Arrangements, duality, and local systems

Alex Suciu

Northeastern University

Joint work with Graham Denham

Workshop on Braids in symplectic and algebraic geometry

ICERM

March 24, 2022

Duality spaces

- The following notion of duality is due to Bieri and Eckmann (1978).
- Let X be a connected, finite-type CW-complex, and set $\pi = \pi_1(X, x_0)$.
- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.
- Davis, Januszkiewicz, Leary, and Okun (2011): Complements of (linear) hyperplane arrangements are duality spaces.

ABELIAN DUALITY SPACES

- We introduces in [Denham–S.–Yuzvinsky 2016/17] an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.
- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- Finitely generated free groups F_n are both duality groups and abelian duality groups.
- Surface groups of genus at least 2 are not abelian duality groups, though they are (Poincaré) duality groups.
- Let $H = \langle x_1, \dots, x_4 \mid x_1^{-2}x_2x_1x_2^{-1}, \dots, x_4^{-2}x_1x_4x_1^{-1} \rangle$ be Higman's acyclic group, and let $\pi = \mathbb{Z}^2 * H$. Then π is an abelian duality group (of dimension 2), but not a duality group.

Theorem (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.

Characteristic varieties

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{ab})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

Theorem (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{C}^*$ satisfies $H^i(X, \mathbb{C}_\rho) \neq 0$, then $H^j(X, \mathbb{C}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

Corollary

Let X be an abelian duality space of dimension n. Then the characteristic varieties propagate, i.e., $\mathcal{V}^1_1(X) \subseteq \cdots \subseteq \mathcal{V}^n_1(X)$.

Resonance varieties

- Let $A^\bullet$ be a graded, graded commutative algebra over $\mathbb{C}$.
- We assume A is connected ($A^0 = \mathbb{C}$) and of finite-type ($\dim A^i < \infty$, for all i).
- For each $a \in A^1$, we have a cochain complex,

$$(A^\bullet, \delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.
- The *resonance varieties* of A are the sets

$$\mathcal{R}_s^i(A) = \{a \in A^1 \mid \dim H^i(A^\bullet, \delta_a) \geqslant s\}.$$

- These sets are *homogeneous* subvarieties of A^1.
- If X is a connected, finite-type CW-complex, we let $\mathcal{R}_s^i(X) := \mathcal{R}_s^i(H^\bullet(X, \mathbb{C}))$.

- We say that the resonance varieties of a graded algebra A *propagate* if $\mathcal{R}^1_1(A) \subseteq \cdots \subseteq \mathcal{R}^n_1(A)$.
- (Eisenbud–Popescu–Yuzvinsky 2003) If X is the complement of a hyperplane arrangement, then its resonance varieties propagate.

Theorem (DSY)

- *Suppose the $\mathbb{C}$-dual of A has a linear free resolution over $E = \bigwedge A^1$. Then the resonance varieties of A propagate.*
- *Let X be a formal, abelian duality space. Then the resonance varieties of X propagate.*
- *Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do* not *propagate.*

Arrangements of smooth hypersurfaces

- Let Y be a smooth, connected complex manifold, and let $\mathcal{A} = \{W_1, \dots, W_m\}$ be a finite collection of smooth, connected, codimension-1 submanifolds of Y.
- Let $D = \bigcup_{i=1}^{m} W_i$ be the corresponding divisor, and let $M(\mathcal{A}) := Y \setminus D$ be the *complement* of the arrangement $\mathcal{A}$.
- We assume that the intersection of any subset of $\mathcal{A}$ is also a smooth manifold, and has only finitely many connected components.
- We also require that, for each point $y \in D$, there is a chart containing y for which each element of the subcollection $\mathcal{A}_y := \{W_i \mid y \in W_i\}$ is defined locally by a linear equation.
- In other words, the hypersurfaces comprising $\mathcal{A}$ have intersections which, locally, are diffeomorphic to hyperplane arrangements.

