Maps of braid groups

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Braids in Symplectic and Algebraic Geometry
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w/ Lei Chen and Kevin Kordek

Executive Summary

We classify homomorphisms:

$$B_n \to B_{2n}$$

and

$$B_n' \to B_n$$

New tools: totally symmetric sets, periodics

Connection to polynomials

$\mathrm{Poly_n}$ = space of monic, square free polynomials of degree n

$$\pi_1(\mathrm{Poly}_n) \cong B_n$$

Our theorems constrain maps

$$\mathrm{Poly}_n \to \mathrm{Poly}_m$$

Connection to polynomials

$\mathrm{Poly_n}$ = space of monic, square free polynomials of degree n

$$\pi_1(\mathrm{Poly}_n) \cong B_n$$

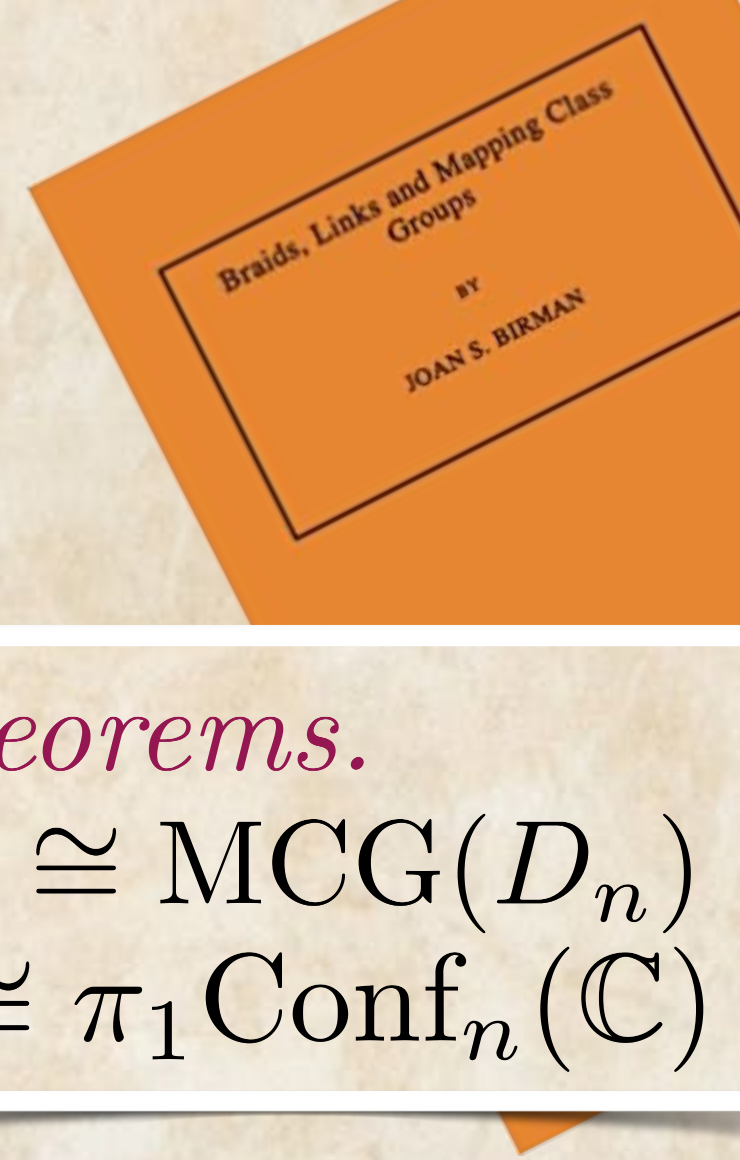
Resolution of the quartic is an algebraic map

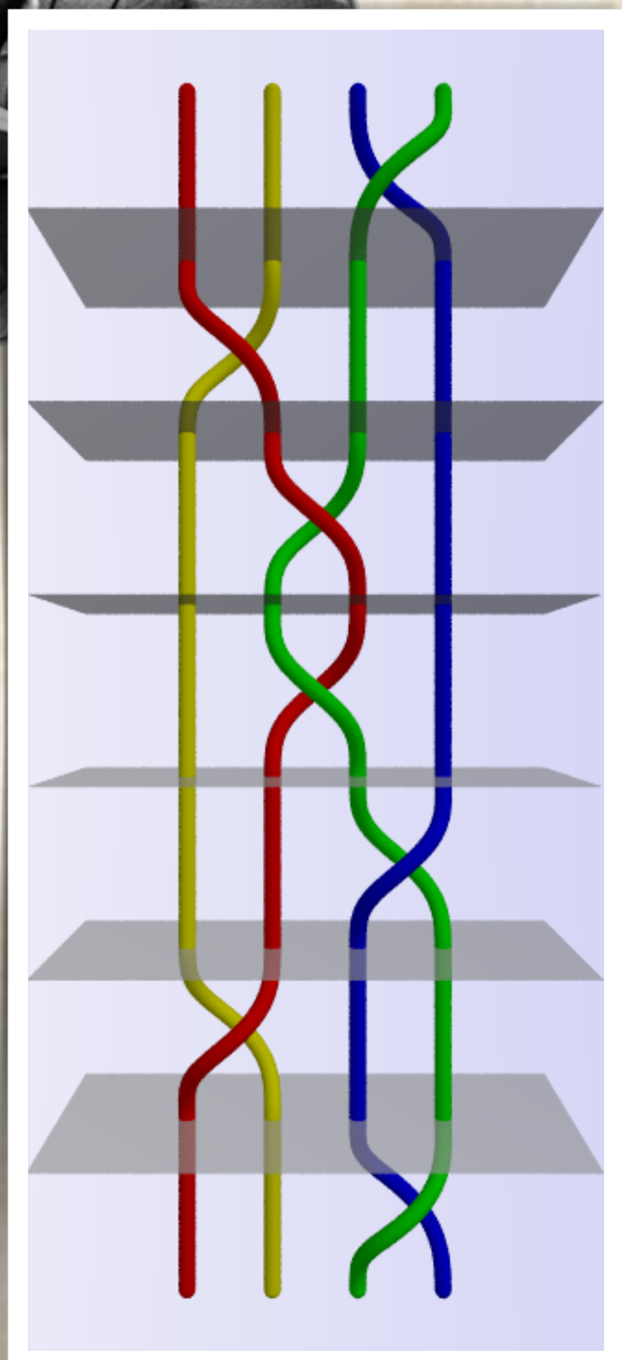
$$\mathrm{Poly}_4 \to \mathrm{Poly}_3$$

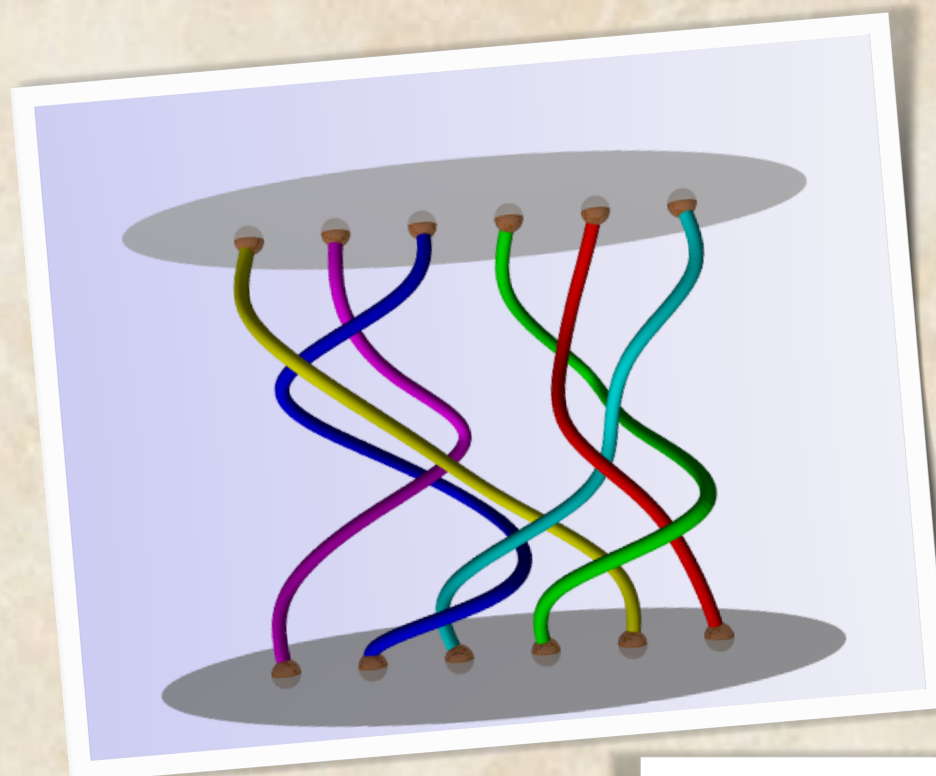
We show every $B_4 \to B_3$ is cyclic or factors through the standard map, so no other maps

