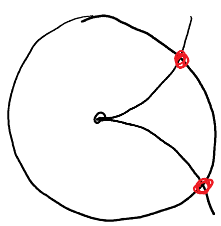
ICERM Braids in Symplectic and Algebraic Geometry: Unexpected Fillings Part I

Wednesday, March 2, 2022 2:51 PM

Joint with Olga Plamenevskaya

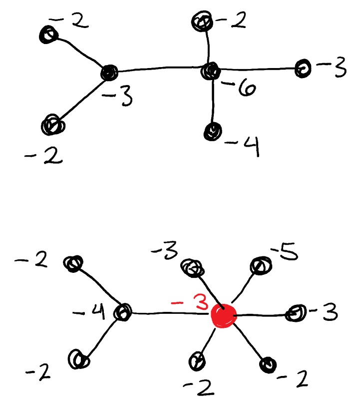
Contact link of a rational surface singularity with reduced fundamental cycle (RFC)

(Y^3, ξ)



Singular $\xrightarrow[\text{resolution}]{\text{deformation}}$ Smooth

Tree, genus 0 components, no bad vertices



Theorem [Ghiggini-Golla-Plamenevskaya]

The contact link of a normal surface singularity is supported by a planar open book if and only if it is a rational singularity with reduced fundamental cycle.

Algebraic deformations of RFC singularities (and more generally sandwiched singularities) were studied in the 90's by de Jong and van Straten:

<u>Theorem</u> [dJvS] Let $(X,0)$ be a RFC singularity. Then there is an associated plane curve singularity $\mathcal{C}$ with weighted components and

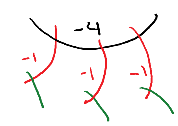
$$\left\{ \begin{array}{c} \text{smoothings} \\ \text{of } (X,0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{picture deformations} \\ \text{of } \mathcal{C} \end{array} \right\}$$

Plane curve deformations are much easier to understand!

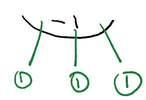
$\underline{(X,0) \to (\mathcal{C}, \ell)}$ weights

<u>Resolution of $(X,0)$</u>

Add -1 curves with <u>"curvettas"</u>

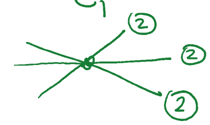


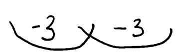
<u>Blowdown</u>

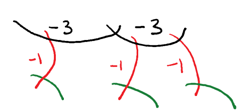


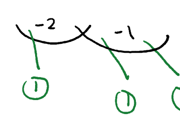
Record # of blowdowns on each curvetta for weights

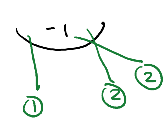
$\mathcal{C}_1$



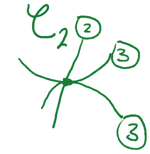








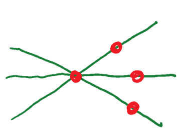
$\mathcal{C}_2$

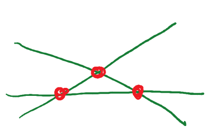


Picture Deformations

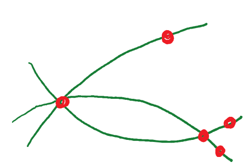
- Algebraic deformation
- "Immediate"
- Singularities are transverse intersections (multipoints ok)
- Weights = # marked points. All intersections are marked

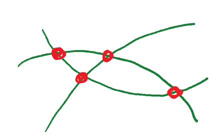
$\mathcal{C}_1$:





$\mathcal{C}_2$:





Symplectic/Stein fillings of the link (Y,ξ) generalize Milnor fibers
(a priori)

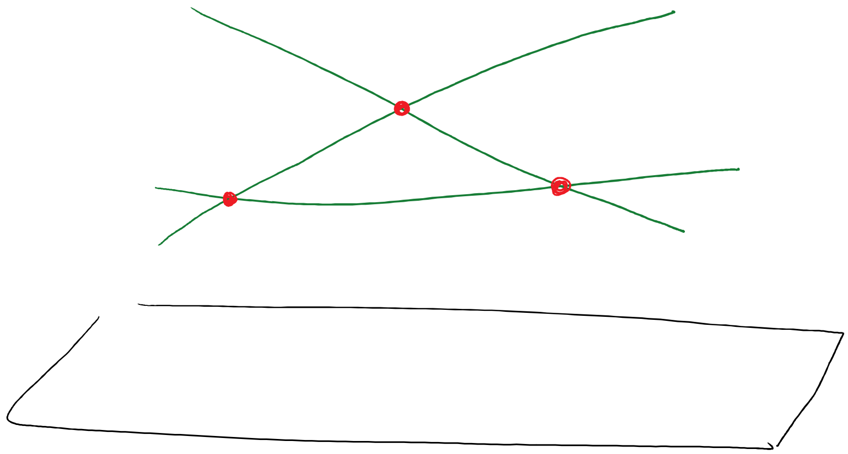
Can we understand all symplectic fillings?