- Let $L(\mathcal{A})$ denote the collection of all connected components of intersections of zero or more of the hypersurfaces comprising $\mathcal{A}$.
- Then $L(\mathcal{A})$ forms a finite poset under reverse inclusion, ranked by codimension. We write $X \leqslant Y$ if $X \supseteq Y$ and $r(X) = \operatorname{codim} X$.
- For every submanifold X in the intersection poset $L(\mathcal{A})$, we let
  - $\mathcal{A}_X = \{W \in \mathcal{A} \mid X \subseteq W\}$: the *closed subarrangement* for X.
  - $\mathcal{A}^X = \{W \cap X \mid W \in \mathcal{A} \backslash \mathcal{A}_X\}$: the *restriction* of $\mathcal{A}$ to X.
- Then $M(\mathcal{A}^X) := X \backslash D_X$, where $D_X = \bigcup_{Z \in L(\mathcal{A}): Z < X} Z$.
- We also let $T\mathcal{A}_X$ be the hyperplane arrangement in the tangent space to Y at a point in the relative interior of X.

Theorem (DSY)

Let $\mathcal{A}$ be an arrangement of hypersurfaces in a compact, smooth manifold Y. Let M be the complement of the arrangement, and let $\mathcal{F}$ be a locally constant sheaf on M. There is then a spectral sequence with

$$E_2^{pq} = \prod_{X \in L(\mathcal{A})} H_c^{p+r(X)}(M(\mathcal{A}^X); H^{q-r(X)}(M(T\mathcal{A}_X), \mathcal{F}_X)),$$

converging to $H^{p+q}(M, \mathcal{F})$, where $\mathcal{F}_X$ is the restriction of $\mathcal{F}$ to $M(T\mathcal{A}_X)$.

Wonderful compactifications

- Let $\mathcal{A}$ be an arrangement of smooth, algebraic hypersurfaces in a smooth, connected complex projective variety Y.
- For each $x \in Y$, there is a linear hyperplane arrangement $T\mathcal{A}_X$ in the $\mathbb{C}$-vector space $V = T_x Y$ tangent to $\mathcal{A}_X$, where $X = \bigcap_{x \in Z \in L(\mathcal{A})} Z$.
- We apply De Concini and Procesi's construction of the wonderful model of a subspace arrangement to $T\mathcal{A}_X \subset V$.
- The construction blows up the arrangement to one with simple normal crossings; let $p\colon \widetilde{V} \to V$ denote the blowup.
- The (total) divisor components are indexed by a 'building set' $\mathcal{G}_X$.
- A subset $S \subseteq \mathcal{G}_X$ indexing divisor components that have non-empty intersection is called a *nested set*.

- The collection of all nested sets forms a simplicial complex, called the nested set complex, $\mathcal{N}(T\mathcal{A}_X)$.
- For a nested set $S \in \mathcal{N}(T\mathcal{A}_X)$ of size r, let D_S denote the corresponding intersection of r divisor components in $\widetilde{V}$.
- For a point z in the relative interior of D_S, let $\mathbb{D}_z$ be a sufficiently small closed polydisc in $\widetilde{V}$ centered at z.
- Set $U_S := \mathbb{D}_z \cap M(T\mathcal{A}_X)$. Then $U_S \simeq (S^1)^r$ and $\pi_1(U_S) \cong \mathbb{Z}^r$.

Lemma (Denham–S.)

For every $X \in L(\mathcal{A})$ and every nested set $S \in \mathcal{N}(T\mathcal{A}_X)$, there is a naturally defined homomorphism $\alpha_{X,S} \colon \pi_1(U_S) \to \pi_1(M(\mathcal{A}))$ which is injective.

Let $G = \pi_1(M(\mathcal{A}))$. Let $C_{S,X}$ be the conjugacy class of the subgroup $\alpha_{X,S}(C_S) < G$; this is a free abelian group of rank $|S|$.