Braid groups

Braid groups



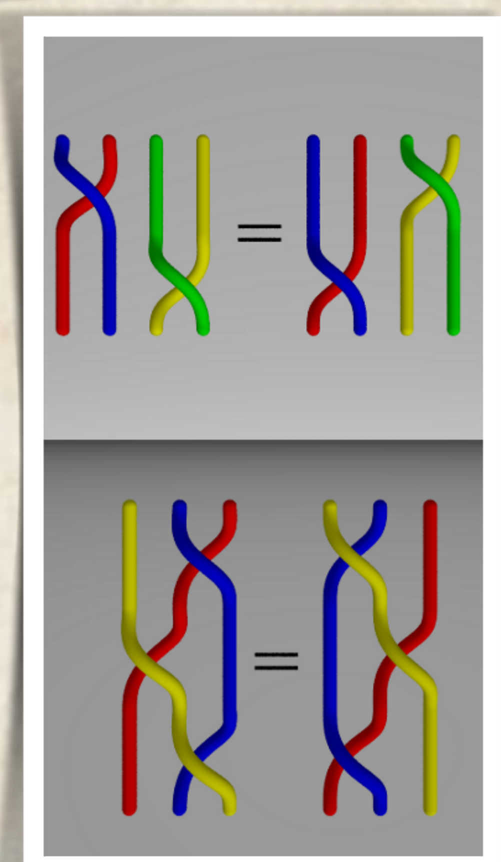




Theorems.

$$B_n \cong \mathrm{MCG}(D_n)$$
$$\cong \pi_1 \mathrm{Conf}_n(\mathbb{C})$$

Braid images:
Ester Dalvit



$$\sigma_i \sigma_j = \sigma_j \sigma_i$$
$$|i - j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i =$$
$$\sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1} \sigma_3 \sigma_1 \sigma_3$$

The squared lantern relation

Math:
Brendle–M

Art:
Buriakova

Maps of braid groups: what was known

Artin's question

In 1947 Artin classified all homomorphisms

$$B_n \to \Sigma_n$$

and asked about automorphisms of B_n.

Uses Bertrand's postulate!

Geometric automorphisms

Sample automorphisms of B_n:

- Conjugation (i.e. inner automorphisms)
- Inversion: $\sigma_i \mapsto \sigma_i^{-1} \quad \forall\, i$

These are geometric: induced by $\mathrm{Homeo}(D_n)$

These generate *all* geometric autos.

Automorphisms are geometric

Thm (Dyer–Grossman '81). All automorphisms of B_n are geometric.

Proof idea:

$$\mathrm{Aut}(B_n) \to \mathrm{Aut}(B_n/\mathbb{Z}) \to \mathrm{Aut}(F_{n-1}) \to \mathrm{Homeo}(D_n)$$

Some Generalizations

Bell–Margalit '06: Injective $B_n \to B_{n+1}$

Castel '08: Homomorphisms $B_n \to B_{n+2}$

Bell–Margalit '07: Automorphisms of PB_n

Childers '17: Automorphisms of hyp. Torelli

McLeay '19: Automorphisms of deeper terms of Johnson filtration

Main Theorem 1:
Expanding the range

Expanding the range

Having a classification of maps $B_n \to B_{n+1}$

we would like to classify maps $B_n \to B_m$

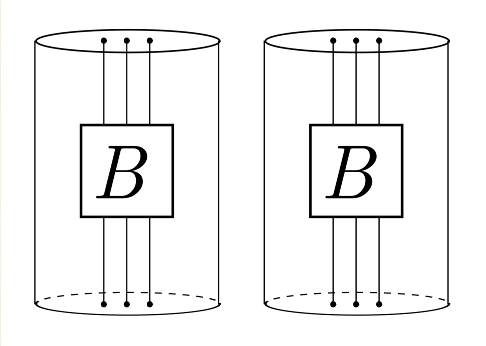
for $m > n + 1$.

When $m=2n$, there are interesting maps...

Some homomorphisms

Some homomorphisms $B_n \to B_{2n}$:

- Diagonal inclusion



- Flip diagonal inclusion
- Many cablings

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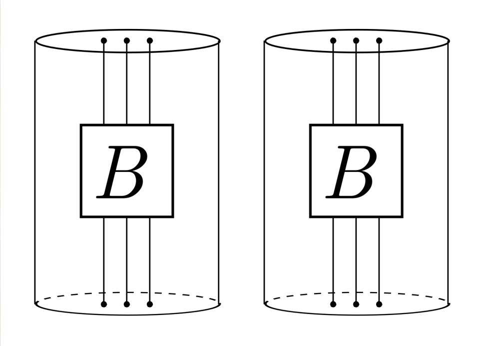

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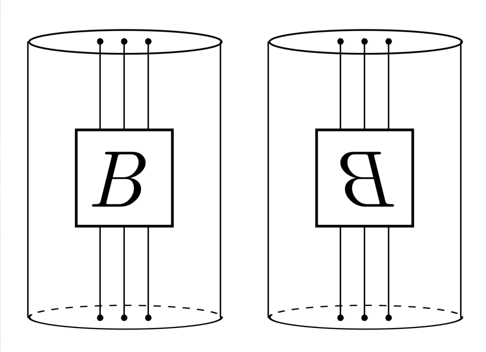

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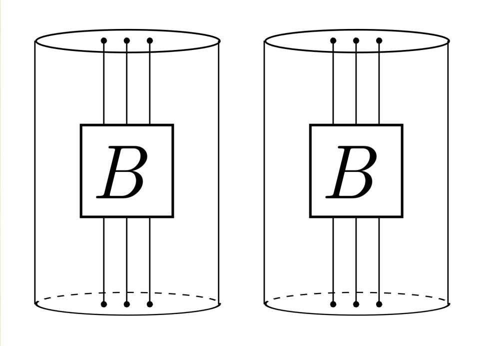

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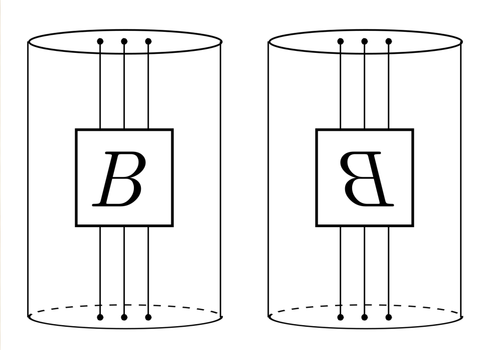

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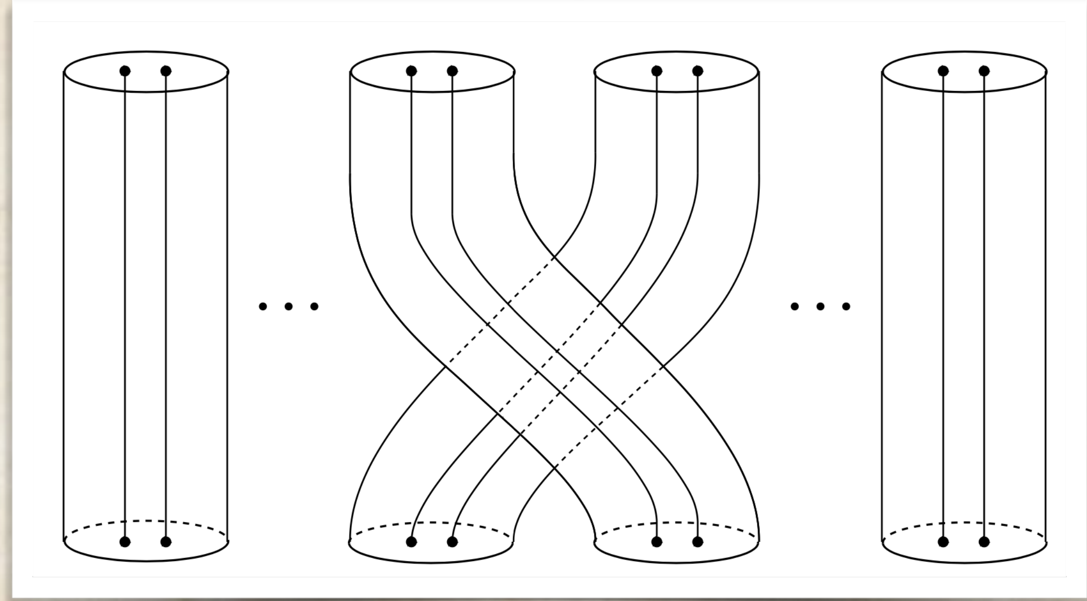

