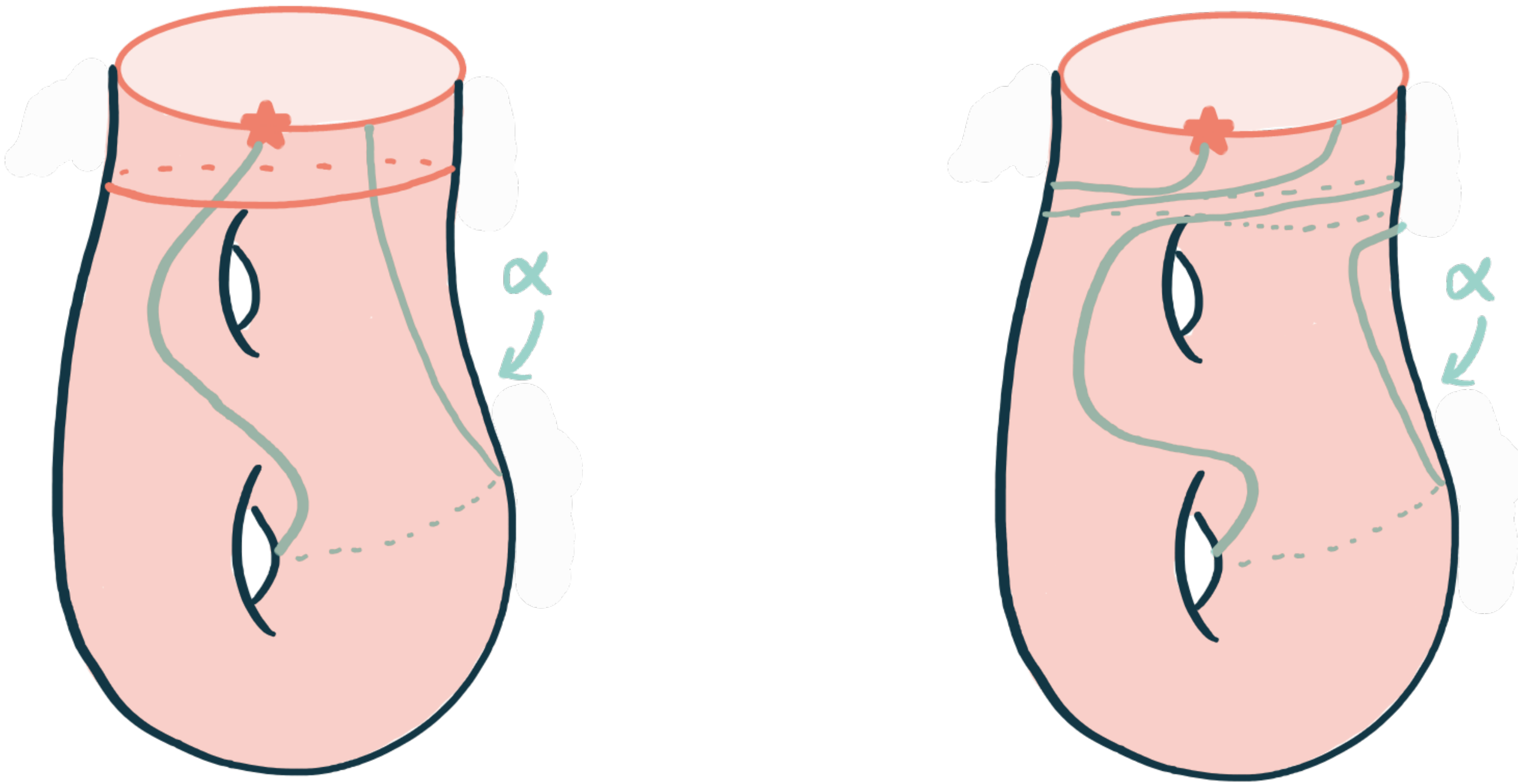
Defining the FDTC



Defining the FDTC

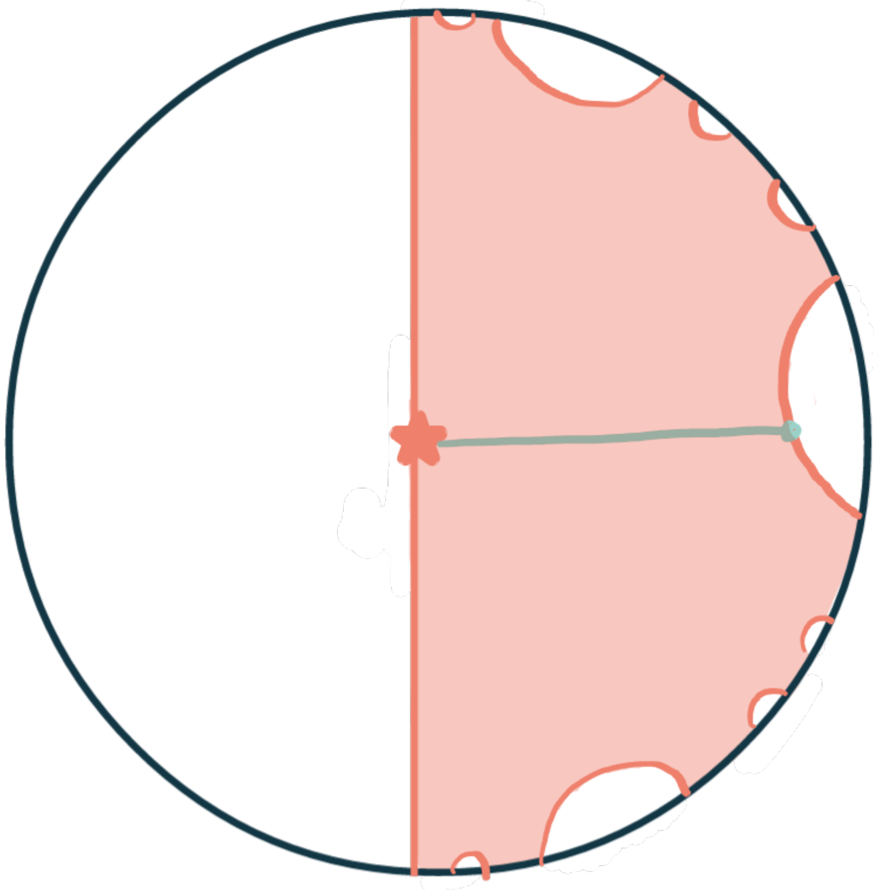


- MCG $\curvearrowright \mathbb{R}$


α
α