<u>Theorem</u> [Plamenevskaya-S.] (Y,ξ) link of $(X,0)$. $(\mathcal{C},\ell)$ dJvS associated curve

$$\left\{\begin{array}{c}\text{symplectic fillings}\\ \text{of } (Y,\xi)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{smooth graphically homotoped}\\ (\mathcal{C},\ell)\end{array}\right\}$$

similar to picture deformations
but do not need to be
algebraic or immediate

<u>Proof</u>: "$\longleftarrow$" (Building a symplectic filling from a curve arrangement)



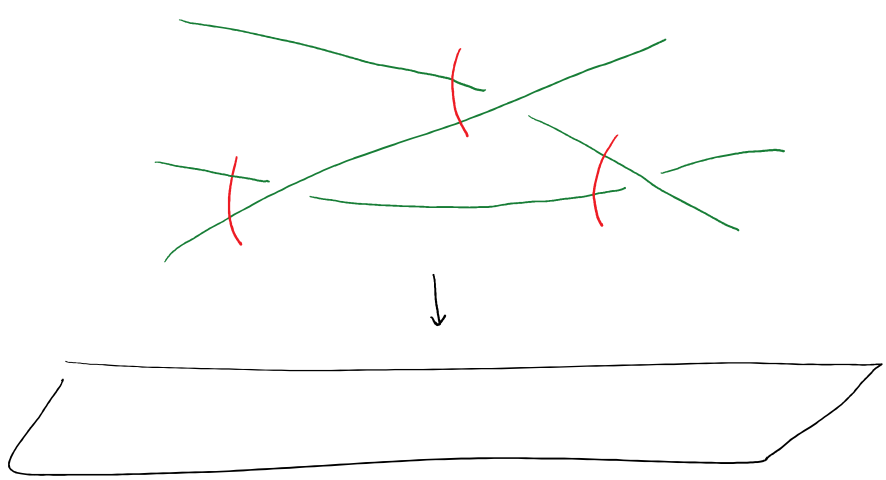
$$\mathbb{C}^2 \setminus \nu(\mathcal{C}) \longrightarrow \mathbb{C}$$

Generic fibers:
$\mathbb{C} \setminus \{m \text{ holes}\}$

Non generic fibers:
$\mathbb{C} \setminus \{k < m \text{ points}\}$

NOT a Lefschetz fibration

blow up at marked points

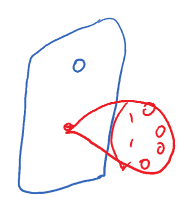


proper transform

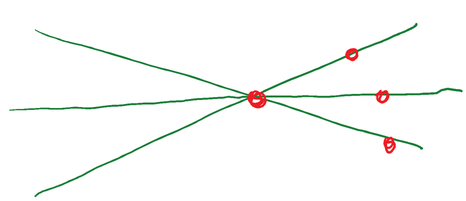
$$\mathbb{C}^2 \#_n \overline{\mathbb{CP}^2} \setminus \tilde{\mathcal{C}} \longrightarrow \mathbb{C}$$

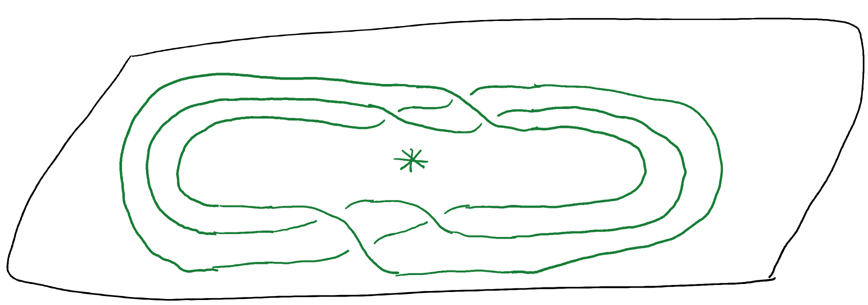
Generic fibers:
$\mathbb{C} \setminus \{m \text{ holes}\}$

Non generic fibers:
Nodal genus 0 w/ m holes



This is a Lefschetz fibration!