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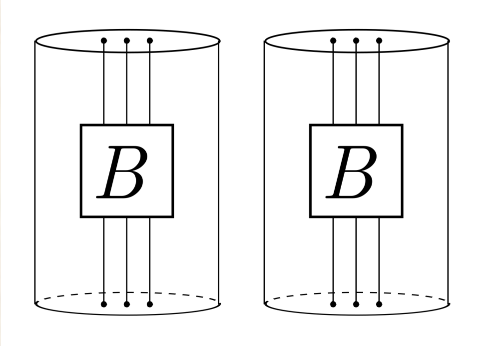

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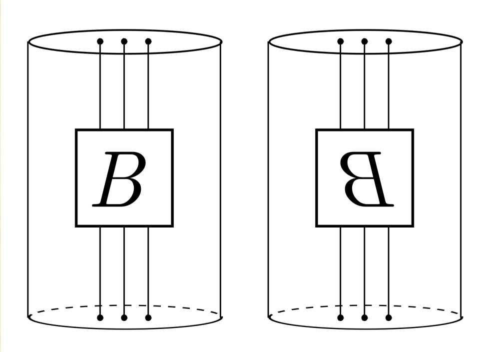

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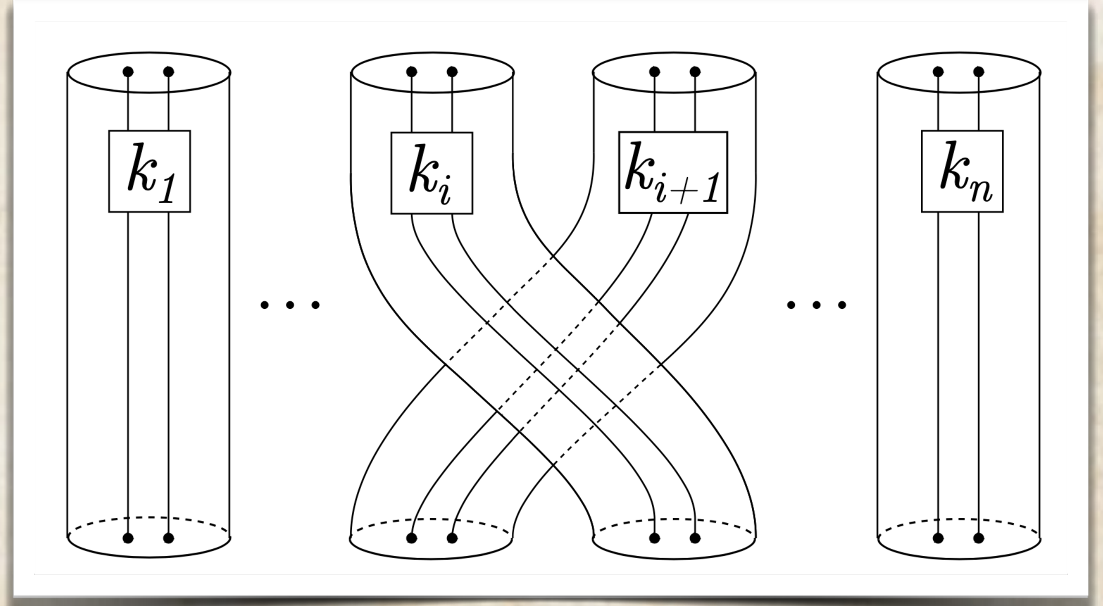
cf. Chen–Salter



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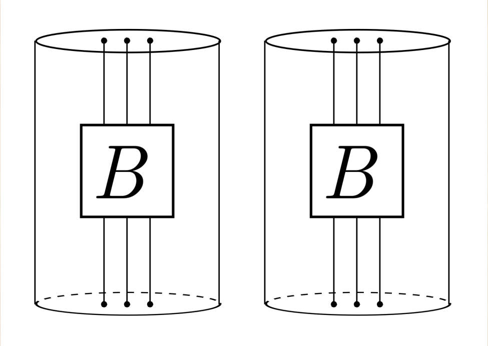

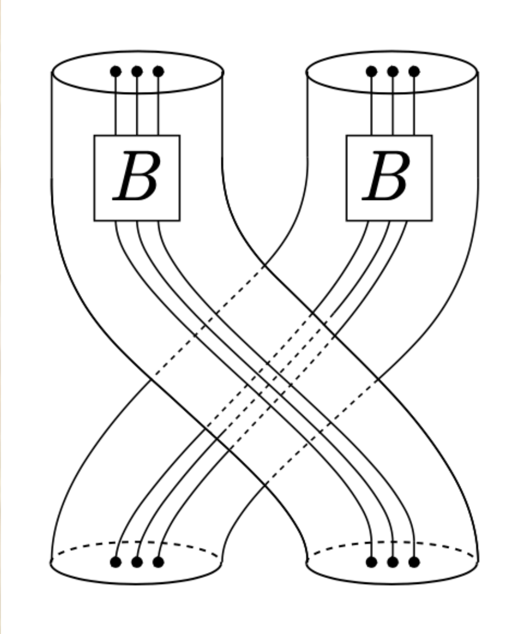

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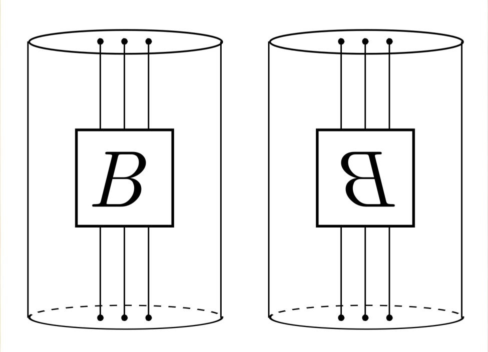

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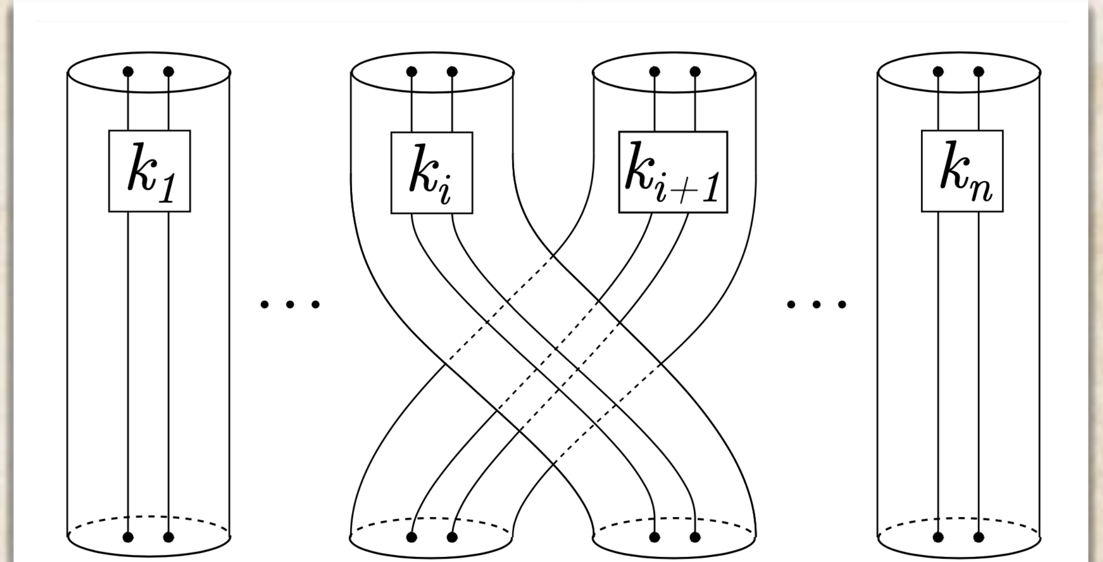

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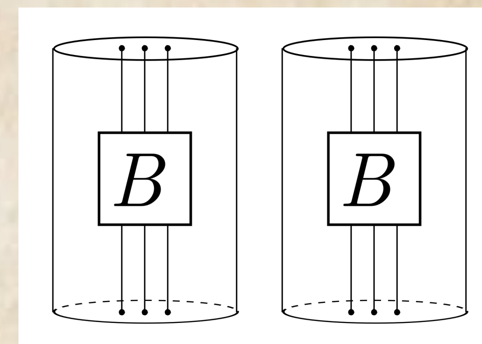