Stein manifolds

- A complex manifold M is said to be a *Stein manifold* if it can be realized as a closed, complex submanifold of some complex affine space.
- Alternatively, holomorphic functions on M separate points, and M is holomorphically convex.
- The Stein property is preserved under taking closed submanifolds and finite direct products.
- A Stein manifold of (complex) dimension n has the homotopy type of a CW-complex of dimension n.

Maximal Cohen–Macaulay modules

Definition

Let $\Bbbk = \mathbb{Z}$ or a field, let $R = \Bbbk[\mathbb{Z}^n]$, and let I be the augmentation ideal of R. We say that an R-module A is a *maximal Cohen–Macaulay (MCM) module* provided that $\operatorname{depth}_R(I, A) \geqslant n$.

Definition (DS)

A (left) $\Bbbk[G]$-module A is a *MCM module* if the restriction of A to each subalgebra $\Bbbk[C_{S,X}]$ is MCM, for all $X \in L(\mathcal{A})$ and all $S \in \mathcal{N}(T\mathcal{A}_X)$.

Theorem (DS)

Suppose that $M(\mathcal{A}^X)$ is Stein for each $X \in L(\mathcal{A})$. Then, for any MCM module A on $M(\mathcal{A})$, we have $H^p(M(\mathcal{A}), A) = 0$ for all $p \neq n$.

The Stein hypothesis in this theorem is indispensable. For instance, let $X = \mathbb{C}^n$, with $n \geqslant 2$, and let $\mathcal{A} = \{0\}$. Then $U = \mathbb{C}^n \backslash \{0\}$ is not Stein, and also not an abelian duality space, since $U \simeq S^{2n-1}$.

Duality and generic vanishing of cohomology

Theorem

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$.*
2. *$M(\mathcal{A}^X)$ is a Stein manifold for each submanifold $X \in L(\mathcal{A})$.*

Then U is both a duality space and an abelian duality space of dimension n.

Consequently, the characteristic varieties of such "recursively Stein" hypersurface complements propagate.

Theorem

Let $G = \pi_1(U)$, and let A be a finite-dimensional representation of G over a field $\Bbbk$. Suppose that $A^{\gamma_g} = 0$ for all g in a building set $\mathcal{G}_X$, where $X \in L(\mathcal{A})$. Then $H^i(U, A) = 0$ for all $i \neq n$.

Consequently, the cohomology groups of U with coefficients in a 'generic' local system vanish in the range below n.

- Let $\ell_2 G$ denote the left $\mathbb{C}[G]$-module of complex-valued, square-summable functions on G.
- Let ${}^{\mathrm{red}}H^i(U, \ell_2 G)$ be the reduced L^2-cohomology groups of U with coefficients in this module.

Theorem

Let U and $G = \pi_1(U)$ be as above. Then ${}^{\mathrm{red}}H^i(U, \ell_2 G) = 0$ for all $i \neq n$.

- Consequently, the ℓ_2-Betti numbers of U are all zero except in dimension n.
- A basic fact about ℓ_2-cohomology is that ℓ_2-Betti numbers compute the usual Euler characteristic. Therefore, we see once again that $(-1)^n \chi(U) \geqslant 0$.

Linear, elliptic, and toric arrangements

Theorem

Suppose that $\mathcal{A}$ is one of the following:

(1) An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;

(2) A non-empty elliptic arrangement in E^n;

(3) A toric arrangement in $(\mathbb{C}^*)^n$.

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

- This theorem extends several previous results:
  - Davis, Januszkiewicz, Leary, and Okun (2011);
  - Levin and Varchenko (2012);
  - Davis and Settepanella (2013), Esterov and Takeuchi (2014).
- Liu, Maxim, and Wang (2018) proved that very affine varieties are abelian duality spaces.