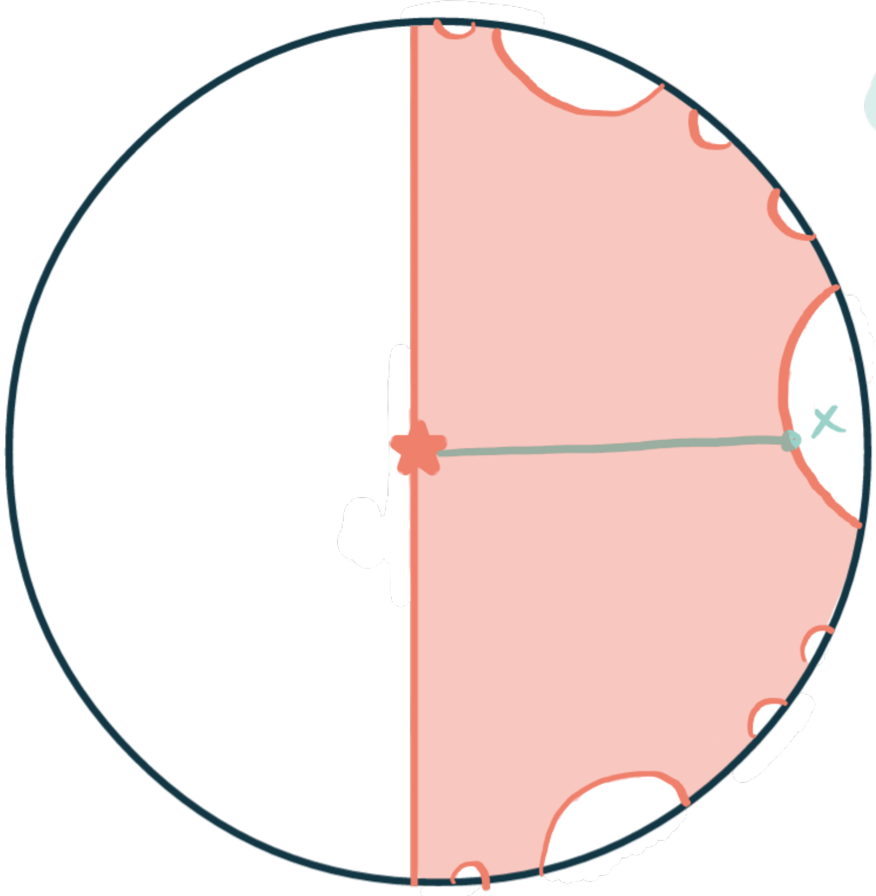
Defining the FDTC

(Ito-Kawamuro)



- MCG $\curvearrowright \mathbb{R}$
- can identify with $\mathbb{R}$ so that $\tilde{T}_c$ is translation by 1.