Monodromy of open book on the boundary is independent of the (compactly supported)

(compactly supported)
deformation/graphical homotopy

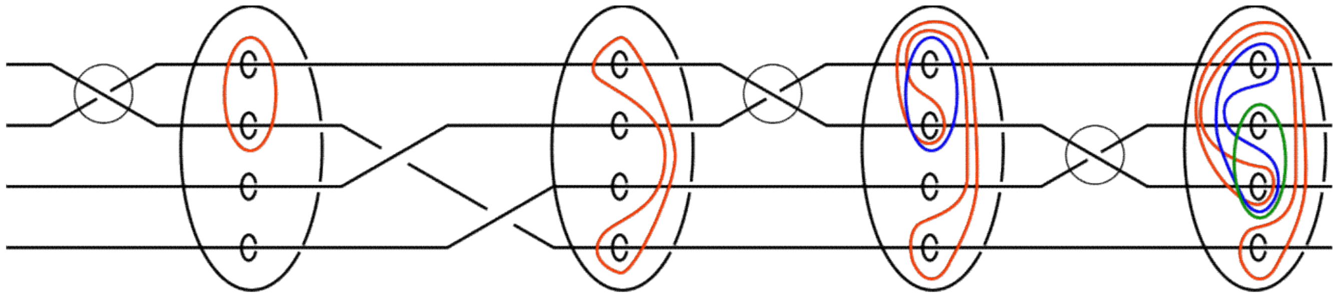
*Punctures vs. holes: check multiplicities -- uses weights

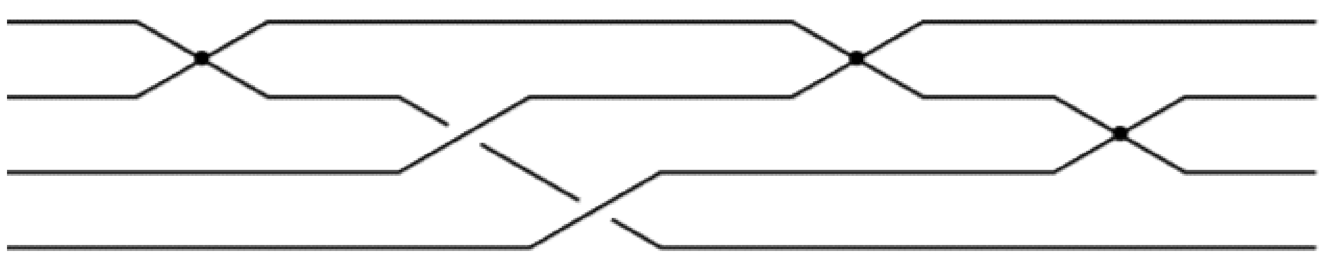
"$\longrightarrow$" (Building a curve arrangement from a symplectic filling)

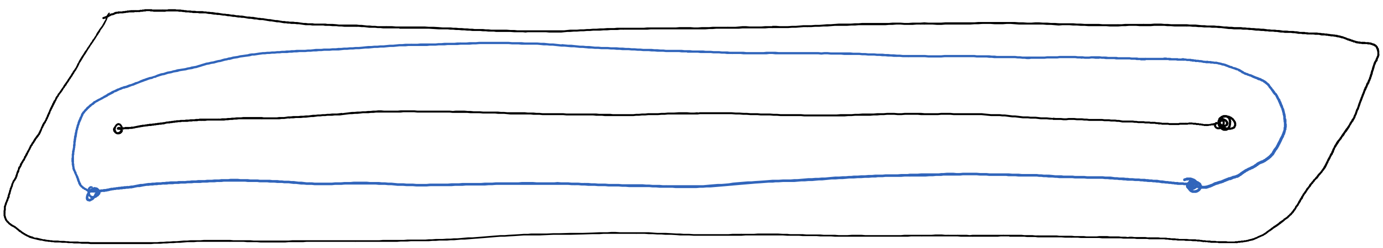
1. All symplectic fillings have planar Lefschetz fibrations with boundary a <u>fixed</u> planar open book ([Wendl, GGP])

   $\rightsquigarrow$ monodromy factorization

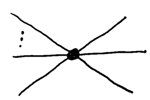
2. Build a <u>braided wiring diagram</u> from monodromy factorization







③ Extend braided wiring diagram to (symplectic) surfaces in $\mathbb{C}^2$

Local model extending  as in complex curves

(interpolate with cutoff fct)

Elsewhere just take product with $\mathbb{R}$

Check this surface is related to $\mathcal{C}$ by graphical homotopy:

- make the boundaries coincide (braids are isotopic by construction)
- then use straight line homotopy (as graphs)