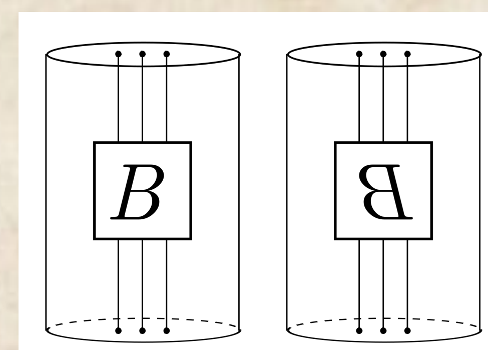

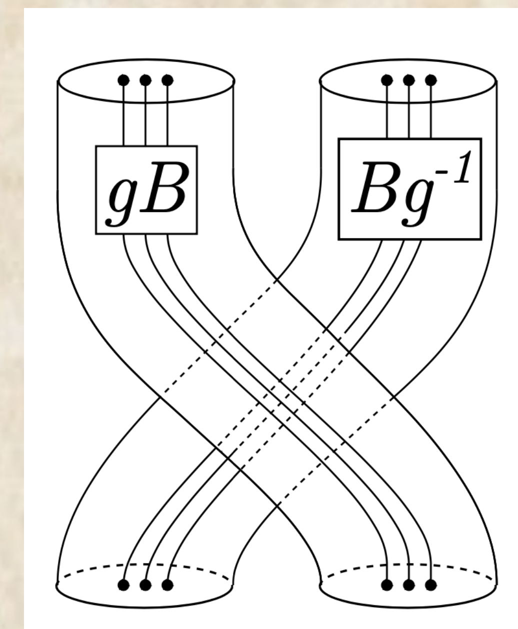

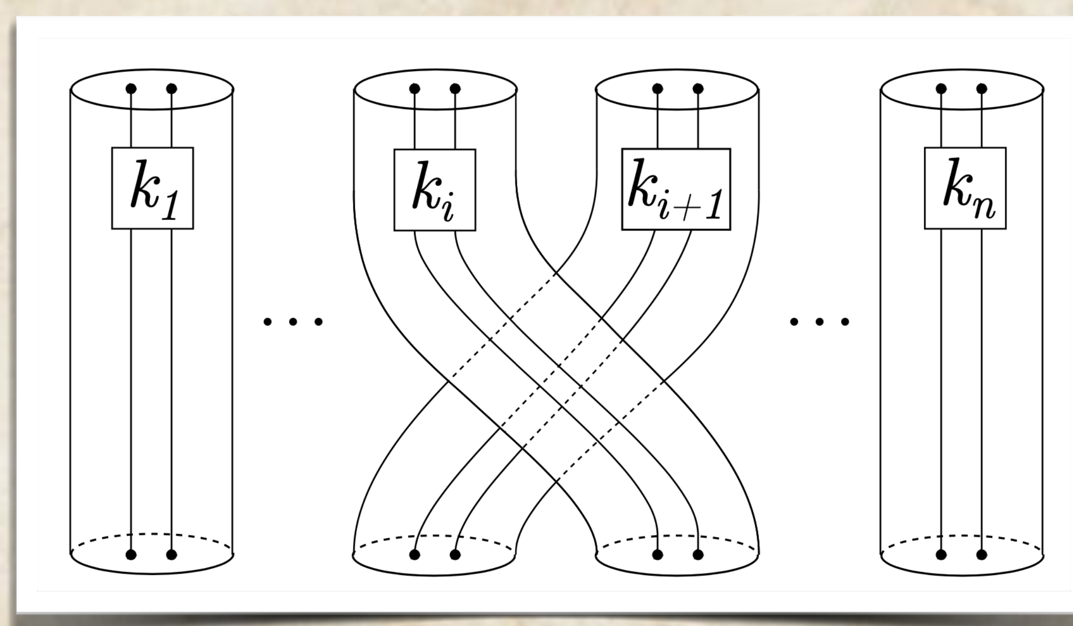

...and more homomorphisms

Given $\rho : B_n \to B_{2n}$ we can:

- Post-compose by $\tau \in \mathrm{Aut}(B_{2n})$
- *Transvect* by $z \in Z(B_{2n})$: Charney–Crisp

$$\rho^z(\sigma_i) = \rho(\sigma_i)z$$

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Standard homomorphisms

- Trivial $\sigma_i \mapsto 1$
- Inclusion $\sigma_i \mapsto \sigma_i$
- Diagonal inclusion $\sigma_i \mapsto \sigma_i \sigma_{n+i}$
- Flip diagonal inclusion $\sigma_i \mapsto \sigma_i \sigma_{n+i}^{-1}$
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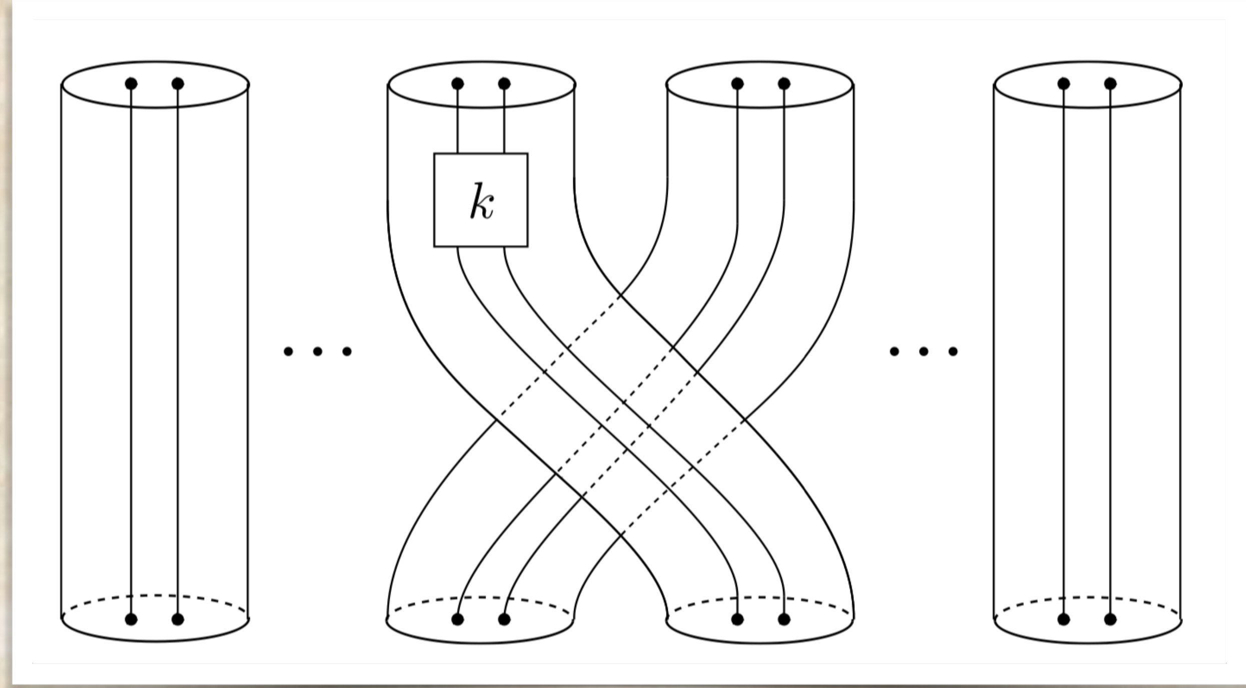

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Main Theorem 1

Theorem (Chen–Kordek–M). Let $n \geq 5$. Any $\rho : B_n \to B_{2n}$ is equivalent* to *exactly* one of the standard homomorphisms.