Orbit configuration spaces

- Let Γ be a discrete group that acts freely and properly discontinuously on a space X.
- The *orbit configuration space* $F_\Gamma(X, n)$ is the subspace of the cartesian product $X^{\times n}$ consisting of n-tuples $(x_1, \dots, x_n)$ for which the Γ-orbits of x_i and x_j are disjoint for all $1 \leqslant i \neq j \leqslant n$.
- If $|\Gamma| = 1$, then $F_\Gamma(X, n) = F(X, n)$, the classical (ordered) configuration space.
- When $X = M$ is a smooth manifold of dimension d and Γ acts by diffeomorphisms, $F_\Gamma(M, n)$ is a smooth manifold of dimension dn.
- Let $M = \Sigma_{g,k}$ be a Riemann surface of genus g with $k \geqslant 0$ punctures, and assume Γ is finite.
- When $k = 0$, the complement in $\Sigma_g^{\times n}$ of $F_\Gamma(\Sigma_g, n)$ is the union of an arrangement of smooth, complex algebraic hypersurfaces.

- Xicoténcatl showed that the classical Fadell–Neuwirth fibration applies in the more general case of orbit configuration spaces:

$$F_\Gamma(\Sigma_{g,k+|\Gamma|}, n-1) \longrightarrow F_\Gamma(\Sigma_{g,k}, n) \longrightarrow \Sigma_{g,k} \;.$$

- Consider the 'tautological' compactification of the orbit configuration space $U = F_\Gamma(\Sigma_{g,k}, n)$, namely $Y = \Sigma_g^{\times n}$.

- The components of the boundary divisor, $D = Y \backslash U$, form an arrangement of hypersurfaces,

$$\mathcal{B}_n := \left\{ H_{ij}^\gamma \mid \gamma \in \Gamma, 1 \leqslant i \neq j \leqslant n \right\} \cup \{ K_{i,l} \mid 1 \leqslant i \leqslant n, 1 \leqslant l \leqslant k \},$$

where H_{ij}^γ is given by the equation $x_i = \gamma \cdot x_j$ and $K_{i,l}$ by $x_i = p_l$, where $p_1, \dots, p_k \in \Sigma_g$ are the punctures of $\Sigma_{g,k}$.

- The intersection poset $L(\mathcal{B}_n)$ can be described in terms of labelled partitions via a slight generalization of the Dowling lattice.

- If $k > 0$, then for each flat $X \in L(\mathcal{B}_n)$, the complement $M(\mathcal{B}_n^X)$ is a Stein manifold.

Theorem

Suppose Γ is a finite group that acts freely on a Riemann surface $\Sigma_{g,k}$ of genus g with k punctures. Let $F_\Gamma(\Sigma_{g,k}, n)$ be the orbit configuration space of n ordered, disjoint Γ-orbits.

1. *If $k > 0$, then $F_\Gamma(\Sigma_{g,k}, n)$ is both a duality space and an abelian duality space of dimension n.*
2. *If $k = 0$, then $F_\Gamma(\Sigma_g, n)$ is a duality space of dimension $n+1$, provided $g \geqslant 1$, and is an abelian duality space of dimension $n+1$ if $g = 1$.*
3. *If $k = 0$, then $F(\Sigma_g, n)$ is neither a duality space nor an abelian duality space if $g = 0$, and it is not an abelian duality space if $g \geqslant 2$.*

Corollary

If Γ is a finite group acting freely on $\Sigma_{g,k}$, the characteristic varieties propagate for the orbit configuration spaces $F_\Gamma(\Sigma_{g,k}, n)$, where either $k \geqslant 1$, or $k = 0$ and $g = 1$.

References

- G. Denham, A.I. Suciu, and S. Yuzvinsky, *Combinatorial covers and vanishing of cohomology*, Selecta Math. **22** (2016), no. 2, 561–594.
- G. Denham, A.I. Suciu, and S. Yuzvinsky, *Abelian duality and propagation of resonance*, Selecta Math. **23** (2017), no. 4, 2331–2367.
- G. Denham, A.I. Suciu, *Local systems on arrangements of smooth, complex algebraic hypersurfaces*, Forum of Mathematics, Sigma **6** (2018), e6, 20 pages.