Defining the FDTC



Def the translation number for $f \in \widetilde{\text{Homeo}}^+(S^1)$ is:

$$\lim_{n\to\infty} \frac{f^n(x)-x}{n}$$

and $\text{FDTC}(\varphi) = \text{translation}\,\#(\tilde{\varphi})$

Does this help compute FDTC?

For braids, there is an algebraic algorithm to determine $\lfloor \beta \rfloor_\alpha$

Q: Does this generalize?

A: Probably not. But yes, in some cases:

- symmetric mapping classes
- braids in S^3
- braids in OBD with page 
- braids in OBD with symmetric monodromy

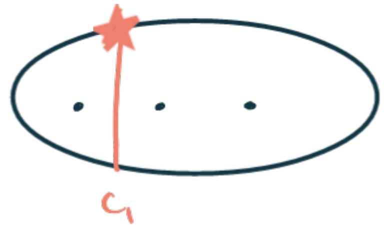
Is this better?

Existing algorithms are geometric:

- need to separate into Nielsen-Thurston type
  - reducible
  - periodic
  - pseudo Anosov
- produce invariant laminations of φ
- understand when arcs are moved to the right

Algebra of Thurston's LO's

For braids:



β moves c_1 to the right
iff there is a rep ω which is σ_1 positive
i.e. σ_1 only occurs with positive exponent

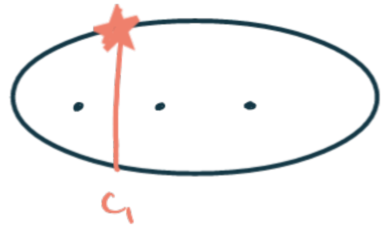
Ex: $\omega_1 := \sigma_1 \sigma_2 \sigma_1^{-1} = \sigma_2^{-1} \sigma_1 \sigma_2 := \omega_2$

not σ_1-postive (or negative)

σ_1-positive

Algebra of Thurston's LO's

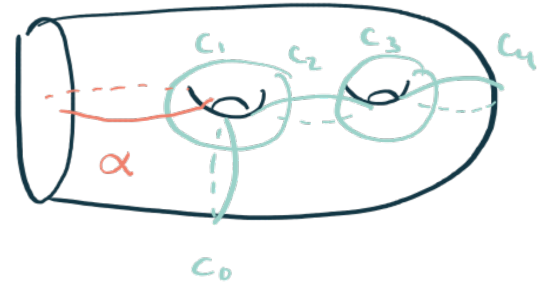
For braids:



Dehornoy gives algorithm to produce σ_1-positive|negative|empty word.

$\Longrightarrow$ algorithm to compute $\lfloor \varphi \rfloor_{c_1}$

Algebra of Thurston's LO's

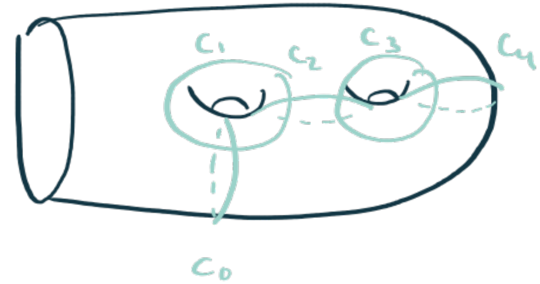


For general mapping classes:

If φ is rep by a c_1-positive word then φ moves α to the right

- not clear that every φ moving α to the right has a c_1-positive word

Algebra of Thurston's LO's



For general mapping classes:

If φ is rep by a c_1-positive word then φ moves c_1 to the right

- not clear that every φ moving c_1 to the right has a c_1-positive word

How would this algorithm work?

Facts:

- $FDTC(\varphi) = k/n$ with n constrained by S (Ito-Kawamuro)
- if $\lfloor \varphi \rfloor_\alpha = k$ then $k \leq FDTC(\varphi) \leq k+1$

Idea:

- there is some $N(S)$ so that $FDTC(\varphi^{N(S)})$ is an integer.
- then $\lfloor \varphi^{2N(S)} \rfloor$ determines $FDTC(\varphi^{2N(S)}) = 2N(S) \cdot FDTC(\varphi)$

Thank you!