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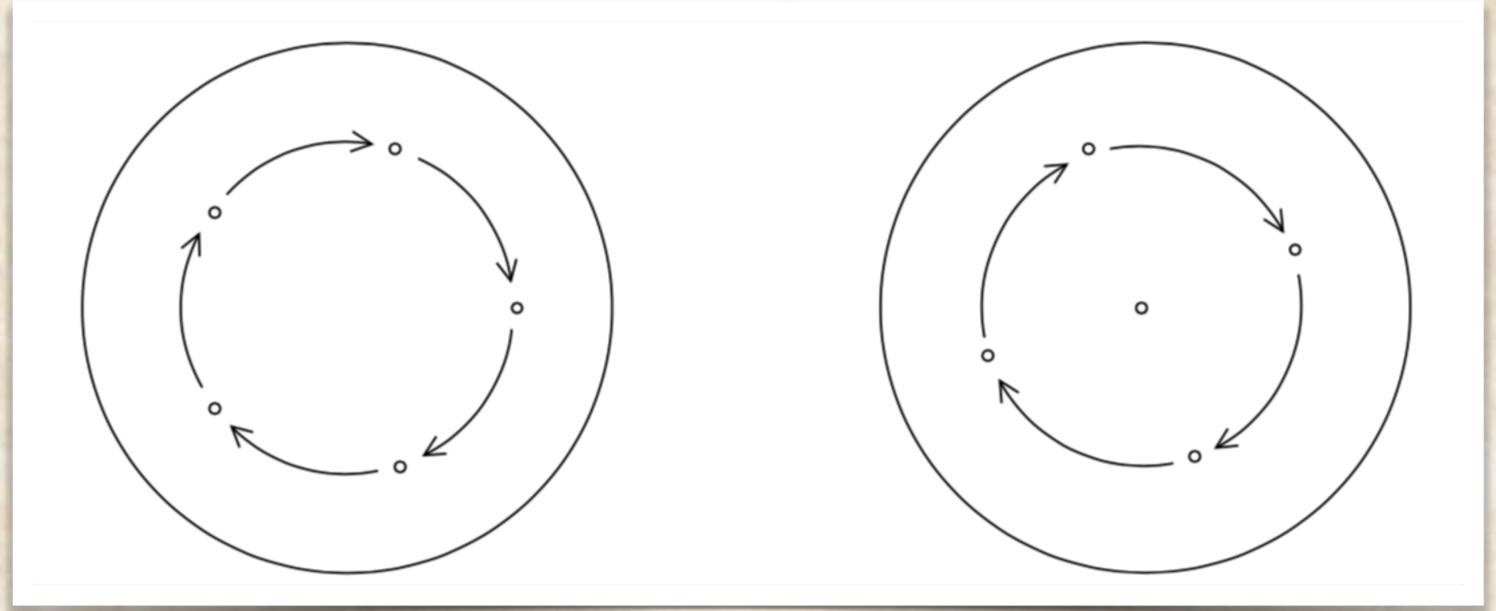
Consequence: classification of

$$B_n \to B_m, \; m \leq 2n$$

Proof of Main Theorem 1: Special maps

Periodic braids



α_5 β_5

$\alpha_n^n = \beta_n^{n-1}$ generates $Z(B_n)$

Special maps

Thm (Lin). A holomorphic $\mathrm{Poly}_n \to \mathrm{Poly}_m$ induces a map $B_n \to B_m$ that is special: periodics map to periodics.

Thm (Lin). If $n(n-1) \nmid m(m-1)$ any special $B_n \to B_m$ is cyclic.

We give a new proof.

Special maps

Sample case. Consider $\rho : B_5 \to B_7$ and say

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So B_5' maps to $Z(B_7)$, and we conclude that B_5' has abelian image, hence ρ is cyclic.

Main Theorem 2:
Shrinking the domain

Shrinking the domain

Having classified maps $B_n \to B_n$

we would like to classify maps $\Gamma \to B_n$

for certain $\Gamma \leqslant B_n$.

The commutator subgroup

Signed word length gives a map

$$L : B_n \to \mathbb{Z}$$

This is the abelianization, so:

$$B_n' = \ker(L)$$

Connection to polynomials

$\mathrm{Poly_n}$ = space of monic, square free polynomials of degree n

$\pi_1(\mathrm{Poly}_n) \cong B_n$

$\mathrm{SPoly_n}$ = subspace with discriminant 1

$\pi_1(\mathrm{SPoly}_n) \cong B_n'$

Lin's questions

Is every endomorphism $B'_n \to B'_n$...

1. injective?

2. an automorphism of B'_n?

3. the restriction of an endomorphism of B_n?

4. the restriction of an automorphism of B_n?

What's harder about B_n'?

Given $B_n \to B_n$, we obtain

$$B_n \to B_n \to S_n$$

Can then apply Artin's theorem. We did not have a classification of maps

$$B_n' \to S_n$$

An equivalence relation

Two homomorphisms

$$\rho : G \to H \quad \text{and} \quad \sigma : G \to H$$

are *equivalent* if there is $\alpha \in \mathrm{Aut}(H)$ with

$$\rho = \alpha \circ \sigma$$

Main Theorem 1

Theorem (Kordek–M). Let $n \geq 7$. *Any*

$$\rho : B_n' \to B_n$$

is either trivial or equivalent to inclusion.

Answers the four questions of V. Lin

Related results

Lin '04. Any $\rho : B_n' \to B_m,\ m < n$ is trivial

Orevkov '17. $\mathrm{Aut}(B_n') \cong \mathrm{Aut}(B_n)$

McLeay '18. New proof of Orevkov's result

Orevkov '20. Extension to n=*4,5,6*

Also...

The 2nd result (almost) follows from 1st:

$$B_{n-2} \hookrightarrow B'_n \to B_n$$

The 1st result restricts the composition...

It is also the case the the proof of the second result can be extended to prove the first (forthcoming work with Caplinger)

Proof of Main Theorem 2: Totally symmetric sets

Totally symmetric sets

$G =$ group

$X = \{x_1, \ldots, x_k\} \subseteq G$ is a totally symmetric set if

- the x_i commute pairwise, and
- any permutation of X is achieved by conjugation in G

Example. $\{\sigma_1, \sigma_3, \ldots\} \subseteq S_n$ or B_n

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Totally symmetric sets

$X = \{x_1, \ldots, x_k\} \subseteq G$ is a TSS if

- the x_i commute pairwise, and
- any permutation of X is achieved by conjugation in G

Fundamental Lemma. Under a homomorphism, X maps to a singleton or a TSS of size k.

The TSS Blueprint

To classify maps $\rho : G \to H$:

1. Find a large TSS $X=\{x_1,\ldots,x_k\}$ in G

2. Classify all large TSSs in H

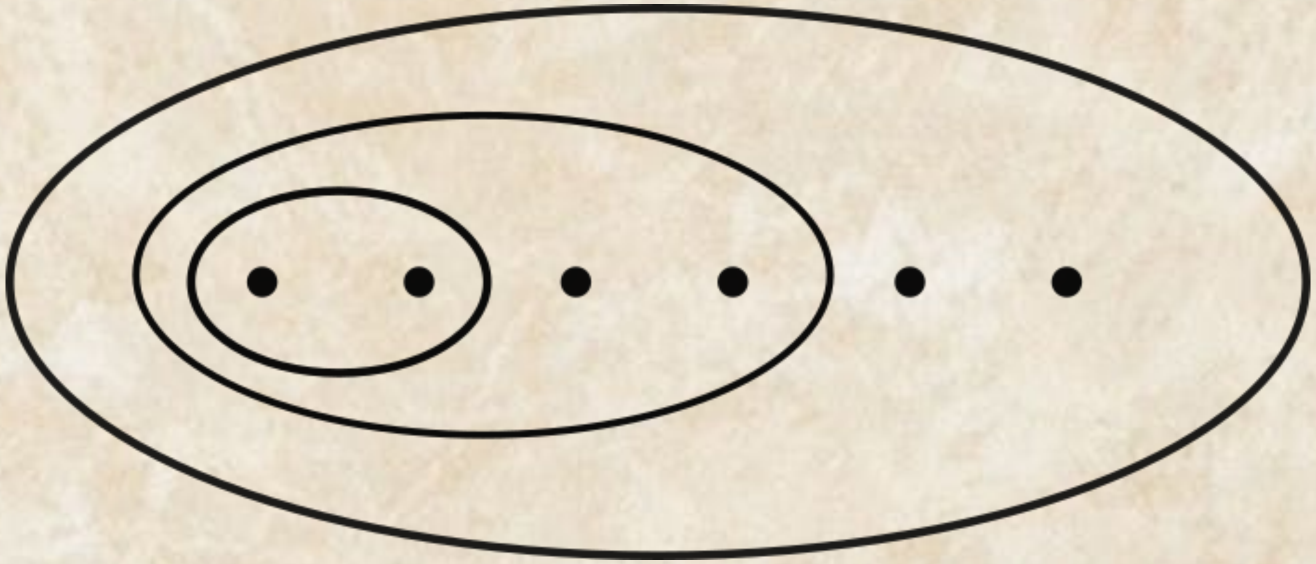
3. If X maps to singleton, then $\langle\langle x_1 x_2^{-1} \rangle\rangle \subseteq \ker\rho$
$\rightsquigarrow$ try to show the kernel is large

4. Otherwise, try to show ρ is standard

Totally symmetric sets in B_n

Classifying TSSs in B_n

Canonical reduction system for a braid:



Birman–
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Each complementary region is either periodic (=rotation) or irreducible.

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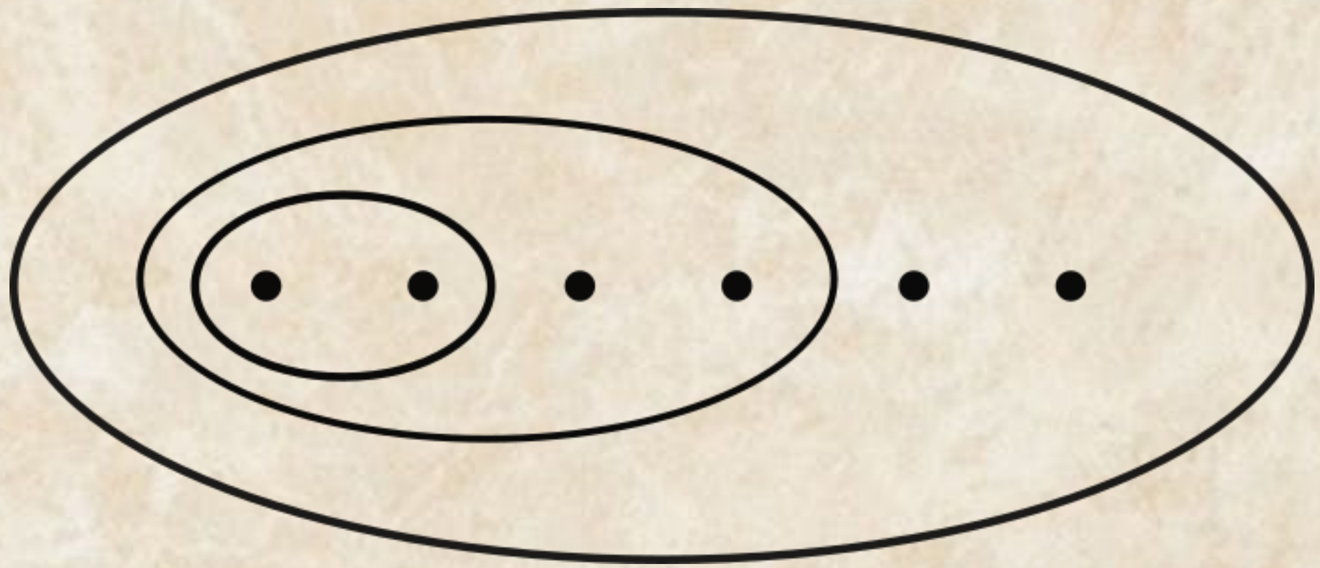

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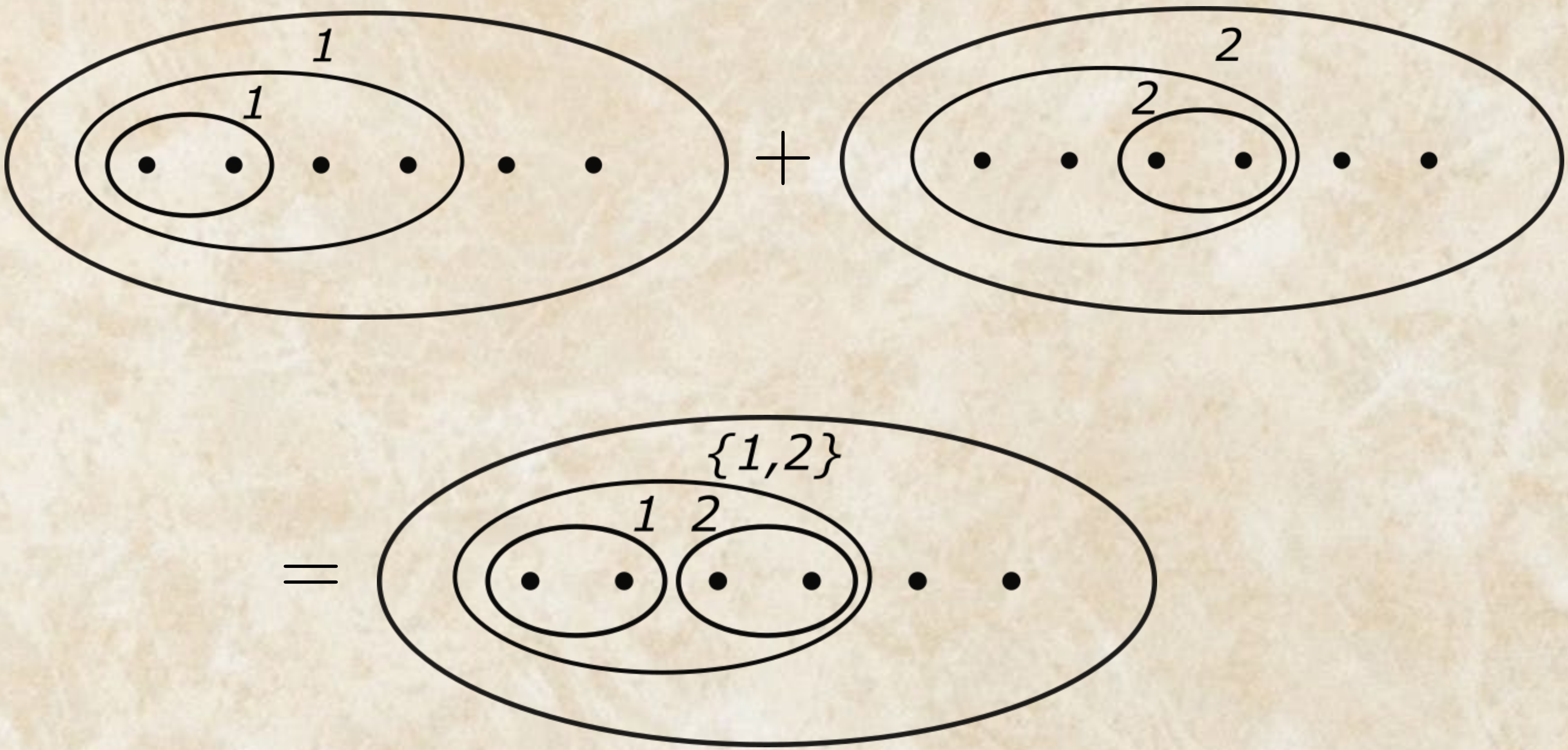
Fact. Commuting $\implies$ disjoint CRSs

Classifying TSSs in B_n

Combining canonical reduction systems:



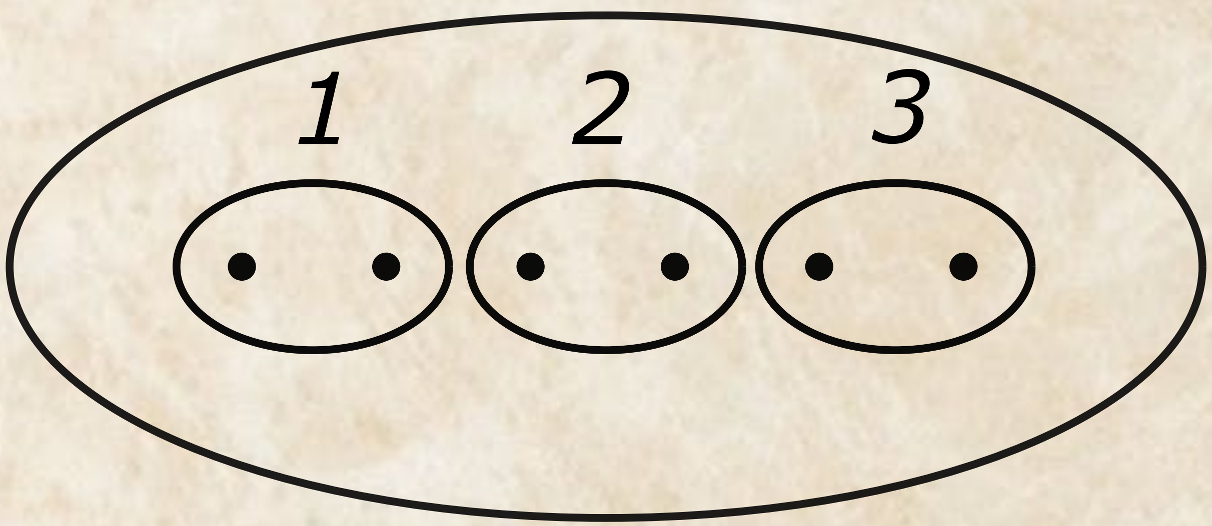
Classifying TSSs in B_n

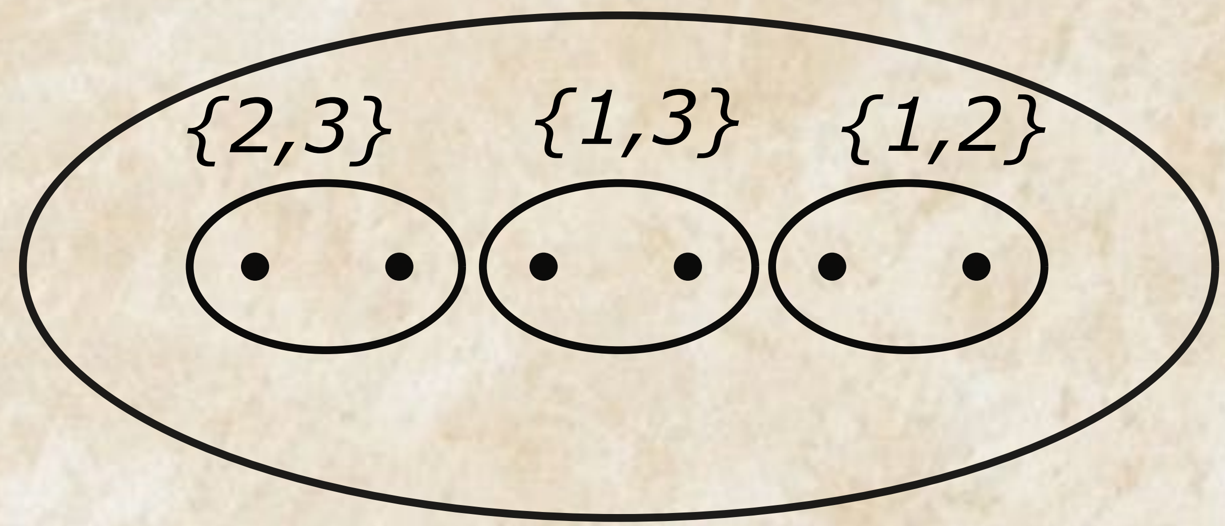
Canonical reduction system functor:

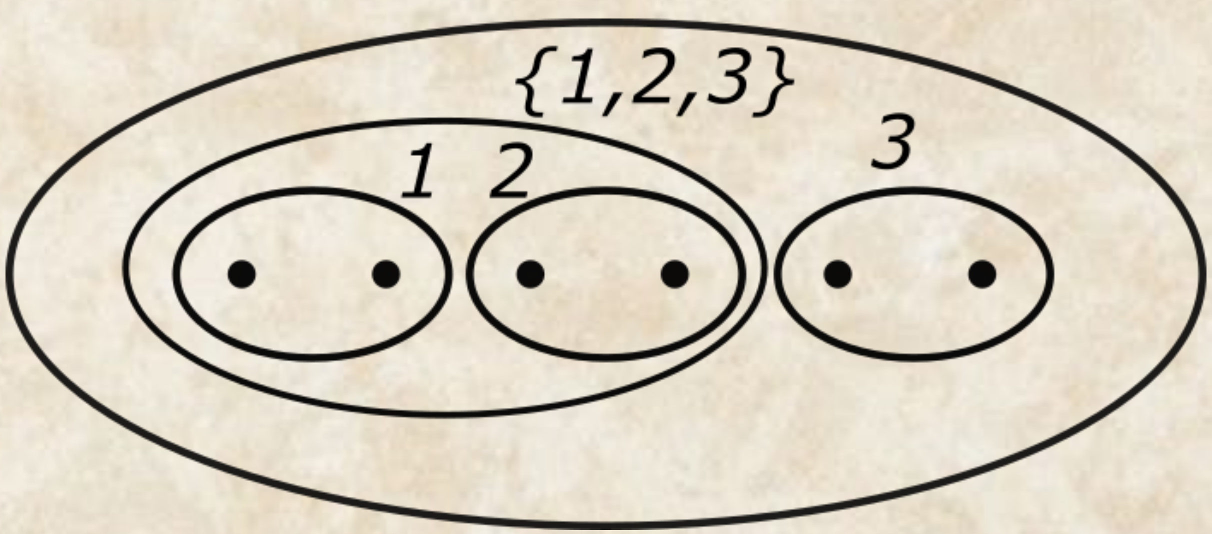
Totally symmetric set $\longrightarrow$ Totally symmetric labeled multicurve

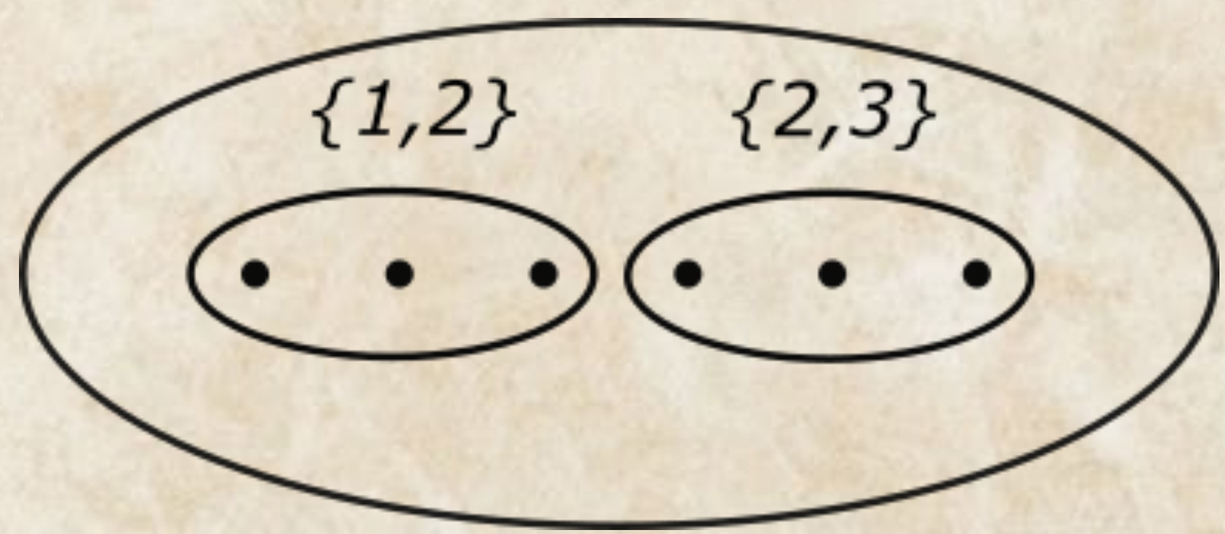
Symmetry: any relabeling induced by Σ_k can be realized by a homeo.

Examples and non-examples of totally symmetric labeled multicurves

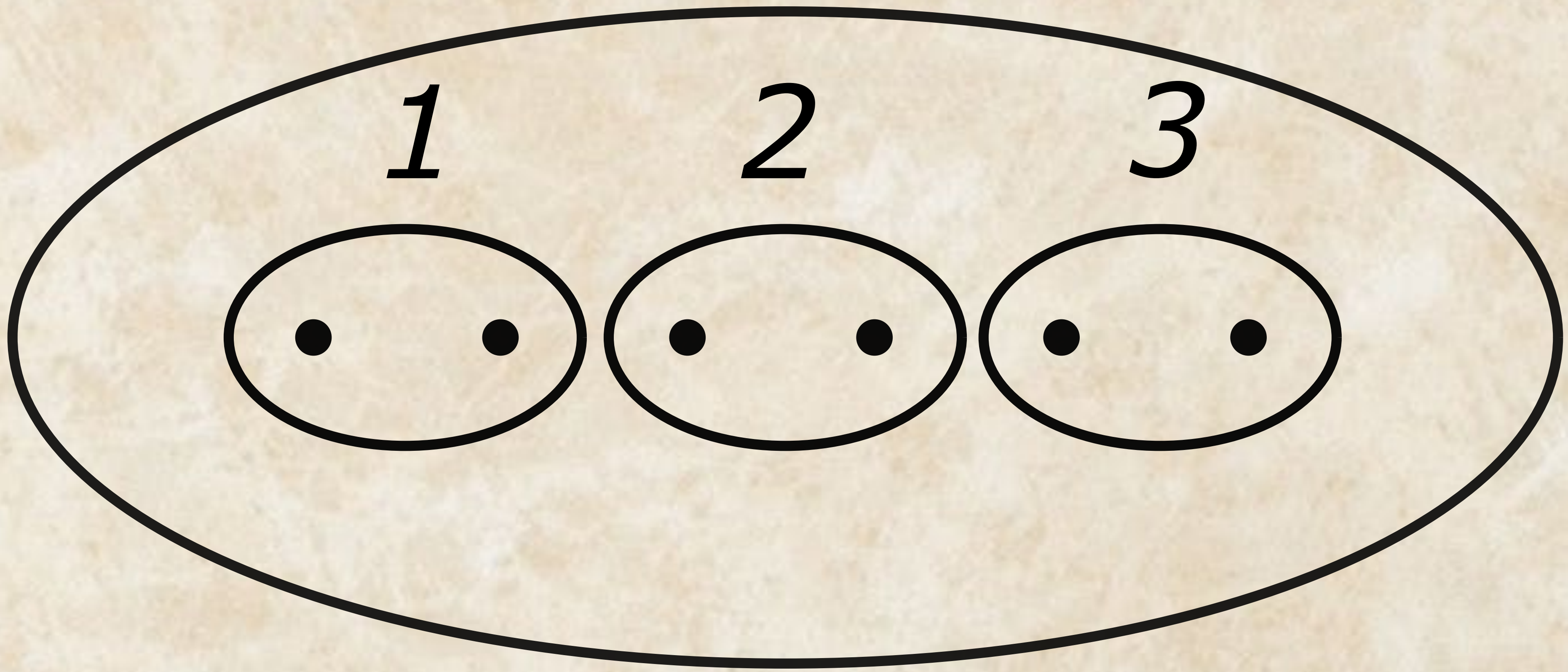








From totally symmetric labeled multicurves to braids



The TSS Blueprint

To classify maps $\rho : B_n \to B_n$:

1. $\{\sigma_1, \sigma_3, \ldots, \ldots\}$ is a TSS in B_n

2. We classify all large TSSs in B_n

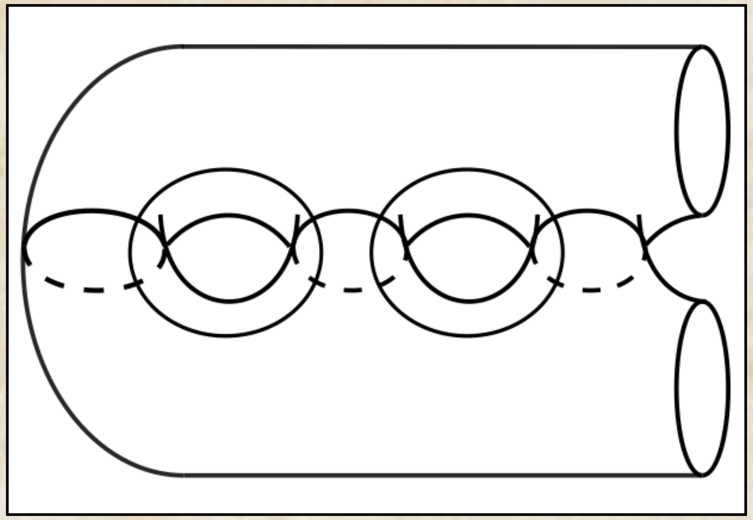
3. If X maps to singleton, $B_n' = \langle\langle \sigma_1 \sigma_3^{-1} \rangle\rangle \subseteq \ker \rho$
$\rightsquigarrow$ so ρ cyclic

4. Otherwise, ρ is equivalent to the identity

Other results on totally symmetric sets

Other results about TSS's

Chen–Mukherjea '20. Classification of maps from B_n to $\mathrm{Mod}(S_g)$ for $g < n\text{-}2$.



cf. Birman–Hilden

Li–Partin '19. Classifications of large TSS's in free groups, dihedral groups, solvable groups, direct products, free products, etc.

Other results about TSS's

Caplinger–Salter '22. TSS's in $\mathrm{GL}_n(\mathbb{Z})$. Friday!

Caplinger '22. New understanding of $\mathrm{Aut}(S_n)$

Uses the TSS: $\{(1\ i)\}$

Other results about TSS's

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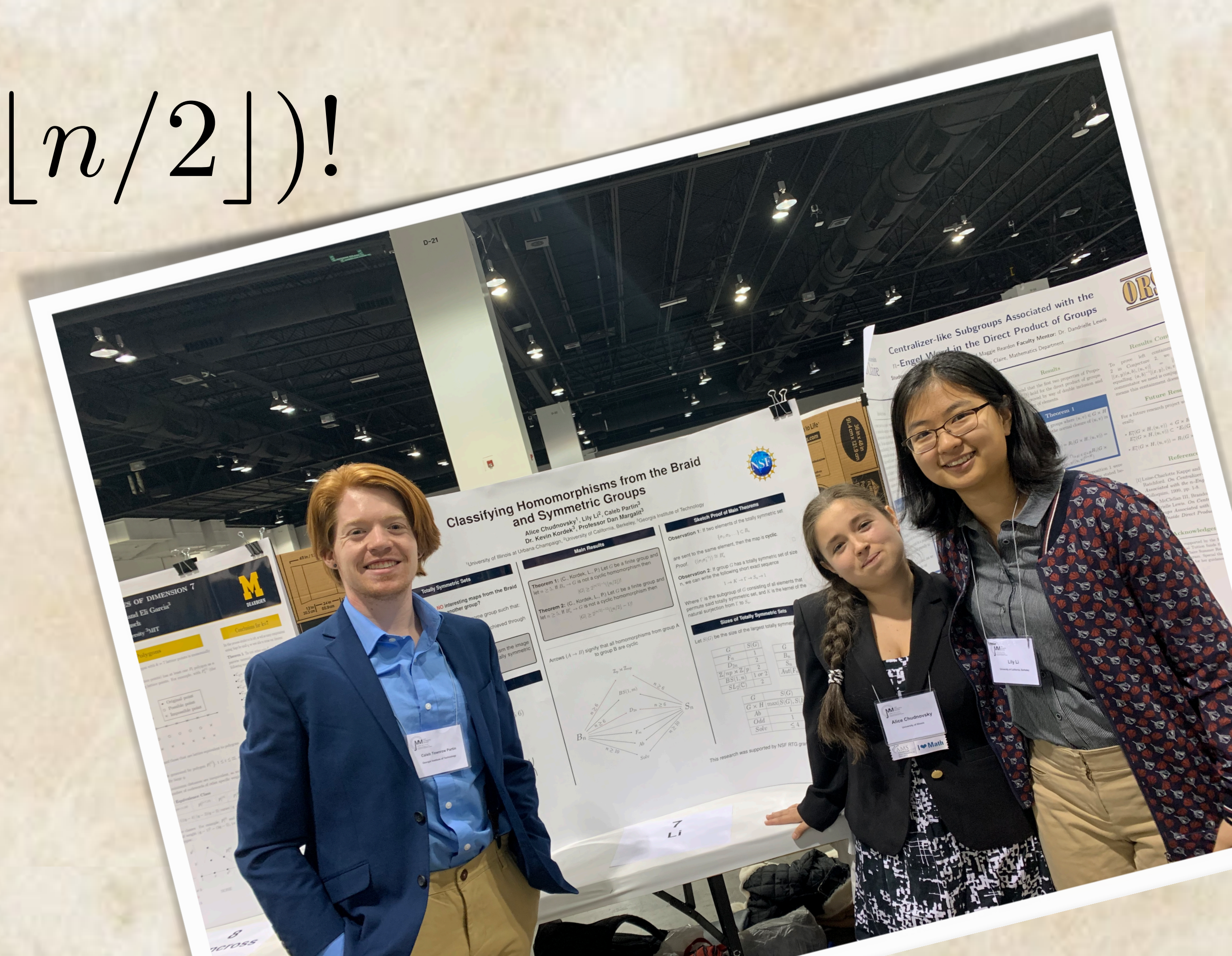

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Caplinger–Kordek '20. Conjecture true for $n{=}5{,}6$.

Kolay '21. The conjecture is true.

Kolay's proof

Theorem (Kolay '21). The smallest non-cyclic quotient of B_n is S_n.

Proof. Induction on n. The standard $\binom{n}{2}$ half-twists satisfy the fundamental lemma. Apply the orbit-stabilizer theorem to $\rho(B_n)$ acting by conjugation on $\rho(\sigma_1)$. Count:

$$\binom{n}{2} \cdot 2 \cdot (n-2)!$$

Some directions

Some Directions

1. Expand the range further. We conjecture that essentially all maps $B_n \to B_m$ are reducible (iterated cablings).

2. Restrict the domain further. Are all maps $G \to B_m$ geometric when $G \leqslant B_n$ is sufficiently rich?

3. Investigate TSS's further.